BASIS PROPERTIES FOR SEMIGROUPS

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A universal algebra A is said to have the *basis property* (BP) if any two minimal generating sets (*bases*) for a subalgebra of A have the same cardinality. This property was studied by the author for inverse semigroups in **[5, 6]**. For instance free inverse semigroups have BP. When treated as universal algebras, a classical theorem of linear algebra states that vector spaces have BP. In this paper we study BP for semigroups.

It turns out that a more amenable property is the *strong basis property* (SBP): for any pair $U \subseteq V$ of subalgebras of A, any two "U-bases" for V have the same cardinality. (A *U-basis* for V is a set minimal with respect to the property that, together with U, it generates V). Then SBP implies BP; in **[5, 6]** necessary and sufficient conditions were found for an inverse semigroup to have SBP.

For all but a very restricted class of semigroups, those having SBP are determined. The main result is the following (Theorem 3.1).

THEOREM. *Let S be a semigroup containing no nontrivial $\mathcal{J}$-class with one-element $\mathcal{D}$-classes. Then S has* SBP *if and only if each non-null principal factor is either* (a) *a left zero or right zero semigroup or a group with* SBP *for subgroups,* (b) *one of the above with adjoined zero or* (c) *isomorphic with B_5, the combinatorial Brandt semigroup with five elements.*

The theorem applies to all regular semigroups, periodic semigroups etc. We conjecture that in fact the main statement is valid for *all* semigroups. This is equivalent to the conjecture (§ 2) that no [0-] simple idempotent-free semigroup on which both $\mathcal{R}$ and $\mathcal{L}$ are trivial has BP.

Comparing the theorem with the results of **[6]** it is notable that there are many inverse semigroups which have SBP when regarded as inverse semigroups but not when regarded as semigroups. On the other hand we show that for groups and completely regular semigroups it does not matter whether they are regarded as type $\langle 2 \rangle$ or type $\langle 2, 1 \rangle$.

Similarly (§ 5) a monoid has SBP (or BP) as a monoid if and only if it does as a semigroup. From the main theorem we may then deduce the result of J. Doyen **[3]** that any periodic $\mathcal{R}$-trivial monoid has BP. It was that result which rekindled the author's interest in the topic of basis properties.

1. Global results. We consider first some generalities from universal algebra which will enable us to correlate basis properties for semigroups, monoids, groups, inverse semigroups etc. The reader is referred to **[1]** for more details.

The lattice of subalgebras of an algebra A always has a least element: if A has no nullary operations (for instance in the cases of semigroups and inverse semigroups) this is the empty subalgebra; otherwise (for instance for monoids and groups) it is the subalgebra generated by the nullary operations. (By this convention it is clear that SBP implies BP on A). The subalgebra *generated* by a subset X is denoted $\langle X \rangle$. If U is a subalgebra of A and X is a subset we will often abbreviate $\langle U \cup X \rangle$ to $\langle U, X \rangle$, and to $\langle U, x, y, \ldots \rangle$ if $X = \{x, y, \ldots\}$.

The key to the utility of the *strong* basis property is the following lemma, implicit in the proof of [**5**, Theorem 2.3] for inverse semigroups. The reader is referred to [**8**] for its proof and for an axiomatic version of SBP—the "weak exchange property".

LEMMA 1.1. *An algebra A has* SBP *if and only if for each subalgebra U of A and x, y, a in A, $\langle U, x, y \rangle = \langle U, a \rangle$ implies either $a \in \langle U, x \rangle$ or $a \in \langle U, y \rangle$.*

From now on we specialize to semigroups. Thus BP and SBP will always refer to subsemigroups, unless otherwise specified. For a semigroup A we may further assume (c.f. the proof of [**5**, Theorem 2.2]) that x, y and a belong to the same $\mathcal{J}$-class of A.

In [**6**, Proposition 2.2] it was shown that, for inverse semigroups, BP and SBP are preserved by Rees ideal quotients. It is easily verified that the proof is valid for semigroups. Since BP and SBP are inherited by subsemigroups the following important "global" result is clear. Recall that the *principal factor* associated with a $\mathcal{J}$-class J of a semigroup S is the Rees quotient S^1JS^1/I, where $I = S^1JS^1 \backslash J$. For this and other elementary semigroup-theoretic notions and definitions the reader is referred to [**2**].

PROPOSITION 1.2. *The* BP *and* SBP *are inherited by principal factors.*

The principal factors of a semigroup are either *null*, 0-*simple* or *simple* (when J is the minimum $\mathcal{J}$-class). In the next section we specialize to these cases. For *inverse* semigroups it was shown [**6**, Theorem 2.3] that for SBP the converse of the proposition holds: if each principal factor has SBP (as an inverse semigroup) then so does the semigroup itself. (For BP the converse fails.) Whether the same holds for semigroups we do not know, though the answer is "yes" if the conjecture of § 2 is true.

One final preliminary observation will be useful. If a semigroup S has BP or SBP then so does the semigroup S^0. Conversely, let S be a semigroup with 0 which has no zero divisors. If S has BP or SBP then so does the subsemigroup $S \backslash \{0\}$.

2. Simple and 0-simple semigroups. The techniques of this section follow a common theme: to prove that a certain class of semigroups does *not* have SBP we show that each of its members contains (an isomorphic copy of) a particular semigroup without SBP. Thus we study counterexamples. First we consider groups (regarded as semigroups).

LEMMA 2.1. *The infinite cyclic group Z does not have* BP.

Proof. As a semigroup (actually as a monoid) Z may be presented as $\langle a, b \mid ab = ba = 1 \rangle$. Two bases are $\{a, b^6\}$ and $\{a^{10}, a^{15}, b^6\}$; for $a = a^{10}a^{15}(b^6)^4$ and the relative primality of 10, 15 and 6 ensures that no two of a^{10}, a^{15} and b^6 generate Z. (The choice of $\{a, b^6\}$ rather than $\{a, b\}$ will be useful later). ■

Thus a group G with BP is *periodic*. In fact this is also true if G has BP for

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subgroups. Since every subsemigroup is then a subgroup it is therefore immaterial whether we regard G as a group or a semigroup as far as basis properties are concerned. For a study of these properties in groups see [**6**].

LEMMA 2.2. *The bicyclic semigroup* $B = \langle a, b \mid ab = 1 \rangle$ *does not have* BP.

Proof. The equation $a^{25}b^{24} = a$ is valid so the two sets $\{a, b^6\}$ and $\{a^{10}, a^{15}, b^6\}$ generate the same subsemigroup. Each is a basis because their namesakes in Z are, by the proof of Lemma 2.1. ■

A similar argument will be applied to two further semigroups A and C later in the section. For now we make use of the theorem of Andersen [**2**, Theorem 2.54] that any [0-] simple semigroup which is not completely [0-] simple and which contains a nonzero idempotent contains an isomorphic copy of B. This proves

PROPOSITION 2.3. *A* [0-] *simple semigroup with* BP *is either completely* [0-] *simple or idempotent-free* (*has no nonzero idempotents*).

By the concluding remarks of § 1, in the case *with* idempotents we only need consider completely simple semigroups and *proper* completely 0-simple semigroups: those with no zero divisors.

THEOREM 2.4. (i) *A completely simple semigroup has* SBP *if and only if it is either a left zero or right zero semigroup or is a group with* SBP.

(ii) *A proper completely* 0-*simple semigroup has* SBP *if and only if it is isomorphic to the five-element combinatorial Brandt semigroup* B_5.

Proof. Sufficiency. Let L be a left zero semigroup. If U and V are subsemigroups with U contained in V then $V \setminus U$ is the unique U-basis for V. So L has SBP. Dually, every right zero semigroup has SBP. The group case follows from the earlier remarks.

The elements of B_5 are $\{x, y, e, f, 0\}$, where $e^2 = e$ and $f^2 = f$. It is readily verified that for any subsemigroup U, the unique U-basis for B_5 is $\{x, y\} \setminus U$. A similar argument is valid for each subsemigroup V which contains x or y (only B_5 contains both). The remaining nonempty subsemigroups V are semilattices, for which SBP is satisfied by [**6**]. Thus B_5 has SBP.

Necessity. Let S be a completely 0-simple semigroup (not necessarily proper) with SBP. We proceed by systematically barring the undesirable configurations of nonzero $\mathscr{H}$-classes until only those corresponding to (i) and (ii) of the theorem remain.

Step 1. *If S has distinct $\mathscr{R}$-related idempotents then it is a right group with adjoined zero.*

Suppose e and g are such idempotents. If S is not as asserted then there is a non-zero idempotent f with $R_f \neq R_e$. Essentially the only two configurations which may arise are those shown in the "egg-box" pictures ([**2**, § 2.1]) of Fig. 1; there the shaded boxes necessarily contain idempotents whilst the others may or may not.

A technically useful observation is that $L^0 (= L \cup \{0\})$ and R^0 $(= R \cup \{0\})$ are subsemigroups for each nonzero $\mathscr{L}$-class L and $\mathscr{R}$-class R.

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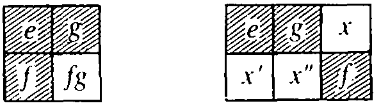

Figure 1

Now in the former configuration $\langle e, f, g\rangle = \langle e, fg\rangle$, since $f = fe = (fg)e$ and $g = eg = e(fg)$. But $e, f \in L_e^0$ and $e, g \in R_e^0$, so $\{f, g\}$ and $\{fg\}$ are each $\langle e\rangle$-bases for $\langle e, f, g\rangle$, contradicting SBP. (Note that if H_{fg} contains an idempotent, BP will not be sufficient, for then $e \in \langle f, g\rangle$).

In the latter configuration let x' and x'' denote the inverses of x in the $\mathscr{H}$-classes shown. Then $\langle e, x, x''\rangle = \langle e, x, g, x'\rangle$, since $x'' = x'g$ and $g = xx''$, $x' = x''e$. Again, $\langle e, x, g\rangle \subseteq R_e^0$ and $\langle e, x, x'\rangle \subseteq L_e^0 \cup L_x^0$, so $\{x''\}$ and $\{g, x'\}$ are $\langle e, x\rangle$-bases for $\langle e, x, x''\rangle$, contradicting SBP.

Thus a completely simple semigroup with SBP is either a right group or, by duality, a left group; a proper completely 0-simple semigroup with SBP is inverse.

Step 2. *A right group R with* SBP *is either a group or a right zero semigroup.*

Suppose R contains distinct idempotents e and f and a nonidempotent x in H_e. Then $xf \in H_f$, $\langle e, xf\rangle = \langle e, x, f\rangle$ since $x = (xf)e$ and $f \in \langle xf\rangle$ (for H_f is periodic, by Lemma 2.1); and $\langle e, x\rangle \subseteq H_e$, $\langle e, f\rangle = \{e, f\}$, so $\{xf\}$ and $\{x, f\}$ are $\langle e\rangle$-bases for $\langle e, xf\rangle$, contradicting SBP.

Step 3. *A proper Brandt semigroup* (*completely* 0-*simple inverse semigroup*) *S with* SBP *has exactly two nonzero idempotents.*

Suppose S contains three nonzero idempotents e, f, g, and let x, y and inverses x', y' be as shown in Fig. 2(a). Then

$$\langle x, y, x', y'\rangle = \langle x, y, y'x'\rangle,$$

since $x' = fx' = y(y'x')$ and $y' = y'f = (y'x')x$. But $\langle x, y, x'\rangle \subseteq R_e^0 \cup R_f^0$ and $\langle x, y, y'\rangle \subseteq L_f^0 \cup L_g^0$, so $\{x', y'\}$ and $\{y'x'\}$ are $\langle x, y\rangle$-bases for $\langle x, y, x', y'\rangle$, contradicting SBP.

Step 4. *A proper Brandt semigroup with* SBP *has trivial subgroups.*

Suppose the nonzero $\mathscr{H}$-classes of S are as shown in Fig. 2(b), where a' is the inverse of a and $b \neq e$ (so $a'b \neq a'$). Then $\langle a, a', b\rangle = \langle a, a'b\rangle$ since $b = eb = a(a'b)$ and

$$a' = a'e = a'b^n = (a'b)b^{n-1} = (a'b)(a(a'b))^{n-1},$$

for some $n \geq 2$, (again using the periodicity of H_e). Again,

$$\langle a, a'\rangle = \{a, a', e, f, 0\}$$

and $\langle a, b\rangle \subseteq R_e^0$, so $\{a', b\}$ and $\{a'b\}$ are $\langle a\rangle$-bases for $\langle a, a', b\rangle$, contradicting SBP.

Combining these steps with the comments after Step 1 completes the proof of the theorem.

(a) (b)

Figure 2

The theorem covers many, if not most, familiar cases. For example, any [0-]simple semigroup which is regular or "group-bound" (some power of each element lies in a subgroup) or, in particular, periodic contains a nonzero idempotent. In fact this is true of "eventually regular" semigroups: those in which some power of each element is regular. (The proof follows that of [**2**, Theorem 2.55]). Further, if the following conjecture is true, it then covers all cases.

CONJECTURE. *No idempotent-free* [0-]*simple semigroup has* BP.

Strong evidence for this conjecture is provided by the following.

THEOREM 2.5. *No idempotent-free* [0-]*simple semigroup in which either* $\mathcal{R}$ *or* $\mathcal{L}$ *is nontrivial has* BP.

Proof. It is sufficient, by duality, to suppose S is an idempotent-free 0-simple semigroup in which $\mathcal{R}$ is nontrivial. By [**7**, Theorem 4.2], S contains an isomorphic copy of one of the two semigroups

$$A = \langle a, b \mid a^2b = a \rangle$$

and

$$C = \langle a, b \mid a^2b = a,\ ab^2 = b \rangle.$$

From the equation $a^2b = a$ it easily follows that $a^{25}b^{24} = a$ in either semigroup. So, as in the proof of Lemma 2.2,

$$\langle a, b^6 \rangle = \langle a^{10}, a^{15}, b^6 \rangle$$

and each of $\{a, b^6\}$ and $\{a^{10}, a^{15}, b^6\}$ is a basis for this subsemigroup (since Z satisfies these relations). ∎

So the conjecture remains to be verified in the [0-]simple $\mathcal{D}$-*trivial* case (when both $\mathcal{R}$ and $\mathcal{L}$ are trivial). Note that such a semigroup is necessarily idempotent-free if it has more than one element (more than two, if it has a zero). The author studied such semigroups in [**7**, § 6]. It seems that analogues, like those quoted above, of Andersen's theorem are needed to complete this case. An important semigroup in this context is

$$U = \langle a, b, c \mid b = ab^2c \rangle.$$

Clearly any 0-simple semigroup contains a quotient of U.

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LEMMA 2.6. *The semigroup U does not have* BP.

Proof. Clearly $U = \langle a, b, c \rangle = \langle a, ab, bc, c \rangle$. Consider the elements $\bar{a} = (1, 0)$, $\bar{b} = (-1, -1)$ and $\bar{c} = (0, 1)$ of the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$. These satisfy $\bar{b} = \bar{a}\bar{b}^2\bar{c}$, so the map $a \to \bar{a}$, $b \to \bar{b}$, $c \to \bar{c}$ extends to a homomorphism. Now it is clear that

$$\{\bar{a}, \bar{a}\bar{b}, \bar{b}\bar{c}, \bar{c}\} = \{(1, 0), (0, -1), (-1, 0), (0, 1)\}$$

is another basis for $\mathbb{Z} \oplus \mathbb{Z}$, so $\{a, ab, bc, c\}$ must be another basis for U, contradicting BP. ■

3. The main theorem. Let S be an arbitrary semigroup and let J be a $\mathcal{J}$-class of S, with associated principal factor P. The only situation in which SBP for P has not been determined is when P is [0-]simple and $\mathcal{D}$-trivial (but with more than one [or two] elements). This is the case precisely when J is nontrivial with one-element $\mathcal{D}$-classes, so unfortunately we must exclude such $\mathcal{J}$-classes from the hypothesis of our main theorem below. If the Conjecture of the previous section is true, this clause may be omitted. However, as noted earlier, many important classes of semigroups are in any case covered.

THEOREM 3.1. *Let S be a semigroup containing no nontrivial $\mathcal{J}$-class with one-element $\mathcal{D}$-classes. Then S has* SBP *if and only if each non-null principal factor is either* (a) *a left zero or right zero semigroup or a group with* SBP, (b) *one of the above with adjoined zero or* (c) *isomorphic to B_5.*

Proof. Under the hypothesis, necessity follows from Proposition 2.3, Theorem 2.5 and Theorem 2.4.

To prove sufficiency we use Lemma 1.1 and the remarks succeeding it. So suppose S satisfies the hypotheses of the theorem, U is a subsemigroup and $\langle U, x, y \rangle = \langle U, a \rangle$, where x, y and a belong to the same $\mathcal{J}$-class J, say, with principal factor P. Note that either $a \in \langle U, x \rangle$ or $a \in \langle U, y \rangle$, as required, or, without loss of generality, a can be expressed as a product $sxtyu$ for some s, t, u in $\langle U, x, y \rangle^1$. Moreover, since a, x and y belong to J so does the product of each substring of $sxtyu$ which involves x or y.

There are four cases to consider.

(1) *P is null,* that is all products of elements in J fall into a lower $\mathcal{J}$-class. Thus no such equation $a = sxtyu$ can occur, so $a \in \langle U, x \rangle$ or $a \in \langle U, y \rangle$.

(2) *P is right zero,* with or without adjoined zero. In either case J itself is a right zero subsemigroup of S. Therefore in a product of the above form $(sxt)(yu) = yu$. If $u \in U^1$ then $a \in \langle U, y \rangle$. Otherwise $u = u_1 z u_2$, where $u_1 \in \langle U, x, y \rangle^1$, $z \in \{x, y\}$ and $u_2 \in U^1$, whence

$$a = (yu)(zu_2) = zu_2 \in \langle U, x \rangle \cup \langle U, y \rangle,$$

as required. A dual argument applies if P is left zero.

(3) *P is a group with* SBP, with or without adjoined zero. Again, J is itself a group, with identity e, say. Now $U_e = eUe \cap J$ is a subsemigroup of J and $\langle U_e, x, y \rangle = \langle U_e, a \rangle$, since $x, y, a \in eSe$. From SBP for J it follows that $a \in \langle U_e, x \rangle \cup \langle U_e, y \rangle$. But from

periodicity of the group G (Lemma 2.1), $e \in \langle x \rangle \cap \langle y \rangle$, so $eUe \subseteq \langle U, x \rangle \cap \langle U, y \rangle$. Thus $a \in \langle U, x \rangle \cup \langle U, y \rangle$.

(4) *P is isomorphic to B_5*. Then J consists of exactly two idempotents and two mutually inverse nonidempotents. Here a combinatorial argument is needed, based on the following observation: if $b \in J$ and $bsb \in J$ for some $s \in S^1$ then $bsb = b$; for $\mathscr{H}$ is trivial on J and by complete 0-simplicity of B_5, $bsb \mathscr{R} b \mathscr{L} bsb$.

Since $a \in \langle U, x, y \rangle$ it can therefore be expressed, without loss of generality, in the form $sxtyu$ where *now* s, t and u are in U^1. Similarly, $y = vaw$ for some v, w in U^1. Now either $a = sxt \in \langle U, x \rangle$ or $a = yu \in \langle U, y \rangle$ or (from the structure of J) sxt and yu are mutually inverse and a is idempotent. But $a = sxt(vaw)u$, which is possible in the last case only if $a = sxtv \in \langle U, x \rangle$, as required.

This completes the proof of the theorem.

The following corollaries are almost immediate. From the remarks of the previous section, the hypotheses apply in each case. We have rephrased the theorem where convenient.

COROLLARY 3.2. (i) *A regular semigroup S has* SBP *if and only if each principal factor has one of the specified forms; S is therefore orthodox.*

(ii) *An inverse semigroup has* SBP *if and only if each $\mathscr{D}$-class is either a group with* SBP *or contains exactly two idempotents and two nonidempotents.*

(iii) *A completely regular semigroup has* SBP *if and only if it is a semilattice of left zero semigroups, right zero semigroups and groups.*

(iv) *A band has* SBP *if and only if it is "singular", that is, each $\mathscr{D}$-class is left or right zero.*

That a regular semigroup with SBP is orthodox follows from the fact **[4]** that if each principal factor of a regular semigroup is orthodox then so is the semigroup itself. For SBP for inverse semigroups, with respect to inverse subsemigroups, see **[6]**.

Notice also that for completely regular semigroups (unions of groups) SBP is the same whether regarded as semigroups or algebras of type $\langle 2, 1 \rangle$ since, by periodicity of the subgroups, every subsemigroup is a completely regular subsemigroup.

In the general periodic case the theorem also applies. In particular if either $\mathscr{R}$ or $\mathscr{L}$ is trivial on S then each non-null principal factor is left or right zero, respectively, (possibly with adjoined zero).

COROLLARY 3.3. *Every periodic semigroup on which either $\mathscr{R}$ or $\mathscr{L}$ is trivial has* SBP.

This is closely related to the result of Doyen mentioned in the introduction. (See § 5).

Finally, if every principal factor is null (or trivial), S clearly has SBP. This is the case when $\mathscr{J}$ is *trivial*. However, this case is immediate from Lemma 1.1 and the succeeding comments. In fact U-bases are unique in such semigroups.

4. The basis property. It is certainly not true that if the principal factors of a semigroup S have BP then so does S. Example 6.3 of **[6]** is a semilattice of two groups,

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each with BP, which does not have BP. Thus the completely [0-]simple case is of less interest. We present the following description of such semigroups having BP, omitting the proof, which pursues more deeply the techniques of the proof of Theorem 2.4.

THEOREM 4.1. (i) *A completely simple semigroup S has* BP *if and only if S has one of the following forms or their duals*:

(a) *S is a group with* BP;

(b) *S is a right group whose maximal subgroups are cyclic of order p^n, $n \geq 0$, for some prime p*;

(c) *S has exactly two $\mathcal{R}$-classes, its maximal subgroups are as in* (b) *and the union of any two $\mathcal{L}$-classes is idempotent-generated.*

(ii) *A proper completely* 0*-simple semigroup S has* BP *if and only if S, or its dual, has exactly two $\mathcal{R}$-classes and each nonzero $\mathcal{L}$-class contains exactly one idempotent and one nonidempotent.*

In [**6**, Theorem 6.1] it was shown that a Brandt semigroup, considered as an inverse semigroup, has BP if and only if each of its maximal subgroups has. For semigroups the situation is obviously more chaotic. Note that, in (c), if S is not just a rectangular band then

$$S \cong \mathcal{M}(G; \{1, 2\}, \{1, 2, \ldots, k\}; P),$$

where $G = \mathbb{Z}_{p^n}$, $n \geq 1$, $k \leq p$ and $P = \begin{pmatrix} 0 & 0 \ldots 0 \\ r_1 & r_2 \ldots r_k \end{pmatrix}^T$, where no two r_i's are congruent modulo p.

5. Basis properties for monoids. In [**3**] J. Doyen proved the following theorem.

RESULT 5.1. [**3**, Theorem 1]. *Any two bases for a periodic $\mathcal{R}$-trivial monoid have the same cardinality.*

Since a submonoid of such a monoid is of the same type, this is equivalent to the assertion that any such monoid has BP, *as a monoid*. We now show that in fact it does not matter whether submonoids or subsemigroups are used for BP and SBP. As a consequence Theorem 5.1 (and SBP as well) is immediate from Corollary 3.3.

For clarity we denote by $\langle X \rangle_S$ and $\langle X \rangle_M$ the subsemigroup and submonoid, respectively, generated by a subset X of a monoid M. Clearly $\langle X \rangle_M = \langle X \rangle_S \cup \{1\}$. Similarly, for $U \subseteq V \subseteq S$, if U and V are subsemigroups then an $\underset{\sim}{S}$-U-basis for V is a U-basis as previously defined, whilst if U and V are submonoids an $\underset{\sim}{M}$-U-basis is a subset which generates V, as a monoid, minimally over U. The $\underset{\sim}{M}$*-BP*, $\underset{\sim}{M}$-SBP, $\underset{\sim}{S}$-BP and $\underset{\sim}{S}$-SBP are then defined in the obvious way.

THEOREM 5.2. *Let M be a monoid. Then M has $\underset{\sim}{M}$*-SBP [*$\underset{\sim}{M}$-BP*] *if and only if it has* $\underset{\sim}{S}$-SBP [$\underset{\sim}{S}$-BP].

Proof. Suppose M has $\underset{\sim}{M}$-SBP. Let U and V be subsemigroups of M and let A, B be $\underset{\sim}{S}$-U-bases for V.

If $1 \notin V$ then A and B are $\underset{\sim}{M}$-$U \cup \{1\}$-bases for $V \cup \{1\}$, so $|A| = |B|$.

If $1 \in V$, first suppose $1 \in A$. Then $A \setminus \{1\}$ is an $\underset{\sim}{M}$-$U \cup \{1\}$-basis for V. Now $V = \langle U \cup A \rangle_S$, so any b in B is expressible as a product of members of $U \cup A$. Further, if $b \neq 1$ then $b \notin U \cup \{1\}$, so the product must involve some $a \neq 1$, in which case all occurrences of 1 may be deleted, that is, $b \in \langle U \cup A \setminus \{1\} \rangle_{\underset{\sim}{S}}$. Thus if $1 \notin B$, then

$$V = \langle U \cup B \rangle_{\underset{\sim}{S}} \subseteq \langle U \cup A \setminus \{1\} \rangle,$$

a contradiction. Hence if $1 \in A$ then $1 \in B$ and $B \setminus \{1\}$ is also an $\underset{\sim}{M}$-$U \cup \{1\}$-basis for V, so $|A| = |B|$.

If $1 \in V$ but $1 \notin A$ then, similarly, $1 \notin B$, and both A and B are $\underset{\sim}{M}$-$U \cup \{1\}$-bases for V. Thus $|A| = |B|$ once more and M has $\underset{\sim}{S}$-SBP.

A similar argument shows that $\underset{\sim}{M}$-BP implies $\underset{\sim}{S}$-BP.

Conversely, suppose M has $\underset{\sim}{S}$-SBP. Let U and V be submonoids of M, $U \subseteq V$. Now since U and V are already subsemigroups and $1 \in U$, any $\underset{\sim}{M}$-U-basis for V is already an $\underset{\sim}{S}$-U-basis for V. Thus M has $\underset{\sim}{M}$-SBP. (A more general theorem is proved in [**8**]).

Finally suppose M has $\underset{\sim}{S}$-BP, let V be a submonoid and suppose A and B are $\underset{\sim}{M}$-bases for V. Then $1 \notin A$, $1 \notin B$ and both $A \cup \{1\}$ and $B \cup \{1\}$ are $\underset{\sim}{S}$-bases for V, whence $|A| = |B|$. Thus S has $\underset{\sim}{M}$-BP. ■

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