COMPACTIFICATIONS OF SEMITOPOLOGICAL SEMIGROUPS

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1. Introduction

Suppose S is a semitopological semigroup. We consider various subspaces of $C(S)$ and determine what topological algebraic structure can be introduced into the spaces of means on the subspaces and into the spectra of the C^*-subalgebras of $C(S)$ they generate.

After establishing terminology and some preliminary (probably known) results in part 2, we consider in parts 3 and 4 left introverted, respectively left m-introverted, subspaces of $C(S)$, which were first introduced and studied by Rao and Witz, respectively Mitchell[2], and make some additions to the theory of these subspaces.

The above-mentioned authors proved the existence and gave a characterization of the greatest left introverted subspace $WLUC(S)$ and the greatest left m-introverted subspace $LMC(S)$. (We use Mitchell's notation.) In part 3 we give alternate characterizations of these greatest subspaces in terms of the topology of pointwise convergence on S. We define multiplication in the space of means of a left introverted subspace and in the spectrum of the C^*-subalgebra of $C(S)$ generated by a left m-introverted subspace in an established way, make some comments on the posssibility of reordering the operations defining the multiplication, and prove a universal mapping property for the spectrum of $LMC(S)$.

In part 4, we continue work, mainly of Mitchell, proving the equality of some subspaces in special cases. In particular, we show $LMC(S) = WLUC(S)$ if S is locally compact (Mitchell has proved $LMC(S) = WLUC(S)$ if S is first countable), $LMC(S) = LUC(S)$, the left uniformly continuous subspace, if S is a *subgroup* of a topological group complete in a left invariant metric or locally compact (Rao, resp. Mitchell, proved $LMC(S) = LUC(S)$ if S is a topological

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[2] The author would like to thank T. Mitchell for the preprint [12].

group complete in a left invariant metric, respectively locally compact), and $LUC(S) = AP(S)$, the almost periodic subspace, if S is compact. A key lemma in the proof of the middle assertion of the last sentence is used to give answers (some only partial) to some questions posed by Burckel [22][3] concerning the extension of weakly almost periodic functions. Among other things we prove that, if S is a dense subgroup of a topological group G, then each weakly almost periodic function on S is the restriction to S of a weakly almost periodic function on G. An example is presented to show how badly this last statement can fail if the subgroup S is not required to be dense in G: there are almost periodic functions (not identically zero) on the subgroup S that are best approximated (uniformly on S) by the zero function, if the approximating functions are required to be restrictions to S of weakly almost periodic functions on G.

We construct a semitopological semigroup S for which all the subspaces mentioned are distinct but two: $LMC(S) = WLUC(S)$. We do not know an example for which $LMC(S) \neq WLUC(S)$.

2. Definitions and preliminaries

A *semitopological semigroup* is a Hausdorff topological space S in which a separately continuous associative multiplication is defined, that is, the operations, *multiplication on the right* $s \to st$ and *multiplication on the left* $s \to ts$, are defined and are continuous mappings from S into $S \, \forall t \in S$, and $(st)u = s(tu)$, $\forall s, t, u \in S$. If the multiplication is jointly continuous, i.e., the map $(s,t) \to st$ from $S \times S$ into S is continuous, S is called a *topological semigroup*. We begin with topological preliminaries; the algebraic structure of S is involved in the following discussion only after Lemma 2.1.

Let $C(S)$ be the C^*-algebra of all continuous bounded complex-valued functions on S; we indicate the supremum norm on $C(S)$ by $\| \quad \|$. Suppose X is a closed linear subspace of $C(S)$ containing the constant functions. Then there is a canonical map e of S into the dual X^* of X: $e(s)(f) = f(s) \, \forall f \in X$. (In case $X = C(S)$, we let e_c denote this map.) e is continuous if X^* is given the w^* topology $\sigma(X^*, X)$, the only topology we consider on dual spaces or subspaces of dual speces without specific mention. We call $\{x \in X^* / \| x \| = 1 = x(1)\}$ the *space of means* on X and denote it by $F(X)$. $F(X)$ is w^*-compact. Members of the convex hull of $e(S) = \{e(s)/s \in S\}$ are referred to as *finite means* on X. It is well known that the finite means are w^*-dense in $F(X)$ and that any $y \in X^*$ can be written as a linear combination of means in $F(X)$. If S is locally compact, we denote by $M(S)$ the space of positive regular Borel measures on S of bound 1; the complex-linear span of $M(S)$ is isometrically isomorphic to $C_0(S)^*$ [3, Theorem 3, p. 256], where

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[3] The author would like to thank a referee for informing him that this lemma essentially answers question 2, p. 81, of [22].

$C_0(S)$ is the C^*-subalgebra of $C(S)$ consisting of all functions vanishing at infinity. $M(S)$ contains all the finite means.

The closure in X^* of $e(S)$ is a compact Hausdorff space which we call $S_0(X)$. It is a *compactification* of S; that is, it is a compact Hausdorff space containing a dense continuous image of S. $S_0(X)$ is homeomorphic in the natural way to the spectrum of the C^*-subalgebra of $C(S)$ generated by X, $C^*(X)$, say; this is because a net in $e(S) \subset X^*$ converges w^* if and only if it converges when considered as a net in $C^*(X)^*$. The "only if" part is all that is not obvious. Suppose that $\{e(s_\alpha)\}$ is a net converging w^* in X^*. We have $\{e(s_\alpha)f\}$ converging $\forall f \in X$. Extending each $e(s_\alpha)$ to a functional on $C^*(X)$ in the natural way (and designating this extension by the same symbol $e(s_\alpha)$), we have $\{e(s_\alpha)f\}$ converging for all f in the algebra generated by X. Hence, by uniform boundedness of the net $\{e(s_\alpha)\}$, $\{e(s_\alpha)f\}$ is convergent $\forall f \in C^*(X)$.

We state without proof the following generalization of the Stone-Čech theorem [10; Theorem 24, p. 153]:

LEMMA 2.1. *Let E be a Hausdorff topological space and let $u(v)$ be a continuous map of E onto a dense subset of a compact Hausdorff space E_1 (E_2). The transpose map u^* (v^*) injects $C(E_1)$ $(C(E_2))$ into $C(E)$. If*

$$v^*(C(E_2)) \subset u^*(C(E_1)),$$

there is a unique continuous map w from E_1 onto E_2 such that $(w \cdot u)(s) = v(s) \, \forall s \in E$.

Suppose $f \in C(S)$ and $s \in S$. The *left* (*right*) *translate* f_s (f^s) of f by s is defined by $f_s(t) = f(st)$ $(f^s(t) = f(ts)) \, \forall t \in S$. A subspace X of $C(S)$ is called *left* (*right*) *translation invariant* if $f_s \in X$ $(f^s \in X) \, \forall f \in X$, $s \in S$; if X is both left and right translation invariant, it is called *translation invariant*. Separate continuity of the multiplication in S ensures that $C(S)$ is translation invariant.

Suppose now that X is a closed linear translation invariant subspace of $C(S)$. We define multiplication in $e(S)$ by $e(s)e(t) = e(st)$ and note that this definition makes sense, namely $e(s_1t_1)f = e(s_2t_2)f \, \forall f \in X$ whenever $e(s_1)f = e(s_2)f$ and $e(t_1)f = e(t_2)f \, \forall f \in X$; this follows directly from translation invariance of X. So $e(S)$ is a semitopological semigroup, again by translation invariance of X, and e is a continuous homomorphism of S onto $e(S)$. In this situation, we prove the following corollaries to Lemma 2.1.

COROLLARY 2.2. *For each $t \in S$, multiplication on the left (right)*

$$e(s) \to e(t)e(s) \, (e(s) \to e(s)e(t))$$

rom $e(S)$ into $e(S)$ extends uniquely by continuity to all of $S_0(X)$.

PROOF. We prove the corollary for multiplication on the left. The proof for multiplication on the right is similar. Since X is translation invariant, so is the C^*-subalgebra of $C(S)$ generated by X. To apply the lemma, let u be the identity injection of $e(S)$ into $S_0(X) = E_1$ and let v be the map $e(s) \to e(t)e(s)$ of $e(S)$ into the closure of $e(t)e(S)$ in $S_0(X)$. The corollary is proved.

COROLLARY 2.3. *Let X_i be a closed linear subspace of $C(S)$, e_i be the canonical continuous map of S into $S_0(X_i)$, $i = 1,2$. If the C^*-subalgebra of $C(S)$ generated by X_2 is contained in that generated by X_1, there is a unique continuous map w from $S_0(X_1)$ onto $S_0(X_2)$ such that*

$$(w \cdot e_1)(s) = e_2(s) \;\forall s \in S.$$

If X_1 and X_2 are translation invariant and multiplication is defined in $e_1(S)$ and $e_2(S)$ as above, then w is a homomorphism of $e_1(S)$ onto $e_2(S)$, and w preserves the continuous extension, given by Corollary 2.2, of multiplication on left (right) by elements of $e_i(S)$ to all of $S_0(X_i)$, $i = 1,2$. If, as well, multiplication on the left (right) by elements of $S_0(X_i)$ is continuous on $e_i(S)$ and hence, extends uniquely by continuity to all of $S_0(X_i)$, $i = 1,2$, w preserves this extension.

REMARK. If one of the continuous extensions mentioned in the last statement of the corollary holds in both $S_0(X_1)$ and $S_0(X_2)$, $S_0(X_1)$ and $S_0(X_2)$ are semigroups and w is a homomorphism of $S_0(X_1)$ onto $S_0(X_2)$.

PROOF OF COROLLARY 2.3. The first statement follows from the lemma. If X_i is translation invariant and multiplication has been defined in $e_i(S)$, so that e_i is a homomorphism, $i = 1,2$, then

$$w(e_1(s)e_1(t)) = (w \cdot e_1)(st) = e_2(st) = e_2(s)e_2(t) = w(e_1(s))w(e_1(t)).$$

The proofs that w preserves the continuous extensions of multiplication (if they exist) are straightforward. We prove that w preserves the continuous extension to all of $S_0(X_i)$ of multiplication on the left by elements of $e_i(S)$, $i = 1,2$. Suppose

$$e_1(s_\alpha) \to s_0 \in S_0(X_1).$$

Then

$$e_2(s_\alpha) = w(e_1(s_\alpha)) \to w(s_0)$$

and, if $t \in S$,

$$e_1(t)e_1(s_\alpha) \to e_1(t)s_0\,;$$

and

$$w(e_1(t)s_0) = w(\lim_\alpha e_1(t)e_1(s_\alpha)) = \lim_\alpha w(e_1(ts_\alpha)) = \lim_\alpha (e_2(t)e_2(s_\alpha)) = w(e_1(t))w(s_0),$$

as required.

DEFINITION. If S_i is a compactification of S (having certain properties), e_i is the continuous map of S onto a dense subset of S_i, $i = 1,2$, and w is a continuous map from S_1 onto S_2 such that $(w \cdot e_1)(s) = e_2(s) \forall s \in S$ (and w preserves the properties), we say S_1 *is greater than* S_2 *in the family of compactifications* (having the certain properties) *of* S.

We discuss briefly two well-known compactifications of a semitopological semigroup S. A function $f \in C(S)$ is called *left almost periodic* (*weakly almost periodic*) if $\{f_s \mid s \in S\}$ is relatively compact (weakly relatively compact) in $C(S)$ and *right* almost periodic (weakly almost periodic) if $\{f^s \mid s \in S\}$ is relatively compact (weakly relatively compact) in $C(S)$. It turns out that f is *left* almost periodic (weakly almost periodic) if and only if it is *right* almost periodic (weakly almost periodic). So, without ambiguity, we let $AP(S)(WAP(S))$ denote the class of almost periodic (weakly almost periodic) functions on S omitting reference to *left* and *right*. $AP(S)(WAP(S))$ is a translation invariant C^*-subalgebra of $C(S)$ whose spectrum $S_0(AP(S))(S_0(WAP(S)))$, $S_a(S_w)$ for short, is canonically a compactification of S called the *almost periodic* (*weakly almost periodic*) *compactification*. $S_a(S_w)$ is the greatest compactification of S that is a topological (semitopological) semigroup; that is, if $e_a(e_w)$ is the canonical continuous homomorphism from S into $S_a(S_w)$ and ϕ is a continuous homomorphism of S onto a dense subset of a compact topological (semitopological) semigroup T, then $\exists$ a unique continuous homomorphism $b_a(b_w)$ from $S_a(S_w)$ onto T such that

$$b_a \cdot e_a(s) = \phi(s)(b_w \cdot e_w(s) = \phi(s)) \forall s \in S.$$

This property of $S_a(S_w)$ is sometimes called the *universal mapping property* of $S_a(S_w)$. If S is a compact topological (semitopological) semigroup, then

$$C(S) = AP(S)(C(S) = WAP(S)).$$

The reader is referred to [1,2,9,16,22] for these results

3. Two kinds of subspaces

In this section we present discussions of two kinds of subspaces of $C(S)$.

DEFINITION. Let X be a closed translation invariant subspace of $C(S)$ containing the constant functions. Then X is called *left introverted* (*left m-introverted*) if, $\forall f \in X$ and $\forall x \in X^*$ ($\forall f \in X$ and $\forall x \in S_0(X)$), the function $s \to x(f_s)$ is in X.

REMARK. It follows from well-known theorems of Banach spaces and commutative C^*-algebras that X^* can be replaced by $F(X)$, $C(S^*)^*$ or $F(C(S))$ in the definition of a left introverted subspace and that $S_0(X)$ can be replaced by $S_c = S_0(C(S))$ in the definition of a left m-introverted subspace.

Rao [18] and Witz [20] (Mitchell [11, 12]) introduced the concept of a left introverted (left m-introverted) subspace of $C(S)$ and proved the existence of a greatest such subspace, $WLUC(S)$ ($LMC(S)$); we use Mitchell's notation. In fact, $WLUC(S) = \{f \in C(S)/$the function $s \to x(f_s)$ is in $C(S)\, \forall x \in C(S)^*\}$ and $LMC(S) = \{f \in C(S)/$ the function $s \to x(f_s)$ is in $C(S)\, \forall x \in S_0(C(S))\}$.

It is easy to see that $LMC(S)$ is a C^*-subalgebra of $C(S)$ and that $WLUC(S)$ is uniformly closed. The question arises: is $WLUC(S)$ a C^*-subalgebra of $C(S)$? In the next section we shall seee that $WLUC(S) = LMC(S)$ in many familiar cases; $WLUC(S)$ is certainly a C^*-subalgebra then.

We give another characterization of these subspaces in terms of the topology of pointwise convergence on S, $\sigma(C(S), e_c(S))$.

THEOREM 3.1. *$WLUC(S) = \{f \in C(S)/$ the family of convex combinations of right translates of f is relatively $\sigma(C(S), e_c(S))$-compact in $C(S)\}$. $LMC(S) = \{f \in C(S)/\{f^s/s \in S\}$ is relatively $\sigma(C(S), e_c(S))$-compact in $C(S)\}$.*

PROOF. We prove the second statement. The proof of the first is similar. Suppose $f \in LMC(S)$ and $\{f^{s_\alpha}\}$ is a net of right translates of f. Let $\{e_c(s_{\alpha_\nu})\}$ be a subnet of $\{e_c(s_\alpha)\} \subset S_c$ converging to $x \in S_c$, say. Then $\{f^{s_{\alpha\nu}}\}$ converges pointwise on S to the function $t \to x(f_t)$ which is in $C(S)$. So $\{f^s/s \in S\}$ is relatively $\sigma(C(S)$, $e_c(S))$-compact.

On the other hand, suppose $\{f^s/s \in S\}$ is relatively $\sigma(C(S), e_c(S))$-compact and $x \in S_c$. Then choose a net $\{e_c(s_\alpha)\} \subset e_c(S)$ converging to x. By hypothesis, a subnet $\{f^{s_{\alpha\nu}}\}$ of $\{f^{s_\alpha}\}$ converges pointwise on S to $h \in C(S)$, say. The function $t \to x(f_t) = \lim_\nu e_c(s_{\alpha_\nu})(f_t) = \lim_\nu f(ts_{\alpha_\nu}) = h(t)$ is in $C(S)$. So $f \in LMC(S)$.

A much-studied example of a left introverted C^*-subalgebra of $C(S)$ is the *left uniformly continuous* subspace, $LUC(S) = \{f \in C(S)/\, \|f_{s_\alpha} - f_s\| \to 0$ whenever $s_\alpha \to s$, all in $S\}$ [7, 8, 12, 14, 18, 19, for example]. This function space is so named because, if S is a topological group G, then a function in $C(G)$ is in $LUC(G)$ if and only if it is uniformly continuous with respect to the left uniformity of G (see [10]). In the next section we shall see that, if S is a compact semitopological semigroup, $LUC(S)$ can be strictly smaller than $C(S)$, each function of which is uniformly continuous with respect to the unique uniformity of S [10; pp. 198, 199]. Other examples of left introverted subspaces are $WAP(S)$ and $AP(S)$.

If X is left introverted (left m-introverted), multiplication can be defined in $F(X)$ ($S_0(X)$) in the usual way [7, 14, 20, among others]: if x, $y \in F(X)$ ($x, y \in S_0(X)$), $f \in X$, then $xy(f) = x(h)$, where $h(t) = y(f_t)\, \forall t \in S$. It is easy to verify that, with this multiplication and the w^* topology, $F(X)$ ($S_0(X)$) is a compact semigroup and that:

(i) the map $x \to xy$ is continuous $\forall y \in F(X)\, (\forall y \in S_0(X))$.

(ii) the map $x \to yx$ is continuous at least if y is a finite mean ($y \in e(S)$).

There is a possibility of reordering the computations defining the product;

for example, if $f \in X$, $x \in F(X)$ and y is a finite mean,

$$y = \sum_{n=1}^{m} a_n e(s_n),$$

where $0 \leqq a_n \leqq 1$, $s_n \in S$, $n = 1, 2, \cdots, m$, and

$$\sum_{n=1}^{m} a_n = 1,$$

then

$$yx(f) = \sum_{n=1}^{m} a_n x(f_{s_n}) = x(g),$$

where $g(t) = y(f^t)$. Not all products can be computed in rearranged order like this; in the first place, the function $t \to y(f^t)$ may fail to be in X, and in the second place, even if the function $t \to y(f^t)$ is in $X\ \forall y \in X^*$ and $\forall f \in X$, it does not follow that the order of computations can be changed. For this would imply separate continuity of the multiplication in $F(X)$, which does not always obtain (see the example after Corollary 3.3). This brings us to a special case.

LEMMA 3.2. *Suppose S is locally compact, X is a left introverted C^*-subalgebra of $C(S)$, $f \in X$, $x \in F(X)$ and $y \in M(S) \subset C_0(S)^*$. Then the function $t \to y(f^t) = g(t)$, say, is in X and $yx(f) = x(g)$.*

PROOF. Glicksberg [5; pp. 205, 207] has proved the following theorems:

Let A and B be locally compact spaces, and f a bounded complex function on $A \times B$ which is separately continuous. Then for $\mu \in C_0(A)^*$,

$$t \to \int f(s,t) d\mu(s)$$

is continuous on B.

If, as well, $v \in C_0(B)^*$, then

$$\iint f(s), t d\mu(s) dv(t) = \iint f(s,t) dv(t) d\mu(s).$$

To apply these results to f in the lemma, we put $A \times B = S \times S_0(X)$ and consider the function $(s,t) \to t(f_s)$ defined on $S \times S_0(X)$.

REMARK. In case S is a locally compact group G, a more direct proof of Lemma 3.2 can be given using the fact that, if U is a neighbourhood of the identity of G and $W \subset G$ is compact, then there a neighbourhood V of the identity such that $tVt^{-1} \subset U\ \forall t \in W$ [13; p. 55].

A mean y on a left invariant subspace Y of $C(S)$ is called a *left invariant* *mean*, abbreviated to *LIM*, if $y(f_s) = y(f)\, \forall f \in Y,\ s \in S$. Lemma 3.2 has an immediate corollary.

COROLLARY 3.3. *Let S be locally compact, let X be a left introverted C^*-subalgebra of $C(S)$ and let x be a LIM on X. Then* $\forall f \in X,\ y \in M(S),\ x(f) = x(h)$, *where* $h(t) = \int f(st)dy(s)$.

REMARK. Corollary 3.3. is a generalization of the known result (see [8; Lemma 2.2.2, p. 27, and line following], for example), that, if S is a locally compact group G, a *LIM* x on $LUC(G)$ is "topologically" left invariant; that is, $x(h * f) = x(f)\, \forall f \in LUC(G)$ and $\forall h \in L^1(G) \cap M(G)$. Granirer [6] and Renaud [26] have generalized this result in other directions.

AN EXAMPLE. Let $S = Z$, the (discrete) group of integers; $C(Z) = LMC(Z)$ and $S_0(LMC(Z)) = S_0(C(Z)) = \beta Z$, the Stone-Čech compactification of Z. Then multiplication on the left in βZ by elements of βZ is not always continuous For, let y_1 be a cluster point in βZ of the sequence,

$$\{e_c(-1), e_c(-2), e_c(-3), \cdots\}$$

and y_2 a cluster point of

$$\{e_c(1), e_c(2), e_c(3), \cdots\}.$$

Let $\{e_c(n_\alpha)\}$ be a net in $e_c(Z)$ converging to y_2. Define f by

$$f(n) = 1 \text{ if } n \geqq 0,\ f(n) = 0 \text{ if } n < 0.$$

Then

$$y_2(f_n) = \lim_{m \to \infty} f(n+m) = 1\, \forall n \in Z.$$

Hence, $y_1 y_2(f) = 1$. However,

$$y_1 e_c(n_\alpha)(f) = \lim_{m \to \infty} f(n_\alpha - m) = 0\, \forall \alpha.$$

So, $y_1 e_c(n_\alpha) \nrightarrow y_1 y_2$. This implies $\beta Z = S_0(LMC(Z))$ is not commutative even though Z is.

It is worth noting that the proof that $S_0(LMC(Z))$ is not a semitopological semigroup amounts to finding a function in $LMC(Z) \backslash WAP(Z)$. This is what one would expect in the light of [16; Theorem 4.3]; namely, $S_0(LMC(Z))$ fails to be a semitopological semigroup if and only if $LMC(Z)$ contains a function that is not in $WAP(Z)$.

THEOREM 3.4. *Let e_m be the canonical continuous homomorphism of S into $S_0(LMC(S))$. $S_0(LMC(S))$ has the following universal mapping property: if v is a continuous homomorphism of S onto a dense subset of a compact semigroup T in which multiplication on the left is continuous at least by elements of $v(S)$ and multiplication on the right is continuous without restriction, then*

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there is a unique continuous homomorphism w of $S_0(LMC(S))$ onto T such that $(w \cdot e_m)(s) = v(s) \forall s \in S$.

PROOF. Let v and T be as stated, let $h \in C(T)$, $f = v^*h \in C(S)$ and let $\{f^{s_\alpha}\}$ be a net of right translates of f. Let $\{v(s_{\alpha_\nu})\}$ be a subnet of $\{v(s_\alpha)\}$ converging to $x \in T$, say. Then $h^x \in C(T)$ and it is not difficult to see that $f^{s_{\alpha_\nu}} \to v^*(h^x) \in C(S)$ pointwise on S. Hence, $v^*(C(T)) \subset LMC(S)$. If we identify T and $S_0(v^*(C(T)))$ in the natural way, Corollary 2.3 yields the desired homomorphism.

4. Inclusion relationships among the subspaces

For any semitopological semigroup S all but the last of the following inclusions are immediate consequences of the definitions: $C(S) \supset LMC(S) \supset WLUC(S) \supset LUC(S) \supset AP(S)$. Also $WLUC(S) \supset WAP(S) \supset AP(S)$. We prove

LEMMA 4.1. *$LUC(S) \supset AP(S)$ for any semitopological semigroup S.*

PROOF. Let $f \in AP(S)$ and let $s_\alpha \to s$, all in S, and let e_a be the canonical continuous homomorphism of S into $S_a = S_0(AP(S))$. Let $h = (e_a^*)^{-1} f \in C(S_a)$. By compactness of S_a and joint continuity of the multiplication there,

$$\| h^{e_a(s_\alpha)} - h^{e_a(s)} \| \to 0;$$

hence,

$$\| f^{s_\alpha} - f^s \| = \| e_a^*(h^{e_a(s_\alpha)} - h^{e_a(s)}) \| \to 0.$$

When S is first countable, Mitchell [12] proved $LMC(S) = WLUC(S)$ using a theorem of Rainwater [17, or 15; p. 33].

PROPOSITION 4.2. *$LMC(S) = WLUC(S)$ whenever S is locally compact.*

PROOF. We need only show that $LMC(S) \subset WLUC(S)$. Suppose $f \in LMC(S)$. Then the function $(s, x) \to x(f_s)$ is separately continuous on $S \times S_0(LMC(S))$. The proof is completed by noting that $LMC(S)$ is a C^*-subalgebra of $C(S)$ and putting $B = S$, $A = S_0(LMC(S))$ in the first result of Glicksberg quoted in the proof of Lemma 3.2.

In [18; Theorem 2] Rao states that he shows $LUC(G) = WLUC(G)$, where G is a topological group complete in an invariant metric. In fact, he proves $LMC(G) = LUC(G)$ for topological groups complete in a left invariant metric.

Mitchell [12] has proved that $LMC(G) = LUC(G)$ for any locally compact group G, using a theorem of Ellis [4].

In a less general setting, that of σ-compact, locally compact groups, we present another proof of this result.

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Let G be a σ-compact, locally compact group. By Lemma 2.31, p. 54, of [13], there is an open subgroup G' of G such that G'/G_0 is compact, where G_0 is the identity component of G. By σ-compactness of G, there are at most countably many left (or right) G'-cosets. We fix coset decompositions of G:

$$G = \bigcup_0^\infty s_n G' = \bigcup_0^\infty G' s_n^{-1}.$$

Suppose $f \in C(G)$. Define f_n' by

$$f_n'(s) = f(s) \text{ if } s \in G's_n^{-1},\ f_n(s) = 0 \text{ otherwise.}$$

Define f_n by $f_n'(s) = f_n'(ss_n^{-1})$; the continuous functions $\{f_n\}$ all vanish off G'. Let the restriction of each f_n to G' also be denoted by f_n. $\exists$ a compact normal subgroup H of G' such that the $\{f_n\}$ are constant on the cosets of H in G' and G'/H is complete in a left invariant metric [13; p. 61 and p. 34]. Continuing, we prove

LEMMA 4.3. *There exists a compact normal subgroup H_1 of G such that $H_1 \subset H$ and G'/H_1 (hence G/H_1) is complete in a left invariant metric.*

PROOF. Put $H_1 = \bigcap_0^\infty s_n H s_n^{-1}$, clearly a compact subgroup of H. H_1 is normal in G. For suppose $s \in G$ and s_k are given. $s^{-1}s_j = s_m t$ for some $t \in G'$ and some m. So

$$sH_1s^{-1} \subset ss_mHs_m^{-1}s^{-1} = ss_mtHt^{-1}s_m^{-1}s^{-1} = ss^{-1}s_jHs_j^{-1}ss^{-1} = s_jHs_j^{-1},$$

using normality of H in G'. This proves H_1 is normal in G.

Let $\{V_m'\}_{m=0}^\infty$ be compact sets forming a basis of neighbourhoods of the identity in G'/H. Let $\{V_m\}$ be their inverse images in G' under the canonical map u: $G' \to G'/H$. Then each V_m is compact and $\bigcap_0^\infty V_m = H$; for, if $t \in V_m\, \forall m$, but $t \notin H$, then $tH \in V_m'\, \forall m$, but $tH \notin H$, contradicting that $\{V_m'\}$ is a basis of neighbourhoods of the identity of G'/H. Consider the countable family of neighbourhoods $\{s_nV_ms_n^{-1}/n,\ m = 0,1,2,\cdots\}$. Reindex these in a single sequence and, by taking finite intersections, form an equivalent monotone decreasing sequence $\{W_m\}_{m=0}^\infty$. Then each W_m is compact and

$$\bigcap_m W_m = \bigcap_{n,m} s_nV_ms_n^{-1} = H_1.$$

If u_1 is the canonical continuous map of G' onto G'/H_1, we show $\{u_1(W_m)\}_{m=0}^\infty$ is a basis of neighbourhoods of the identity in G'/H_1. Each open set in G'/H_1 containing the identity is the image under u_1 of an open set in G' containing H_1. So let V be open in G' and $H_1 \subset V$. Then $F = G' \setminus V$ is closed and

$$F \cap H_1 = F \cap \bigcap_m W_m = \bigcap_m (F \cap W_m)$$

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is empty. It follows that $F \cap W_{m_0}$ is empty for some m_0; hence $W_{m_0} \subset V$, as desired.

Thus G'/H_1 has a countable basis at the identity; reference to [13; p. 34] completes the proof of the lemma.

Since a translate of a function is constant on cosets of a normal subgroup if the function is,

$$f = \sum_n f_n^{s_n}$$

is constant on the cosets of H_1. Thus f determines a function

$$h \in C(G/H_1)\colon h(sH_1) = h(t)$$

for any $t \in sH_1$.

We now suppose as well that $f \notin LUC(G)$. It follows directly from the definitions of left uniform continuity and quotient topology that $h \notin LUC(G/H_1)$. By Rao's result and Theorem 3.1, there is a net $\{h^{t_\alpha H_1}\}$ of right translates of h converging pointwise on G/H_1 to a function discontinuous on G/H_1. The net of lifts to G, $\{f^{t_\alpha}\}$, must converge pointwise on G, and the limit cannot be continuous. We have proved

THEOREM 4.4. *If G is a σ-compact, locally compact group, then $LMC(G) = LUC(G)$.*

We have seen that $LMC(G) = LUC(G)$ if G is a topological group complete in a left invariant metric or locally compact. Completeness of the group, though essential in the proof, is not a necessary condition for the result, as will be seen in a corollary of the next theorem.

LEMMA 4.5. *If S is a dense subgroup of a topological group G and $f \in LMC(S)$, then f extends to a function continuous on G.*

PROOF. Suppose $f \in C(S)$, $\{t_\alpha\}$, $\{t_\beta\} \subset S$, $t \in G/S$, $t_\alpha \to t$, $t_\beta \to t$ and $\lim_\alpha f(t_\alpha)$, $\lim_\beta f(t_\beta)$ exist and differ; that is, f does not extend to a function continuous on G We prove $f \notin LMC(S)$.

Let a subnet $\{e_c(t_{\alpha_\nu})\}$ of $\{e_c(t_\alpha)\}$ converge to $x \in S_0(C(S))$, say. (Here e_c is the canonical continuous homomorphisms of S into $S_0(C(S))$.) Then $t_\beta t^{-1} \to e$ and $t_\alpha^{-1} t_{\alpha_\nu} \to e$ jointly, by which we mean, for example:

given a neighbourhood V of e, $\exists \alpha_0 = \alpha_0(V)$, $\beta_0 = \beta_0(V)$ such that, if $\alpha \geqq \alpha_0$, $\beta \geqq \beta_0$, then $t_\beta t_\alpha^{-1} \in V$.

If the function $s \to x(f_s)$ is to be continuous on S, then, as $t_\beta t_\alpha^{-1} \to e$, $x(f_{t_\beta t_{\alpha_\nu}^{-1}})$ should approach

$$x(f_e) = x(f) = \lim_\nu f(t_{\alpha_\nu}) = \lim_\alpha f(t_\alpha).$$

But $x(f_{t_\beta}t_\alpha^{-1}) = \lim_\nu f(t_\beta t_\alpha^{-1}t_{\alpha_\nu})$ is close to $f(t_\beta)$ for all large enough ν since $f \in C(S)$ and $t_\alpha^{-1}t_{\alpha_\nu} \to e$. This implies that

$$x(f_{t_{\beta}t_\alpha^{-1}}) \to \lim_\beta f(t_\beta) \text{ as } t_\beta t_\alpha^{-1} \to e$$

and we have the desired conclusion: $f \notin LMC(S)$.

THEOREM 4.6. *Let S be a dense subgroup of a topological group G. Then $LMC(S) = LMC(G)|_S$.*

PROOF. By Lemma 4.5, $LMC(S)$ is (canonically isomorphic to) a C^*-subalgebra of $C(G)$. Hence, if e_c is the canonical continuous homomorphism of G into $S_0(C(G))$ and $\forall s \in G\ v(s)$ is the restriction of $e_c(s)$ to $LMC(S)$, the map $s \to v(s)$ is a continuous homomorphism of G onto a dense subset of $S_0(LMC(S))$. By Theorem 3.4, v factors canonically through S_0 $(LMC(G))$: there is a continuous homomorphism of $S_0(LMC(G))$ onto $S_0(LMC(S))$ taking the canonical image of s in $S_0(LMC(G))$ onto $v(s) \in S_0(LMC(S))\ \forall s \in S$. This implies that $LMC(S)$ is (canonically isomorphic to) a C^*-subalgebra of $LMC(G)$. Since it is clear that the restriction of a member of $LMC(G)$ to S is a member of $LMC(S)$, we are done.

COROLLARY 4.7. *Suppose S is (a topological group homeomorphic and isomorphic to) a subgroup of a topological group G, which is complete in a left invariant metric or locally compact. Then $LMC(S) = LUC(S)$.*

PROOF. We may assume, without loss, that S is dense in G. We know already that $LUC(G) = LMC(G)$ and, by the theorem, $LMC(S) = LMC(G)|_S$.

The following theorems are proved in [22; pp. 31, 42, 44]:

(a) *If G is a locally compact group, then every function in $WAP(G)$ is (left and right) uniformly continuous.*

(b) *Let G be a commutative topological group, S a dense subgroup. Then $WAP(S) = WAP(G)|_S$.*

(c) *If G is a locally compact group, then $AP(G) = WAP(G)$ implies that G is compact.*

Burckel asks [22; p. 81] if the local compactness hypothesis is necessary in (a) and (c) and suspects that the commutativity hypothesis is unnecessary in (b). We can shed some light on these matters.

THEOREM 4.8. *If S is (homeomorphic and isomorphic to) a subgroup of a topological group which is complete in a left invariant metric or locally compact, then all functions in $WAP(S)$ are (left and right) uniformly continuous.*

PROOF. In this setting $LMC(S) = LUC(S)$ (Corollary 4.7). But $WAP(T) \subset LMC(T)$ for any semitopological semigroup T. Hence, each $f \in WAP(S)$ is left uniformly continuous. It is not hard to see that such an f is also right uniformly continuous. (One way to do this uses the facts that inversion in S, the map $s \to s^{-1}$, induces isometries of $C(S)$ onto $C(S)$ and of $LUC(S)$ onto $RUC(S)$, the space of right uniformly continuous functions on S.)

REMARK. We do not know if $WAP(S)$ consists only of uniformly continuous functions for every topological group S. We do know that $LUC(S)$ need not contain $WAP(S)$ if S is only required to be a semitopological semigroup (see Corollary 4.12 and Lemma 4.13).

We now present a variant of a theorem of Berglund, [22; p. 42] or [21; Proposition 4], which states that, *if*

(α) *S is a dense subsemigroup of a semitopological semigroup T, and*

(β) $WAP(S) \subset C(T)\big|_S$,

then $WAP(S) = WAP(T)\big|_S$ *and* $AP(S) = AP(T)\big|_S$.

We strengthen hypothesis (α) and drop hypothesis (β). Theorem 4.9 also contains the generalization of theorem (b) mentioned above that Burckel [22; p. 81] expected.

THEOREM 4.9. *If S is a dense subgroup of a topological group G, then*

(i) $WAP(S) = WAP(G)\big|_S$, *and*

(ii) $AP(S) = AP(G)\big|_S$.

PROOF. Each $f \in WAP(S)(AP(S))$ has a continuous extension to G, since $WAP(S)$ $(AP(S)) \subset LMC(S)$. The proof can be completed by referring directly to Berglund's theorem or by proceeding as in the proof of Theorem 4.6, using the universal mapping property of the weakly almost periodic (almost periodic) compactification instead of Theorem 3.4.

The next corollary answers Burckel's question concerning theorem (c) above.

COROLLARY 4.10. *Let S be a totally bounded topological group. Then* $AP(S) = WAP(S)$.

PROOF. Weil [27] has shown that a totally bounded topological group S is homeomorphic and isomorphic to a dense subgroup of a compact topological group G. (G may be regarded as the completion of S with respect to the left uniformity of S, which is the same as the right uniformity of S, since S is totally bounded). We then have $C(G) = AP(G) = WAP(G)$, and an application of Theorem 4.9 completes the proof.

Corollary 4.10 leads us to ask a question whose answer we conjecture is *no*.

QUESTION. *Can $AP(S) = WAP(S)$ if S is a topological group that is not totally bounded*? One can exclude groups that have a totally bounded neighbourhood of the identity; for Weil [27] has shown that such a group is a dense subgroup of a locally compact (non-compact) group and Theorem 4.9 and theorem (c) quoted above yield the result, $AP(S) \neq WAP(S)$.

One might wonder what happens to the result of Theorem 4.9 if the subgroup S is not required to be dense in G. If G is locally compact and abelian and S is closed, or if G is locally compact and S is normal and open, the result comes through unscathed. $WAP(S) = WAP(G)|_S$ [22; pp. 47, 49] and hence $AP(S)$ $AP(G)|_S$. However, the following example shows that the conclusion of Theorem 4.9 can not always be made if the subgroup S is not dense in G. This example is often cited to show that a positive definite function on a (closed, normal) subgroup need not extend to a function positive definite on the containing group [23; 13.11.4, 24; p. 204, 25; p. 22].

Example. Let G be the group of pairs $\{(x,y) \mid x,y \in R, x > 0\}$ with multiplication $(x,y)(a,b) = (xa, xb + y)$. Then no non-trivial character of the closed abelian normal subgroup $S = \{(1,y) \mid y \in R\}$ extends to a (left and right) uniformly continuous function on G.

Proof. Let $(1,y) \to e^{iy_0y}$ be such a character, $y_0 \neq 0$. Suppose $(x,y) \to f(x,y)$ is a uniformly continuous extension of it to G, i.e.,

$$f(1,y) = e^{iy_0y} \forall y \in R.$$

Take any fixed

$$V_m = \left\{(x,y) \,\Big|\, |1-x| < \frac{1}{m}, |y| < \frac{1}{m}\right. ,$$

where m is a positive integer. (The family $\{V_m\}$ forms a basis for neighbourhoods of the identity of G.) We find

$$A, B, C \in G \ni B^{-1}A \in V_m,\ AC^{-1} \in V_m \text{ and } |f(B) - f(C)| = 2.$$

This will complete the proof.

We need only choose $n \in R \ni n + 1 > m$, i.e., $1 - n/(n+1) < 1/m$, and put

$$A = \left(\frac{n}{n+1}, \frac{n\pi}{y_0}\right),\ B = \left(1, \frac{n\pi}{y_0}\right),\ C = \left(1, \frac{(n+1)\pi}{y_0}\right).$$

The calculations are trivial.

Remark. Examination of this example reveals that, if a function h on S is to extend to a function uniformly continuous on G, it must oscillate "more and more slowly" as $|y| \to \infty$. To be more precise, if the function $(1,y) \to h(y)$ is such that for some $\varepsilon > 0$ and all $\delta > 0$,

$$\exists y_1 \in R \text{ and } y \in [(1-\delta)y_1, (1+\delta)y_1] \text{ with } |h(y) - h(y_1)| \geqq \varepsilon,$$

then h does *not* extend to a function uniformly continuous on G. We conclude

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that, if h is a non-trivial character on S and g is the restriction to S of a function uniformly continuous on G, then $\|f - g\| \geqq 1$.

The following two lemmas enable us to see that the conclusion $LMC(S) = LUC(S)$ can be impossible if the multiplication in S is not jointly continuous. The first is a generalization of a lemma of Namioka [14; Lemma 1.3].

LEMMA 4.11. *If S is a compact semitopological semigroup, then $LUC(S) = AP(S)$.*

PROOF. By Lemma 4.1 we only have to show $LUC(S) \subset AP(S)$. Suppose $\{f^{s_\alpha}\}$ is a net of right translates of $f \in LUC(S)$. A subnet of $\{s_\alpha\}$ converges to $s \in S$, say, and the corresponding subnet of $\{f^{s_\alpha}\}$ converges uniformly to f^s.

COROLLARY 4.12. *If the multiplication in S is not jointly continuous, then $LUC(S) = AP(S) \neq WAP(S) = C(S) = LMC(S)$.*

A more general situation in which $LUC(S) \not\supset WAP(S)$ is presented in the following lemma.

LEMMA 4.13. *Suppose that S is a semitopological semigroup, that $\exists$ nets $\{s_\alpha\} \subset S$, $s_\alpha \to s \in S$, and $\{t_\beta\} \subset S$, $t_\beta \to t \in S$, and that $\exists$ a function $f \in WAP(S)$ such that $f(st) = 1$ and $f(s_{\alpha_\nu} t_{\beta_\nu}) = 0$ $\forall s_{\alpha_\nu} t_{\beta_\nu}$ in a subnet $\{s_{\alpha_\nu} t_{\beta_\nu}\}$ of the product net $\{s_\alpha t_\beta\}$. (This implies that the multiplication in S is not jointly continuous.) Then $f \notin LUC(S)$.*

PROOF. The subnets $\{s_{\alpha_\nu}\}$ and $\{t_{\beta_\nu}\}$ converge to s and t respectively, and $f(st_{\alpha_\nu}) \to f(st) = 1$ by separate continuity of multiplication. Hence,

$$\|f_{s_{\alpha_\nu}} - f_s\| \geqq |f(s_{\alpha_\nu} t_{\beta_\nu}) - f(st_{\beta_\nu})| \to 1; \; f \notin LUC(S).$$

AN EXAMPLE. Let S_1 be a compact semitopological semigroup with multiplication not jointly continuous. Let $S = S_1 \cup R$ be the set-theoretic union of S_1 and R, and let us regard the images of S_1 and R in $S_1 \cup R$ as being open and closed there. S is made a semigroup with the following multiplication:

$xy = xy$ if x and y are both in S_1 or both in R;

$xy = yx = y$ if $x \in S_1$, $y \in R$.

Then $C(S) \neq LMC(S) \neq LUC(S) \neq AP(S)$, $WLUC(S) \neq WAP(S) \neq AP(S)$, and $LUC(S) \not\supset WAP(S)$, $WAP(S) \not\supset LUC(S)$. However, S is locally compact, so $LMC(S) = WLUC(S)$ by Proposition 4.2.

QUESTION. *Is there a semigroup S for which $LMC(S) \neq WLUC(S)$?*

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