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A MULTIPLIER RULE IN SET-VALUED OPTIMISATION

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A multiplier rule is given as a necessary optimality condition for proper minimality in set-valued optimisation. We use derivatives in the sense of the lower Dini derivative for the objective set-valued map and the set-valued maps defining the constraints.

1. INTRODUCTION

Let X, Y, W and Z_i, where $i \in I := \{1, 2, \dots, n\}$, be (real) normed spaces, let the space Y be partially ordered by a pointed convex cone C, let $C_i \subset Z_i$ be convex sets with nonempty interior, let $Q \subset X$ and let $h \in W$. Let $F : X \rightrightarrows Y$, $G_i : X \rightrightarrows Z_i$ and $H : X \to W$ be given set-valued maps. Let $\mathcal{K}$ be the set of all pointed closed convex cones $K \subset Y$ such that $C \backslash \{0_Y\} \subseteq \operatorname{int}(K)$ (interior of K). We use 0_E to represent the zero element of some space E.

We consider the following set-valued optimisation problem ((P_n) for brevity):

$$\text{minimize}\, F(x) \quad \text{subject to} \quad x \in S := \bigl\{x \in Q \mid h \in H(x),\, G_i(x) \cap -C_i \neq \emptyset \ \forall\, i \in I\bigr\}.$$

We are interested in the local proper minimisers. A point $(\overline{x}, \overline{y}) \in X \times Y$ is called a local proper minimiser to (P_n), if there is a neighbourhood U of $\overline{x}$ such that for some $K \in \mathcal{K}$ we have $\bigl(F(S \cap U) - \overline{y}\bigr) \cap \bigl(-K \backslash \{0_Y\}\bigr) = \emptyset$ where $F(S \cap U) := \bigcup_{x \in S \cap U} F(x)$ and $\overline{y} \in F(\overline{x})$. It is a local minimiser to (P_n), if $\bigl(F(S \cap U) - \overline{y}\bigr) \cap \bigl(-C \backslash \{0_Y\}\bigr) = \emptyset$ and a local weak minimiser to (P_n), if $\bigl(F(S \cap U) - \overline{y}\bigr) \cap \bigl(-\operatorname{int}(C)\bigr) = \emptyset$ provided that $\operatorname{int}(C) \neq \emptyset$. If $n = 1$, C_1 is a cone and H is single-valued we shall denote (P_n) by (P_1) and if additionally $H \equiv 0_W$, then by (P_0).

In recent years a great deal of attention has been given to the characterisation of the weak-minimality for (P_0) and (P_1) by employing various notions of derivatives for set-valued maps, see **[2, 3, 4, 7, 8, 11, 12, 13]** and the references therein. A common strategy adopted in these works is to use direct arguments, based on the derivatives chosen for the set-valued maps involved, to verify a claim that the images of the derivatives do not intersect with certain open cones. Here it should be noted that such a disjunction is given

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in the image spaces. Further, this disjunction is then combined with some separation arguments and multiplier rules are obtained. In the context of the classical nonlinear programming, this approach is best comparable with the methodology of [**14**]. However, there is an another approach, often termed the Dubovitskii and Milyutin approach, in nonlinear programmings. This has been systematically investigated by Dubovitski and Milutin [**6**], Halkin [**9**], Craven [**5**] among others, and is based on expressing the optimality as the disjunction of some cones in the domain space. Then the use of some separation arguments combined with some Farkas type lemmas lead to the existence of multipliers. Although this approach has been very useful in scalar and vector optimisation, hitherto there is no counterpart in set-valued optimisation.

The aim of this short note is to show how the ideas in the Dubovitski and Milutin approach can be employed to obtain a multiplier rule for (P_n). Our results show that the lower Dini derivative, introduced by Penot [**15**], is more suited for this approach.

We organise this paper in four sections. In Section 2 we collect some preliminaries. Section 3 contains a multiplier rule for (P_n) which is the main result of this paper. The proof of this result in divided into several lemmas which are of interest in their own right. The paper concludes with some remarks about this approach.

2. Preliminaries

First we recall some results about tangent cones (see [**1, 16, 17**]). Set $\mathbb{P} := \{t \in \mathbb{R} \mid t > 0\}$.

Definition 2.1: Let Z be a normed space, let $S \subset Z$ and let $\overline{z} \in \mathrm{cl}(S)$ (the closure of S).

(a) The contingent cone $T(S,\overline{z})$ of S at $\overline{z}$ is the set of all $z \in Z$ such that there are sequences $(\lambda_n) \subset \mathbb{P}$ and $(z_n) \subset Z$ with $\lambda_n \downarrow 0$, $z_n \to z$ and $\overline{z} + \lambda_n z_n \in S$ for every $n \in \mathbb{N}$.

(b) The interiorly contingent cone $IT(S,\overline{z})$ of S at $\overline{z}$ is the set of all $z \in Z$ such that for any sequences $(\lambda_n) \subset \mathbb{P}$ and $(z_n) \subset Z$ with $\lambda_n \downarrow 0$ and $z_n \to z$, there exists an integer $m \in \mathbb{N}$ such that $\overline{z} + \lambda_n z_n \in S$ for every $n \geqslant m$.

Remark 2.1. It is known that $T(S,\overline{z})$ is a closed cone possessing the isotony property, that is, for subsets S_1 and S_2 such that $S_1 \subset S_2$, the inclusion $T(S_1,\overline{z}) \subset T(S_2,\overline{z})$ holds for every $\overline{z} \in \mathrm{cl}(S_1) \cap \mathrm{cl}(S_2)$. On the other hand the interiorly contingent cone $IT(S,\overline{z})$ is an isotone open cone. Concerning the relationship between $T(S,\overline{x})$ and $IT(S,\overline{z})$, we have $IT(S,\overline{z}) = Z \backslash T(X \backslash S,\overline{z})$. A useful implication of this relationship is that the cones $T(S,\overline{z})$ and $IT(S,\overline{z})$ form an admissible pair, that is, for every pair of sets $S_1, S_2 \subset Z$ with $S_1 \cap S_2 = \emptyset$, we have $T(S_1,\overline{z}) \cap IT(S_2,\overline{z}) = \emptyset$ for every $\overline{z} \in Z$. Also for arbitrary sets $S_1, S_2 \subset Z$ we have $IT(S_1 \cap S_2,\overline{z}) = IT(S_1,\overline{z}) \cap IT(S_2,\overline{z})$ for every $\overline{z} \in S_1 \cap S_2$. In general, this important property is not shared by the contingent cones. For some $S \subset Z$, the identities $T(S,\overline{z}) = T(\mathrm{cl}(S),\overline{z})$ and $IT(S,\overline{z}) = IT(\mathrm{int}(S),\overline{z})$ hold. Moreover, for a convex solid set S, we have $\mathrm{cl}(IT(S,\overline{z})) = T(S,\overline{z})$ and $\mathrm{int}(T(S,\overline{z})) = IT(S,\overline{z})$.

Next we collect some definitions for set-valued maps. Let X and Y be normed spaces. Let $F : X \rightrightarrows Y$ be a set-valued map, that is, for each $x \in X$, we have $F(x) \subset 2^Y$ (the power set of Y). The (effective) domain and the graph of F are defined by $\mathrm{dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}$ and $\mathrm{gph}(F) := \{(x,y) \in X \times Y \mid y \in F(x)\}$, respectively. We shall say that F is *strict* if $\mathrm{dom}(F) = X$. Given a convex cone $C \subset Y$, which induces a partial ordering in Y, the profile map $F_+ : X \rightrightarrows Y$ is given by: $F_+(x) := F(x) + C$ for every $x \in \mathrm{dom}(F)$. Now the epigraph of F can be defined as the graph of F_+, that is, $\mathrm{epi}(F) = \mathrm{gph}(F_+)$. The map F is called convex, if $\mathrm{gph}(F)$ is convex and C-convex, if $\mathrm{epi}(F)$ is a convex set. Finally, we define the *weak-inverse image* $F[S]^-$ of F with respect to a set $S \in Y$ as $F[S]^- := \{x \in X \mid F(x) \cap S \neq \emptyset\}$.

Now, let X^* be dual of X and let $M \subset X$. The negative dual of M, denoted by $M^\star$, is a subset of X^* defined by: $M^\star = \{l \in X^* : \ l(x) \leqslant 0 \text{ for every } x \in M\}$. It is known that if $M_1 \subseteq M_2$ then $M_2^\star \subseteq M_1^\star$. Additionally, $M \subset (M^\star)^\star$ with equality if and only if M is a closed convex cone. Also, the positive dual is then the set defined by $M^* = -M^\star$. Both the positive and the negative duals are closed convex cones. Moreover, the properties just mentioned for the negative dual hold for the positive dual as well.

The following definition of the derivative of a set-valued map is due to Aubin (see [**1**]).

Given a set-valued map $F : X \rightrightarrows Y$ and a point $(\overline{x},\overline{y}) \in \mathrm{gph}(F)$, the contingent derivative of F at $(\overline{x},\overline{y})$ is the set-valued map $D_cF(\overline{x},\overline{y}) : X \rightrightarrows Y$ defined by:

$$D_cF(\overline{x},\overline{y})(x) := \Big\{y \in Y \mid (x,y) \in T\big(\mathrm{gph}(F),(\overline{x},\overline{y})\big)\Big\}.$$

Another notion of derivative for set-valued maps which turns out to be great importance in the present approach is the so-called lower Dini derivative introduced by Penot [**15**].

We recall that given a set-valued map $F : X \rightrightarrows Y$ and a point $(\overline{x},\overline{y}) \in \mathrm{gph}(F)$, the lower Dini derivative of F at $(\overline{x},\overline{y})$ is the set-valued map $D_lF(\overline{x},\overline{y}) : X \rightrightarrows Y$ defined by:

$$D_lF(\overline{x},\overline{y})(x) := \liminf_{(t,z)\to(0_+,x)} \frac{F(\overline{x}+tz)-\overline{y}}{t}.$$

Equivalently $y \in D_lF(\overline{x},\overline{y})(x)$ if and only if for every $(\lambda_n) \subset \mathbb{P}$ and for every $(x_n) \subset X$ with $\lambda_n \downarrow 0$ and $x_n \to x$ there are a sequence $(y_n) \subset Y$ with $y_n \to y$ and an integer $m \in \mathbb{N}$ such that $\overline{y} + \lambda_n y_n \in F(\overline{x} + \lambda_n x_n)$ for every $n \geqslant m$.

Finally we conclude this section by recalling the following important result.

Lemma 2.1. ([**6**]) *Let $C_0, C_1, \dots, C_n$ be non-empty convex cones in a normed space X and let C_i, for $i \in I := \{1,2,\dots,n\}$, be open. Then $\bigcap_{i=0}^{n} C_i = \emptyset$ if and only if there exist $f_j \in C_j^\star$, $j \in \{0\} \cup I$, not all zero, such that: $f_0 + f_1 + \dots + f_n = 0$.*

3. A MULTIPLIER RULE

We begin by introducing the following definitions.

DEFINITION 3.1: Let X and Y be normed spaces and let $R : X \rightrightarrows Y$ be a set-valued map. The map R is called locally convex at $(\overline{x}, \overline{y}) \in \operatorname{gph}(R)$, if the lower Dini derivative $D_l R(\overline{x}, \overline{y})$ of R at $(\overline{x}, \overline{y})$ is a convex set-valued map. The map R is called regular at $(\overline{x}, \overline{y}) \in \operatorname{gph}(R)$, if additionally $D_l R(\overline{x}, \overline{y})$ is strict.

Given $A \subset \mathbb{R}$ and $b \in \mathbb{R}$, the inequality $A \geqslant b$ means that $a \geqslant b$ for every $a \in A$. With this convention in mind we are ready to give the promised multiplier rule.

THEOREM 3.1. *Let $(\overline{x}, \overline{y}) \in \operatorname{gph}(F)$ be a local proper minimiser of (P_n) and let $\overline{z}_i \in G_i(\overline{x}) \cap (-C_i)$ where $i \in I := \{1, 2, \dots, n\}$. Let there exist an open convex cone $L \subset IT(Q, \overline{x})$ and a closed convex cone $M \subseteq T\big(H[h]^-, \overline{x}\big)$. Let F be regular at $(\overline{x}, \overline{y})$ and let G_i be regular at $(\overline{x}, \overline{z}_i)$. Then there exist functionals $s \in L^*$, $t \in C^*$, $u_i \in C_i^*$, $v \in M^*$, not all zero, such that $u_i(C_i - \overline{z}_i) \geqslant 0$. Moreover, the following inequality holds for every $x \in X$:*

$$t \circ D_l F(\overline{x}, \overline{y})(x) + u_1 \circ D_l G_1(\overline{x}, \overline{z}_1)(x) + \cdots + u_n \circ D_l G_n(\overline{x}, \overline{z}_n)(x) \geqslant s(x) + v(x). \tag{1}$$

In addition, if C_i is a cone, then the complementary slackness condition $u_i(\overline{z}_i) = 0$ holds.

We shall divide the proof in several lemmas. We begin with the following.

LEMMA 3.1. *Let $(\overline{x}, \overline{y}) \in \operatorname{gph}(F)$ be a local minimiser to (P_n). Then:*

$$U \cap Q \cap F\big[\overline{y} - C \backslash \{0_Y\}\big]^- \bigcap_{i=1}^{n} G_i[-C_i]^- \cap H[h]^- = \emptyset \tag{2}$$

where U is a neighbourhood of $\overline{x}$ used in the definition of the local minimality.

PROOF: Assume that there exists $x \in U \cap Q \cap F\big[\overline{y} - C \backslash \{0_Y\}\big]^- \bigcap_{i=1}^{n} G_i[-C_i]^- \cap H[h]^-$. Now, from $x \in U \cap Q \bigcap_{i=1}^{n} G_i[-C_i]^- \cap H[h]^-$ we notice that x is feasible and from $x \in F\big[\overline{y} - C \backslash \{0_Y\}\big]^-$ we obtain $F(x) \cap \big(\overline{y} - C \backslash \{0_Y\}\big) \neq \emptyset$ which is in contradiction to the local minimality of $(\overline{x}, \overline{y})$. ∎

LEMMA 3.2. *Let $(\overline{x}, \overline{y}) \in \operatorname{gph}(F)$ be a local minimiser to (P_n). Then:*

$$IT(Q, \overline{x}) \cap IT\Big(F\big[\overline{y} - C \backslash \{0_Y\}\big]^-, \overline{x}\Big) \bigcap_{i=1}^{n} IT\big(G_i[-C_i]^-, \overline{x}\big) \cap T\big(H[h]^-, \overline{x}\big) = \emptyset.$$

PROOF: This assertion follows from Lemma 3.1 and the properties of the interiorly contingent cones and the contingent cones mentioned in Remark 2.1. ∎

LEMMA 3.3. *Let X and Y be normed spaces, let $F : X \rightrightarrows Y$ be a set valued map and let $(\overline{x}, \overline{y}) \in \operatorname{gph}(F)$. Let K be a pointed convex cone with $\operatorname{int}(K) \neq \emptyset$. Then:*

$$D_l F(\overline{x}, \overline{y})\big[-\operatorname{int}(K)\big]^- \subseteq IT\Big(F\big[\overline{y} - \operatorname{int}(K)\big]^-, \overline{x}\Big).$$

PROOF: Let $x \in D_l F(\overline{x}, \overline{y})\big[-\operatorname{int}(K)\big]^{-}$. Then there exists $y \in D_l F(\overline{x}, \overline{y})(x) \cap -\operatorname{int}(K)$. Let $(x_n) \subset X$ and $(\lambda_n) \subset \mathbb{P}$ be arbitrary sequences such that $x_n \to x$ and $\lambda_n \downarrow 0$. It suffices to show that there exists $m \in \mathbb{N}$ such that $\overline{x} + \lambda_n x_n \in F\big[\overline{y} - \operatorname{int}(K)\big]^{-}$ for every $n \geqslant m$. By the definition of $D_l F(\overline{x}, \overline{y})(\cdot)$, there exist $(y_n) \subset Y$ with $y_n \to y$ and $n_1 \in \mathbb{N}$ such that $\overline{y} + \lambda_n y_n \in F(\overline{x} + \lambda_n x_n)$ for every $n \geqslant n_1$. Since $y \in -\operatorname{int}(K)$ and $y_n \to y$, there exists $n_2 \in \mathbb{N}$ such that $\lambda_n y_n \in -\operatorname{int}(K)$ for every $n \geqslant n_2$. This implies that $\overline{y} + \lambda_n y_n \in F(\overline{x} + \lambda_n x_n) \cap \big(\overline{y} - \operatorname{int}(K)\big)$ for $n \geqslant m := \max\{n_1, n_2\}$. Hence for the sequences (x_n) and (λ_n) we have $\overline{x} + \lambda_n x_n \in F\big[\overline{y} - \operatorname{int}(K)\big]^{-}$ for $n \geqslant m$. This is equivalent to saying that $x \in IT\Big(F\big[\overline{y} - \operatorname{int}(K)\big]^{-}, \overline{x}\Big)$. The proof is complete. ∎

LEMMA 3.4. *Let X and Z be normed spaces, let $G : X \rightrightarrows Z$ be a set valued map and let $(\overline{x}, \overline{z}) \in \operatorname{gph}(G)$. Let $A \subset Z$ with $\operatorname{int}(A) \neq \emptyset$. Then the following holds:*

$$D_l G(\overline{x}, \overline{z})\big[IT(-A, \overline{z})\big]^{-} \subseteq IT\big(G[-A]^{-}, \overline{x}\big).$$

PROOF: Let $u \in D_l G(\overline{x}, \overline{z})\big[IT(-A, \overline{z})\big]^{-}$ be arbitrary. Let $(u_n) \subset X$ and $(\lambda_n) \subset \mathbb{P}$ be arbitrary sequences such that $u_n \to u$ and $\lambda_n \downarrow 0$. It suffices to show that there exists $m \in \mathbb{N}$ such that $\overline{x} + \lambda_n u_n \in G[-A]^{-}$ for every $n \geqslant m$. Since $u \in D_l G(\overline{x}, \overline{z})\big[IT(-A, \overline{z})\big]^{-}$, there exists $v \in D_l G(\overline{x}, \overline{z})(u) \cap IT(-A, \overline{z})$. Therefore, there are a sequence $(v_n) \subset Z$ and an integer $n_1 \in \mathbb{N}$ such that $v_n \to v$ and $\overline{z} + \lambda_n v_n \in G(\overline{x} + \lambda_n u_n)$ for every $n \geqslant n_1$. Because of the containment $v \in IT(-A, \overline{z})$ there exists $n_2 \in \mathbb{N}$ such that $\overline{z} + \lambda_n v_n \in -A$ for every $n \geqslant n_2$. Therefore we have $\overline{z} + \lambda_n v_n \in G(\overline{x} + \lambda_n u_n) \cap (-A)$ for every $n \geqslant m := \max\{n_1, n_2\}$. Consequently $u \in IT\big(G[-A]^{-}, \overline{x}\big)$. ∎

LEMMA 3.5. *Let X and Y be normed spaces, let $D \subseteq X$ be convex and let $A \subset Y$ be a solid closed convex cone. Let $T : D \rightrightarrows Y$ be a A-convex set-valued map. If $T\big[-\operatorname{int}(A)\big]^{-} \neq \emptyset$, then for every $p \in P^\star$ where $P := T[-A]^{-}$, there exists $t \in A^*$ such that*

$$t \circ T(x) \geqslant p(x) \quad \textit{for every} \quad x \in D.$$

If $T\big[-\operatorname{int}(A)\big]^{-} = \emptyset$, then there exists $t \in A^* \backslash \{0_{Y^*}\}$ such that

$$t \circ T(x) \geqslant 0 \quad \textit{for every} \quad x \in D.$$

PROOF: We begin with the case when the set $T\big[-\operatorname{int}(A)\big]^{-}$ is nonempty. Then the (negative) dual $P^\star$ of $P := T[-A]^{-}$ is also nonempty. We choose $p \in P^\star$ arbitrarily and define a set $E := \Big\{\big(y, p(x)\big) \in Y \times \mathbb{R} \mid y \in T(x) + A,\ x \in D\Big\}$. In view of the assumptions that D is convex, T is A-convex and $p \in Y^*$, we deduce that E is a convex set. Indeed, let $(y_1, z_1), (y_2, z_2) \in E$ be arbitrary. Then by the definition of E, for $i = 1, 2$, there exists $x_i \in X$ with $z_i = p(x_i)$ and $y_i \in T(x_i) + A$. For $\lambda \in (0, 1]$, we have $\lambda z_1 + (1 - \lambda) z_2 = p\big(\lambda x_1 + (1 - \lambda) x_2\big)$. Further, in view of the A-convexity of T, we have $\lambda y_1 + (1 - \lambda) y_2 \in \lambda T\big(x_1\big) + (1 - \lambda) T\big(x_2\big) + A \subseteq T\big(\lambda x_1 + (1 - \lambda) x_2\big) + A$. This, in view of the convexity of the set D, implies that $\lambda(y_1, z_1) + (1 - \lambda)(y_2, z_2) \in E$.

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Next, we claim that $E \cap \big(-\operatorname{int}(A) \times \mathbb{P}\big) = \emptyset$. In fact, if this is not the case, then there exists $(x, y) \in X \times Y$ such that $y \in \big(T(x) + A\big) \cap \big(-\operatorname{int}(A)\big)$ and $p(x) > 0$. Let $w \in T(x)$ be such that $y \in w + A$. Then $w \in y - A \subset -\operatorname{int}(A) - A = -\operatorname{int}(A)$. This however contradicts that $p \in P^\star$. Therefore $E \cap \big(-\operatorname{int}(A) \times \mathbb{P}\big) = \emptyset$ and hence by a separation theorem (see [**10**]), we get the existence of $(f, g) \in Y^* \times \mathbb{R}\backslash\{0_{Y^*}, 0\}$ and a real number α such that we have

$$(3) \qquad f(u) + g(v) \geqslant \alpha \quad \text{for every} \quad (u, v) \in E$$

$$(4) \qquad f(c) + g(d) < \alpha \quad \text{for every} \quad (c, d) \in -\operatorname{int}(A) \times \mathbb{P}.$$

Since A is a cone, we can set $\alpha = 0$ in (3) and (4). By taking $d \in \mathbb{P}$ arbitrary close to 0 and $c \in -\operatorname{int}(A)$ arbitrary close to 0_Y, we obtain $f \in A^*$ and $g \leqslant 0$, respectively. We claim that $g < 0$. Indeed, if $g = 0$, we get $f(c) < 0$ for every $c \in -\operatorname{int}(A)$ and $f(u) \geqslant 0$ for every $u \in T(D) + A$. This, however is impossible because we have $\big(T(D) + A\big) \cap \big(-\operatorname{int}(A)\big) \neq \emptyset$. Therefore $g < 0$. Moreover, from (3), for every $x \in D$ we have $f \circ (T + A)(x) \geqslant -(g \cdot p)(x)$. By setting $t = (-f/g) \in A^*$ and noticing that $0_Y \in A$, we finish the proof of the first part.

For the second part, we notice that if $T\big(-\operatorname{int}(A)\big) = \emptyset$, we have $T(D) \cap -\operatorname{int}(A) = \emptyset$ and hence by the arguments similar to those given above we can prove the existence of $t \in A^*\backslash\{0_Y\}$ such that $t \circ T(x) \geqslant 0$ for every $x \in D$. ∎

Proof of Theorem 3.1 Set $\Phi := D_l F(\overline{x}, \overline{y})\big[-\operatorname{int}(K)\big]^-$ and $\Psi_i := D_l G_i(\overline{x}, \overline{z}_i)\big[IT(-C_i, \overline{z}_i)\big]^-$. We shall prove the theorem by analysing the three possibilities, namely:

(i) $\Phi = \emptyset$;

(ii) $\Psi_i = \emptyset$ for some $i \in I$;

(iii) $\Phi \neq \emptyset$ and $\Psi_i \neq \emptyset$ for every $i \in I$.

We begin with case (i). Let $\Phi = \emptyset$. Then it follows from Lemma 3.5 that there exists $t \in C^*\backslash\{0_{Y^*}\}$ such that $t \circ D_l F(\overline{x}, \overline{y})(x) \geqslant 0$ for every $x \in X$. By choosing $s = 0_{X^*}$, $v = 0_{W^*}$ and $u_i = 0_{Z_i^*}$, for every $i \in I$, we obtain the desired result. For case (ii), let there exist $i \in I$ such that $\Psi_i = \emptyset$. Then again by invoking Lemma 3.5, we obtain $u_i \in \big(T(C_i, -\overline{z}_i)\big)^*\backslash\{0_{Z_i^*}\}$ such that $\big(u_i \circ DG_i(\overline{x}, \overline{z}_i)\big)(x) \geqslant 0$ for every $x \in X$. By setting $s = 0_{X^*}$, $v = 0_{W^*}$ and $u_j = 0_{Z_j^*}$, $i \neq j \in I$, we obtain (1). For $u_i(C_i - \overline{z}_i) \geqslant 0$, it suffices to notice that in view of the convexity of C_i, we have $T(C_i, -\overline{z}_i) \supseteq C_i + z_i$ and hence $u_i(z + z_i) \geqslant 0$ for every $z \in C_i$. If C_i is a cone then by choosing $z = 0_{Z_i}$ and $z = -2z_i$ we obtain that $u(z_i) = 0$.

Finally, we consider the case (iii). Since $(\overline{x}, \overline{y})$ is a proper minimiser of (P_n), it has to be a minimiser of (P_n) with respect to some $K \in \mathcal{K}$. Therefore, it follows from Lemma 3.2 and the imposed conditions that for such $K \in \mathcal{K}$, we have

$$L \cap \Phi \bigcap_{i=1}^{n} \Psi_i \cap M = \emptyset.$$

Since L, M, Φ and Ψ_i, $(i \in I)$ are all nonempty, we can apply Lemma 2.1 to assure the existence of

$$l \in L^\star,\ l_0 \in \Big(D_l F(\overline{x},\overline{y})\big[-\operatorname{int}(K)\big]^-\Big)^\star$$

and

$$l_i \in \Big(D_l G_i(\overline{x},\overline{z}_i)\big[IT(-C_i,\overline{z}_i)\big]^-\Big)^\star$$

and

$$l_{n+1} \in M^\star$$

such that

$$l + l_0 + l_1 + l_2 + \cdots + l_n + l_{n+1} = 0. \tag{5}$$

Now, in view of Lemma 3.5, we get the existence of functionals $t \in C^*$ and $u_i \in T(C_i, -\overline{z}_i)^*$ such that for all $x \in X$, the following inequalities hold

$$\big(t \circ D_l F(\overline{x},\overline{y})\big)(x) \geqslant l_0(x);$$
$$\big(u_i \circ D_l G_i(\overline{x},\overline{z}_i)\big)(x) \geqslant l_i(x), \quad i \in I.$$

Combining of the above inequalities with (5) and setting $s = -l$ and $v = -l_{n+1}$ yield (1). The proof for $u_i(C_i - \overline{z}_i) \geqslant 0$ and the complementary slackness is the same as in part (ii). ∎

4. Concluding Remarks

It is clear that in Theorem 3.1 we have not imposed any differentiability assumption on the map H. Thus it would be of interest to obtain a variant of the well-known theorem of Lyusternik ([**10**]), so that the cone $M^\star$ contain information about some derivative of H. In fact, this is completely true if H is single-valued and sufficiently smooth ([**17**]). Moreover, we can also define a variant of the generalised contingent epiderivative (see [**3**, **8**, **11**]) by taking the minimal points of $D_l(F + C)(\overline{x},\overline{y})$ with respect to the cone C. We mention that our results will remain valid for such an epiderivative.

References

[1] J.P. Aubin and H. Frankowska, *Set valued analysis* (Birkhäuser, Boston, 1990).

[2] G. Bigi and M. Castellani, 'K-epiderivatives for set-valued functions and optimization', *Math. Methods Oper. Res.* **55** (2002), 401–412.

[3] G.Y. Chen and J. Jahn, 'Optimality conditions for set-valued optimization problems', *Math. Methods Oper. Res.* **48** (1998), 187–200.

[4] H.W. Corley, 'Optimality conditions for maximization of set-valued functions', *J. Optim. Theory Appl.* **58** (1988), 1–10.

[5] B.D. Craven, 'Nonlinear programming in locally convex spaces', *J. Optim. Theory Appl.* **10** (1972), 197–210.

[6] A.Y. Dubovitskii and A.A. Milyutin, 'Extremal problems in presence of constraints', *Comp. Math. Math. Phys.* **5** (1965), 395–453.

[7] F. Fabin Flores-Bazàn, 'Optimality conditions in non-convex set-valued optimization', *Math. Methods Oper. Res.* **53** (2001), 403–417.

[8] A. Götz and J. Jahn, 'The Lagrange multiplier rule in set-valued optimization', *SIAM J. Optim.* **10** (1999), 331–344.

[9] H. Halkin, 'A satisfactory treatment of equality and operator constraints in the Dubovitskii-Milutin optimization formalism', *J. Optim. Theory Appl.* **6** (1970), 138–149.

[10] J. Jahn, *Mathematical vector optimization in partially ordered linear spaces* (Peter Lang, Frankfurt, 1986).

[11] J. Jahn and A.A. Khan, 'Generalized contingent epiderivatives in set-valued optimization', *Numer. Func. Anal. Optim.* **27** (2002), 807–831.

[12] J. Jahn and R. Rauh, 'Contingent epiderivatives and set-valued optimization', *Math. Methods Oper. Res.* **46** (1997), 193–211.

[13] D.T. Luc, *Theory of vector optimization*, Lecture Notes Econ. Math. Science **319** (Springer-Verlag, Berlin, Heidelberg, New York, 1988).

[14] Y. Nagahisa and Y. Sakawa, 'Nonlinear programming in normed spaces', *J. Optim. Theory Appl.* **4** (1969), 182–190.

[15] J.P. Penot, 'Differentiability of relations and differential stability of perturbed optimization problems', *SIAM J. Control Optim.* **22** (1984), 529–551.

[16] J.P. Penot, 'Variations on the theme of nonsmooth analysis: Another subdifferential', in *Nondifferentiable optimization*, Lecture Notes Econom. Math. Systems **255**, 1985, pp. 41–54.

[17] L. Rigby, 'Contribution to Dubovitskiy and Milyutin's optimization formalism', in *Optimization Techniques*, Lecture Notes Computer Science **41**, 1976, pp. 438–453.

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