PROPERTIES OF HARMONIC FUNCTIONS OF THREE REAL VARIABLES GIVEN BY BERGMAN-WHITTAKER OPERATORS

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1. Introduction. Let $\mathfrak{L}$ be a closed rectifiable curve, not going through the origin, which bounds a domain Ω in the complex ζ-plane. Let $X = (x, y, z)$ be a point in three-dimensional euclidean space E^3 and set

$$v(X, \zeta) = Z\zeta^2 + x\zeta + Z^*, \qquad Z = \tfrac{1}{2}(iy + z),\ Z^* = \tfrac{1}{2}(iy - z). \tag{1}$$

The Bergman-Whittaker operator defined by

$$H(X) = B(g) \equiv \frac{1}{2\pi i}\int_{\mathfrak{L}} g(v, \zeta)\,\frac{d\zeta}{\zeta}, \tag{2}$$

transforms analytic functions of two complex variables v and ζ into harmonic functions H of three variables defined in a certain domain of E^3 (the domain of association); H can be continued analytically and thus we obtain a mapping of the analytic function g into a harmonic function $\mathfrak{H}$ defined (in general) in a domain which multiply covers E^3. Thus we have the following steps in mapping by this method

$$g(v, \zeta) \rightarrow B(g) \rightarrow \mathfrak{H}(X),$$

the first step being obtained by an integral formula and the second by analytic continuation. Different classes of functions g such as rational or algebraic, the integral of an algebraic function or $g = f\zeta^m$ where f is a meromorphic function of one complex variable and m a non-negative integer have been shown by Bergman and others to lead to different classes of harmonic functions (**1**; **2**; **3**; **4**; **7**).

An important problem in the theory of integral operators consists in the study of various properties of the function $\mathfrak{H}$ such as the location and type of its singularities. In this paper we consider the problem when

$$g(v, \zeta) = f(v, \zeta)p(v), \tag{3}$$

where p is a meromorphic function of v with an infinity of poles, none of which is the origin, and f/ζ is an entire function of the complex variables v and ζ. In §2 the properties of (2) with g given by (3) are discussed, including a study of the number of algebraic singular lines of H as $X \rightarrow \infty$ in different directions. It is found that (2) represents a multiple-valued harmonic function H in the

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domain of association of the integral operator (2); H can be extended analytically except for an exceptional set of lower dimension over a Riemann domain (the domain of existence of $\mathfrak{H}$), which multiply covers E^3. $\mathfrak{H}$ branches along a denumerable set of circles of increasing radii and all passing through the origin, where it has algebraic singularities of a pole-like type and an essential singularity on the negative (or positive) x-axis. Using known results on the minimum modulus of entire functions the growth properties of (2) are studied in §3 when the denominator of p is an entire function of finite order and $\mathfrak{L}$ is the unit circle. In §4 a representation is obtained for H when p in (3) is represented as the limit of a series summable in a star domain by the Mittag-Leffler method.

Suppose p has poles at e_k and

$$0 < |e_1| < |e_2| < \dots . \tag{4}$$

Set

$$\begin{aligned} E(v, r) &= (1 - v)e^{p_r(v)} \\ p_r(v) &= v + \tfrac{1}{2}v^2 + \dots + \frac{1}{r}v^r. \end{aligned} \tag{5}$$

By the Weierstrass factor theorem it is known that

$$p(v) = h(v) \Big/ \prod_{k=1}^{\infty} E(v/e_k, r_k - 1), \tag{6}$$

where h is an entire function and $\{r_k\}$ is a set of positive integers such that the infinite series

$$\sum \left| \frac{v}{e_k} \right|^{r_k}$$

converges for all v. We set

$$\begin{aligned} f_1(v, \zeta) &= f(v, \zeta)h(v) \\ p_1(v) &= 1 \Big/ \prod_{k=1}^{\infty} E(v/e_k, r_k - 1) \end{aligned}$$

so that $g = f_1 p_1$ and use the normalized function p_1 in place of p, dropping the subscripts.

2. Properties of (2).

1. *Explicit representation for H.* If v in p is replaced by its value in (1), a function $P(X, \zeta)$ is obtained which has poles in the ζ-plane for $Z \neq 0$ at

$$\begin{aligned} \zeta_1^{(k)} &= \frac{-x + R_k(X)}{2Z} \\ \zeta_2^{(k)} &= \frac{-x - R_k(X)}{2Z}, \end{aligned} \tag{1a}$$

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where

(1b) $$R_k^2(X) = R^2 + 4Ze_k, \quad R^2 = x^2 + y^2 + z^2.$$

If $R_k(X) = 0$ and $Z \neq 0$, then

(1c) $$\zeta_1^{(k)} = \zeta_2^{(k)} \equiv \zeta_0 = -x/2Z.$$

If $Z = 0$, $(x \neq 0)$, P has poles at

(1d) $$\zeta^{(k)} = e_k/x$$

and, if $X = 0, v(0, \zeta) = 0$ for all ζ, so that $P(0, \zeta) = 1$.

Let

(2) $$b_k^1 = [X | R_k(X) = 0, \qquad X \neq 0]^*, \quad b^1 = \bigcup_{k=1}^{\infty} b_k^1,$$

and c^1 be the x-axis. Separated into real and imaginary parts $R^2(X) = 0$ becomes

$$x^2 + (y - b_k)^2 + (z + a_k)^2 = |e_k|^2$$
$$a_k y + b_k z = 0,$$

where $a_k = \text{Re } e_k$, $b_k = \text{Im } e_k$, so that b_k^1 is a circle lying on the plane $a_k y + b_k z = 0$ with the point $X = 0$ omitted. The sets $b_k^1 \cap c^1$ and $b_k^1 \cap b_j^1$ $(j \neq k)$ are empty.

We remark that v in (1.1) may be replaced by $v - \alpha\zeta$, α a complex number, in which case $R_k^2(X) = 0$ becomes

$$(x - \alpha)^2 + y^2 + z^2 + 2e_k(iy + z) = 0$$

and analogous results are obtained.

It is seen from formulae (1) by a computation that for fixed $X \neq 0$ $|\zeta_\mu^{(k)}| \to \infty$ as $k \to \infty$ $(\mu = 1, 2)$, so that only a finite number of poles of P lie inside $\mathfrak{L}$. Also, if f is an entire function of v and ζ, then $F(X, \zeta) = f(v(X, \zeta), \zeta)$ is an entire function of X and ζ. It is also convenient to assume that f has a factor ζ so that the integrand has no singularity at $\zeta = 0$. Assuming that no pole lies on $\mathfrak{L}$ we get by the residue theorem

$$H(X) = \sum \text{residue at } \zeta_\mu^{(k)}$$

summed over all $\zeta_\mu^{(k)}$ in Ω. Since we have assumed that all e_k are distinct, the value of the residue at $\zeta_1^{(k)}$ for $X \notin b_k^1 \cup c^1$ is

(3) $$A_k(X, \zeta_1^{(k)}) \equiv -e_k A_k F(X, \zeta_1^{(k)}) / \zeta_1^{(k)} R_k(x),$$

where A_k is a non-zero constant equal to

$$Q(X, \zeta) = p[v(X, \zeta)][1 - v(X, \zeta)e_k^{-1}]$$

*The superscript on b_k^1 refers to the dimension of the set in E^3.

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when $\zeta = \zeta_1^{(k)}$, $v(X, \zeta_1^{(k)})$ being equal to e_k. Similarly the residue at $\zeta_2^{(k)}$ is $-A_k(X, \zeta_{2(k)})$. Thus for $X \notin b_k^1 \cup c^1$ for all k for which $\zeta_\mu^{(k)} \in \Omega$

(4) $$H(X) = \sum_{\zeta_\mu^{(k)} \in \Omega} \pm A_k(X, \zeta_\mu^{(k)}).$$

Since f/ζ is an entire function of v and ζ,

$$f_0(v, \zeta) \equiv \frac{f(v, \zeta)}{\zeta} = \sum_{m,n=0}^{\infty} a_{mn} v^m \zeta^n$$

for $|v| < \infty$, $|\zeta| < \infty$. For any $X \in E^3$ and $|\zeta| < \infty$, v is finite, hence for all $X \in E^3$

$$F_0(X, \zeta) \equiv \frac{F(X, \zeta)}{\zeta} = \sum_{m,n=0}^{\infty} a_{mn} v^m(X, \zeta)\zeta^n.$$

Since $v(X, \zeta_1^{(k)}) = e_k$, $F_0(X, \zeta_1^{(k)}) = F_0(\zeta_1^{(k)})$ is a function of $\zeta_1^{(k)}$ only and has the series representation

$$\sum_{n=0}^{\infty} b_n (\zeta_1^{(k)})^n$$

for $Z \neq 0$ from which it is seen that $F_0(\zeta_1^{(k)})$ has a singularity of an essential type on the negative x-axis; $F_0(\zeta_2^{(k)})$ has an analogous singularity on the positive x-axis. Thus the function represented by formula (4) is a multiple-valued function of X which has algebraic singularities of a pole-like type along the curves b_k^1, which are analogous to singularities obtained by Bergman **(3)**, essential-type singularities on the positive or negative x-axis (or both) and is undefined at $X = 0$.*

2. *Exceptional cases to formulae* (3) *and* (4). Exceptional cases arise when (i) the roots of $v(X, \zeta) = e_k$ coincide, (ii) $Z = 0$, (iii) $X = 0$, and (iv) the integrand is undefined.

(i) If $\zeta_\mu^{(k)} \in \Omega$ and $X \in b_k^1$ the integral operator (1.2) gives a different function. In this case $v(X, \zeta) = e_k$ has coincident roots ζ_0 given by (1c) and the residue at ζ_0 is

$$B_k(X, \zeta_0) = -e_k[F_{0\zeta}(\zeta_0)Q(X, \zeta_0) + F_0(\zeta_0)Q_\zeta(X, \zeta_0)]/Z.$$

As we have seen $Q(X, \zeta_0)$ equals the constant A_k and similarly $Q_\zeta(X, \zeta_0)$ equals a constant B_k. Thus $B_k(X, \zeta_0)$ is a single-valued function with a singularity on the x-axis $(x \neq 0)$ and

(5) $$H(X) = \sum_{\substack{\zeta_\mu^{(j)} \in \Omega \\ j \neq k}} \pm A_j(X, \zeta_\mu^{(j)}) + B_k(X, \zeta_0).$$

*If f does not have a common factor ζ and $0 \in \Omega$, then the function represented by (4) also has algebraic singularities along the half-lines $y = 2b_k$, $z = -2a_k$ $(x > 0)$ given by $\zeta_1^{(k)} = 0$. Also, H is increased by a function with a simple pole on each line $y = 2b_k$, $z = -2a_k$.

(ii) If $Z = 0$ and $x \neq 0$, $v(X, \zeta) = x\zeta$ so that the residue at $\zeta = \zeta^{(k)}$ is $-e_k A_k x^{-1} F(e_k x^{-1})$ and on c^1

$$H(X) = -x^{-1} \sum_{\zeta^{(k)} \in \Omega} e_k A_k F(e_k x^{-1})$$

which is a single-valued function of X with an essential singularity at $x = 0$.

(iii) If $X = 0$, $v(X, \zeta) \equiv 0$, $p(v) \equiv 1$ and $H(0) = 0$.

(iv) The set of points in E^3, where the associate g is undefined. For fixed ζ the equation $v(X, \zeta) = e_k$ or

$$\text{(6a)} \qquad \begin{aligned} 2x\mathrm{Re}\zeta - y\mathrm{Im}\zeta^2 + z(\mathrm{Re}\zeta^2 - 1) &= 2a_k \\ 2x\mathrm{Im}\zeta + y(\mathrm{Re}\zeta^2 + 1) + z\mathrm{Im}\zeta^2 &= 2b_k \end{aligned}$$

represents a line $l_k^1(\zeta)$ in E^3. Hence

$$\text{(6b)} \qquad B_k^2(\mathfrak{L}) = \bigcup_{\zeta \in \mathfrak{L}} l_k^1(\zeta)$$

is a ruled surface and

$$\text{(6c)} \qquad B^2(\mathfrak{L}) = \bigcup_{k=1}^{\infty} B_k^2(\mathfrak{L})$$

a family of ruled surfaces. Now $X \in B^2(\mathfrak{L})$ implies that there exists k and $\zeta_1 \in \mathfrak{L}$ such that $X \in l_k^1(\zeta_1)$, which implies that equation $v(X, \zeta_1) = e_k$ is satisfied. But then ζ_1 is one of the poles $\zeta_\mu^{(k)}$ of the function $p(v)$. Consequently the surfaces $B_k^2(\mathfrak{L})$, which are referred to as surfaces of separation **(3)**, subdivide E^3 into a denumerable number of cells (called domains of association) in each of which the number of singularities inside $\mathfrak{L}$ remains constant. Call these cells $D_p^3 (p = 1, 2, \ldots)$. As X crosses from one cell to another it meets a surface $B_k^2(\mathfrak{L})$ and for this X the integral operator (1.2) is undefined. Thus for $X \in D_p^3 - b^1 \cup c^1$, (1.2) defines a branch of a complex harmonic function given by (3) and (4), which we shall call $H^{(p)}$.

We summarize these results in

THEOREM 1. *Let the function g in the integral operator* (1.2) *be given by* (1.3). *For* X *in the set* $D_p^3 - b^1 \cup c^1$ $(p = 1, 2, \ldots)$ (1.2) *represents a branch,* $H^{(p)}$, *of a complex harmonic function given by formulae* (3) *and* (4). *For* $X \in b_k^1$ $(k = 1, 2, \ldots)$ *it represents the function* (5) *and on* c^1 *it represents a single-valued harmonic function with an essential singularity at* $x = 0$. *The integral operator is undefined on* $B^2(\mathfrak{L})$.

Remark. Since in general $H^{(p)}$ cannot be continued into the function represented by (1.2) when $X \in b^1 \cup c^1$, in order to get a general harmonic function $\mathfrak{H}$ by analytic extension we consider only the set of functions

$$\text{(7)} \qquad \mathfrak{H} = \{H^{(p)}\} \quad (p = 1, 2, \ldots);$$

$H^{(p)}$ being represented by (1.2) when $X \in D_p^3 - b^1 \cup c^1$. H refers to the harmonic function represented by (1.2) when $X \in E^3 - B^2(\mathfrak{L})$.

3. *The Riemann domain R^3 on which $\mathfrak{H}$ is single-valued.*

If g is a rational function of $u = \zeta^{-1}v$ and ζ and hence of ζ and X:

$$G(X, \zeta) = P(X, \zeta)/Q(X, \zeta),$$

where P and Q are polynomials in ζ and X, we know that the Riemann domain on which the corresponding harmonic function is defined and single-valued has the equation

$$Q(X, \zeta)/A_{2N} = 0$$

(A_{2N} the leading coefficient of Q) **(2)**. Similarly the Riemann domain R^3 on which $\mathfrak{H}$ is single-valued has the equation

$$S(X, \zeta) = \prod_{k=1}^{\infty} E[v(X, \zeta)/e_k, r_k - 1] = 0. \tag{8}$$

In order to study this domain we consider at first the equation

$$S_p(X, \zeta) = \prod_{k=1}^{p} E[v(X, \zeta)/e_k, r_k - 1] = 0. \tag{9}$$

The Riemann domain R_p^3 defined by (9) has $2p$ sheets given by

$$S_{2k}: \zeta = \zeta_1^{(k)}(X), S_{2k+1}: \zeta = \zeta_2^{(k)}(X) \quad (X \neq 0) \quad (k = 1, \ldots, p),$$

where $\zeta_j^{(k)}$ are given by (1). S_{2k} and S_{2k+1} are connected at the branch curves b_k^1 (see (1) and (2)) and the x-axis c^1. The Riemann domain R^3 given by (8) is the limiting case of R_p^3 as $p \to \infty$. Hence R^3 has an infinite number of sheets $\{S_n\}$ $(n = 1, 2, \ldots)$, the sheets being connected in pairs S_{2k}, S_{2k+1} along the branch curves b_k^1 and c^1. If i and j are not consecutive integers of the form $2k$, $2k + 1$ the sheets S_i and S_j are not connected. As k increases the spheres on which b_k^1 lie are of increasing radii $|e_k|$ and all passing through the origin. Infinity is a singularity of higher order which is not assumed to lie on R^3. We state as a corollary to Theorem 1:

COROLLARY. *Under the hypothesis of* Theorem 1 *the function*

$$\mathfrak{H} = \{H^{(p)}\} \quad (p = 1, 2, \ldots),$$

represented by (1.2) *when* $X \in E^3 - B^2(\mathfrak{L}) \cup b^1 \cup c^1$, *is a complex harmonic function which is single-valued on the Riemann domain* R^3. $H^{(p)}$ *has a finite number of algebraic singularities of a pole-like type on* b^1 *and a singularity on the positive or negative x-axis* (*or both*) *of an essential type.*

4. *The number of algebraic singular lines possessed by H.*

Disregarding the essential type singularity of H on the positive or negative x-axis, let $n(x, y, z)$ be the number of algebraic singular lines possessed by H for $\mathfrak{L}$ a given closed curve. As we have seen by (3) and (4) H has a finite number of algebraic singular lines $\subset b^1$. Furthermore for $X \in D_p^3$ $n(x, y, z)$ is a non-negative integer. However, the number of singularities may become infinite as $X \to \infty$ in certain directions, thus giving a different type of singularity for X infinite in this direction. For example, let $X \to \infty$ along the negative z-axis, $\mathfrak{L}$ be the unit circle, and e_k real and positive. If $x = y = 0$, $z \neq 0$ in (1), then

$$\zeta_\mu^{(k)} = \pm[z^2 + 2ze_k]^{\frac{1}{2}}/z.$$

If $z > 0$, then $|\zeta_\mu^{(k)}| > 1$ and there are no algebraic singularities inside $\mathfrak{L}$ so that $n(0, 0, z) = 0$. If $z < 0$ and also $|z| > e_k$, then $|\zeta_\mu^{(k)}| < 1$ and all such singularities lie inside $\mathfrak{L}$; for fixed z the number $n(0, 0, z)$ is finite but increasing monotonically as $|z|$ increases since e_k is a monotone non-decreasing sequence. Hence $\lim_{z\to-\infty} n(0, 0, z) = \infty$, whereas $\lim_{z\to\infty} n(0, 0, z) = 0$.

3. Growth properties of H. Using certain results on the minimum modulus of an entire function of finite order ρ in the theory of functions of one complex variable **(8)** we have

THEOREM 2. *Let the function g in* (1.2) *be given by* (1.3), *where p^{-1} is an entire function of finite order ρ and f entire of finite order ρ with respect to v on $|\zeta| = 1$. Let $\mathfrak{L}$ be the unit circle $|\zeta| = 1$. If $\sigma > \rho$ and ϵ are arbitrary positive numbers then for all X on the sphere S_R^2: $x^2 + y^2 + z^2 = R^2$, $R = R(\sigma, \epsilon)$, provided X does not belong to a certain set $C^3(\mathfrak{L})$* (see (5)),

$$|H(X)| \leqslant Me^{R^{\rho+\epsilon_1}}, \tag{1}$$

M a positive constant and $\epsilon_1 > \epsilon$.

Proof. From the theory of entire functions of one complex variable it is known that for a canonical product of order ρ, if $\sigma(>\rho)$ and ϵ are positive numbers, then for all sufficiently large $r = r(\sigma, \epsilon)$

$$\log |p^{-1}(v)| > -r^{\rho+\epsilon}, \tag{2}$$

where $|v| = r$, provided v lies outside circles of centre e_k and radius $|e_k|^{-\sigma}$ **(8)**. Consequently

$$|p(v)| < e^{r^{\rho+\epsilon}}. \tag{3}$$

Also

$$|f(v, \zeta)| = 0(e^{r^{\rho+\epsilon}}) \tag{4}$$

on $|\zeta| = 1$.

Now for $X \in S_R^2$ and $\zeta = e^{i\theta}$

$$|v|^2 = |\zeta|^2|Z\zeta + x + Z^*\zeta^{-1}|^2 = x^2 + r_0^2\cos^2(\theta - \phi),$$

where $y = r_0 \cos \phi$, $z = r_0 \sin \phi$. Hence

$$|v|^2 \leqslant x^2 + r_0^2 = R^2.$$

Thus for such X and ζ (3) and (4) hold with r replaced by R from which (1) follows.

The hypothesis

$$|v - e_k| > |e_k|^{-\sigma}$$

for each $\zeta \in \mathfrak{L}$ implies that X must satisfy the condition

$$|\zeta||Z\zeta + x + (Z^* - e_k)\zeta^{-1}| > |e_k|^{-\sigma},$$

that is

$$[x - a_k \cos\theta - b_k \sin\theta]^2 + [(y - b_k)\cos\theta + (z + a_k)\sin\theta]^2 > |e_k|^{-2\sigma}.$$

Now

$$y_k = y_k(\theta) \equiv (y - b_k)\cos\theta + (z + a_k)\sin\theta$$

is one of the equations of rotation by θ in the yz-plane about the point $(b_k, -a_k)$. Thus the set of excluded points

(5a) $$C_k^3(\theta, \mathfrak{L}) = [X|(x - a_k\cos\theta - b_k\sin\theta)^2 + y_k^2 \leqslant |e_k|^{-2\sigma}]$$

is an infinite right cylinder with circular cross-section of radius $|e_k|^{-\sigma}$ and centre

$$x = A_k = A_k(\theta) \equiv a_k\cos\theta + b_k\sin\theta,\ y_k = 0.$$

Its axis is the line perpendicular to the y_k-axis and x-axis and going through the point $(A_k, b_k, -a_k)$. For $0 \leqslant \theta \leqslant 2\pi$ the excluded set is the one parameter family of cylinders

(5b) $$C_k^3(\mathfrak{L}) = \bigcup_{0 \leqslant \theta \leqslant 2\pi} C_k^3(\theta, \mathfrak{L}).$$

Also set

(5c) $$C^3(\mathfrak{L}) = \bigcup_{k=1}^{\infty} C_k^3(\mathfrak{L}).$$

The surface $C_k^3(\mathfrak{L})$ consists of one infinite right cylinder of circular cross-section and radius $|e_k|^{-\sigma}$ in each direction θ, measured from the line $z = -a_k$ about the point $(b_k, -a_k)$ and lying in the plane $x = A_k$.

Now fix $y = y_0, z = z_0$. For each k there exists $\theta = \theta_0$ such that (y_0, z_0) lies on the line $y_k(\theta_0)$, since these lines cover the whole yz-plane. For this value of θ, $x \in C_k^3(\theta_0, \mathfrak{L})$ satisfies the inequality

$$A_k(\theta_0) - |e_k|^{-\sigma} \leqslant x \leqslant A_k(\theta_0) + |e_k|^{-\sigma}.$$

Hence on any line $y = y_0, z = z_0$ the set of points x removed is contained in a set whose total length is

$$2\sum_{k=0}^{\infty} |e_k|^{-\sigma}.$$

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But for $\sigma > \rho$ this series is convergent and hence the set of excluded points is contained in a segment of finite length and the remaining points lie outside the excluded region $C^3(\mathfrak{L})$. This remark is valid for every line perpendicular to the yz-plane. Hence $E^3 - C^3(\mathfrak{L})$ is a three-dimensional set of points.

We must also consider the set of points $B^2(\mathfrak{L})$ where (1.2) is undefined; $X \in B^2(\mathfrak{L})$ implies there exist k and ζ such that the equation $v(X, \zeta) = e_k$ is satisfied. (See (2.6).) Hence $X \in C_k^3(\mathfrak{L})$ and $B_k^2(\mathfrak{L}) \subset C_k^3(\mathfrak{L})$, $B^2(\mathfrak{L}) \subset C^3(\mathfrak{L})$. This completes the proof of Theorem 2.

Similarly other known results on the minimum modulus of entire functions **(5)** may be used to obtain inequalities for the function $H(X)$ represented by (1.2).

Remark. The envelope E_k^2 of the family $\{B_k^2(\theta)\}(0 \leqslant \theta \leqslant 2\pi)$, where $B_k^2(\theta)$ is the boundary of the surface $C_k^3(\theta, \mathfrak{L})$ is found by eliminating θ between the equation of $B_k^2(\theta)$ and of its partial derivative with respect to θ. For fixed (y, z) the distance between the top and bottom sheets of the envelope is $2|e_k|^{-\sigma}$, which is less than 1 for k sufficiently large. The excluded surfaces $C_k^3(\theta)$ lie between the top and bottom sheets of the envelope.

4. Mittag-Leffler summability for H.

1. *Representation for H obtained by using Mittag-Leffler summability method for p.* In this section it is convenient to replace v in (1.1) by

$$u = \zeta^{-1}v \tag{1}$$

and take $\mathfrak{L}$ as the unit circle. We also assume that p in (1.3) is an analytic function of u in a star domain with centre at the origin and f has the series representation

$$f(u, \zeta) = \sum_{p,q=0}^{\infty} c_{pq} u^p \zeta^q, \tag{2}$$

where for u and ζ independent variables the convergence is uniform in any closed domain such that $|u| < \infty$, $|\zeta| < \infty$.

Bergman has shown that there exists a set of homogeneous polynomials $\{\Gamma_{p\kappa}\}$ $(p = 0, 1, 2, \ldots; \kappa = 0, 1, \ldots, 2p)$, $\Gamma_{p\kappa}$ being given by the integral operator (1.2) when the associate is $u^p\zeta^{-p+\kappa}$ **(3)**.

THEOREM 3. *Let the associate g in the operator* (1.2) *equal fp where f has the series representation* (2) *and p is an analytic function of u in a star domain with centre at the origin whose function element is $\sum_{n=0} a_n u^n$. From the representation*

$$p(u) = \lim_{\sigma\to 0} \sum_{n=0}^{\infty} a_n u^n / n^{\sigma n} \tag{3}$$

follows the representation

$$H(X) = \lim_{\sigma\to 0} \sum_{n=0}^{\infty} (a_n / n^{\sigma n}) H_n(X), \tag{4}$$

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where

$$H_n(X) = \sum_{p=0}^{\infty} \sum_{q=0}^{n+p} c_{pq}\Gamma_{n+p,n+p+q}(X).$$

If a is a singularity of p with Re $a \neq 0$, *then* (4) *is not valid when X belongs to the set*

(5) $D_a^3 = [X | y^2 + z^2 \geqslant A^2x^2, x \geqslant \text{Re}\, a \text{ if } \text{Re}\, a > 0, x \leqslant \text{Re}\, a \text{ if } \text{Re}\, a < 0]$

($A = \text{Im}\, a/\text{Re}\, a$). (For the case Re $a = 0$ see paragraph 2.)

Proof. From the theory for one complex variable it is known that if p satisfies the hypothesis of the theorem, then the series

$$\sum_{n=0}^{\infty} a_n u^n / n^{\sigma n} \tag{6}$$

represents an entire function and converges uniformly to the function p in every finite domain inside the star domain as $\sigma \to 0$ through positive values **(6)**. If u is replaced by (1) in series (2) the series will converge uniformly in any closed set in the ζ-plane not containing the origin; for series (6) we show in paragraph 2 that the convergence is uniform on $|\zeta| = 1$ for any fixed X, not belonging to the set D_a^3 given by (5). Hence replacing f and p by their representations (2) and (3) in the integral operator (1.2), we can interchange the order of integration and the limiting operation to obtain

$$H(X) = \lim_{\sigma\to 0} \sum_{n=0}^{\infty} a_n/n^{\sigma n} \sum_{p,q=0}^{\infty} c_{pq}(1/2\pi i) \int_{|\zeta|=1} u^{n+p}\zeta^q \frac{d\zeta}{\zeta}.$$

By the residue theorem the integral on the right which equals

$$(1/2\pi i) \int_{|\zeta|=1} (Z\zeta^2 + x\zeta + Z^*)^{n+p}\zeta^{q-1-n-p} d\zeta$$

has the value 0 unless $q \leqslant n + p$. If $q \leqslant n + p$, its value is $\Gamma_{n+p,n+p+q}(X)$. Thus H has the representation (4).

2. *Excluded sets.* From the theory for one complex variable if a is a singularity of p, then on the half-line arg u = arg a the set $|u| \geqslant |a|$ is excluded. (Note that $a \neq 0$ by hypothesis.) Now arg u = arg a implies that

$$\begin{aligned} y'(\theta)\text{Re}\, a &\equiv (y \cos\theta + z \sin\theta)\text{Re}\, a = x\, \text{Im}\, a, \\ x &> 0 \text{ if } \text{Re}\, a > 0, x < 0 \text{ if } \text{Re}\, a < 0, \end{aligned} \tag{7}$$

which is the equation of a half-plane $\Pi^2(\theta)$. As θ traces the unit circle the surface

$$\Pi^3 = \bigcup_{0\leqslant\theta\leqslant 2\pi} \Pi^2(\theta) \tag{8}$$

is obtained.

Remark. If Re $a = 0$, arg $a = \pi/2$ or $3\pi/2$ and $\Pi^2(\theta)$ is one-half the yz-plane so that Π^3 degenerates into a two-dimensional surface (the yz-plane).

The surface Π^3 does not cover all of E^3 since the points satisfying $y^2 + z^2 < A^2x^2$ do not lie on it.

Proof. $X \in \Pi^3$ implies that there exists θ such that (7) is satisfied. Expressing $\cos\theta$ in terms of $\sin\theta$, squaring and solving for $\sin\theta$, we get

$$\sin\theta = (Axz \pm |y|[y^2 + z^2 - A^2x^2]^{\frac{1}{2}})/(y^2 + z^2), \tag{9}$$

from which the conclusion follows. The cone $y^2 + z^2 = A^2x^2$ is the envelope of the family of planes (7) $(0 \leqslant \theta \leqslant 2\pi)$.

The excluded set in the u-plane on the half-line $\arg u = \arg a$ is the set for which $|u| \geqslant |a|$. In E^3 for ζ on the unit circle $|u| \geqslant |a|$ becomes the surface and exterior of the cylinder $C^2(\theta)$ whose equation is

$$x^2 + y'^2(\theta) = |a|^2.$$

The half-plane $\Pi^2(\theta)$ intersects $C^2(\theta)$ in a line $l^1(\theta)$ parallel to the $z'(\theta)$-axis (the axis perpendicular to the x- and $y'(\theta)$-axes) and through the point $(x = \operatorname{Re} a, y'(\theta) = \operatorname{Im} a, z'(\theta) = 0)$. It intersects the exterior of the cone given by $x^2 + y'^2(\theta) = k^2|a|^2$, $k^2 > 1$ for fixed k in a line parallel to the $z'(\theta)$-axis and on the opposite side of $l^1(\theta)$. Thus for fixed θ the excluded area is that part of $\Pi^2(\theta)$ which lies on the opposite side of $l^1(\theta)$ to the $z'(\theta)$-axis. Call this piece of plane plus the line $l^1(\theta)$, $\Pi_1^2(\theta)$. The complete set of excluded points is

$$\Pi_1^3 = \bigcup_{0 \leqslant \theta \leqslant 2\pi} \Pi_1^2(\theta).$$

Now show that if $\operatorname{Re} a \neq 0$, $D_a^3 = \Pi_1^3$. If $X \in D_a^3$ show there exists θ such that $X \in \Pi_1^2(\theta)$. This means that equation (9) must be satisfied, that is, at least one of the values of $\sin\theta$ in (9) must not exceed one in absolute value. $X \in D_a^3$ implies that the equation $y^2 + z^2 = A^2k^2x^2$ is satisfied for some $k^2 \geqslant 1$ and hence we must show that

$$-1 \leqslant (z \mp |y|[k^2 - 1]^{\frac{1}{2}})/Ak^2x \leqslant 1.$$

But this follows by a careful analysis of all possible cases. Thus $D_a^3 \subset \Pi_1^3$. Conversely $X \in \Pi_1^3$ implies that X satisfies (7) for some value of θ and hence $\sin\theta$ is given by (9), whence $y^2 + z^2 - A^2x^2 \geqslant 0$ so that $X \in D_a^3$. Thus $D_a^3 = \Pi_1^3$.

The total set of excluded points is

$$D^3 = \bigcup_{\{a\}} D_a^3$$

plus the exterior of the circle $y^2 + z^2 = |a|^2$ in the yz-plane if $\operatorname{Re} a = 0$ for any a. Consequently the set where Mittag-Leffler summability holds is the complement of D^3, namely

$$I^3 = \bigcap_{\{a\}} [X | y^2 + z^2 < A^2x^2] \cup [X | y^2 + z^2 \geqslant A^2x^2,$$
$$x < \operatorname{Re} a \quad \text{if} \quad \operatorname{Re} a > 0, x > \operatorname{Re} a \text{ if } \operatorname{Re} a < 0]$$

plus the disk $y^2 + z^2 \leqslant |a|^2$ in the yz-plane (or the intersection of such disks) if $\text{Re}\, a = 0$ for any a. I^3 is not empty since by hypothesis the function element of $p(u)$ has a positive radius of convergence R_0 so that $|a| \geqslant R_0$ for all a. Any point X in the interior of the sphere $x^2 + y^2 + z^2 = R_0^2$ belongs to I^3, since if we set $y = r_0 \cos \phi$, $z = r_0 \sin \phi$, then X satisfies the inequality $x^2 + r_0^2 < R_0^2$. If X is such that $r_0^2 < A^2x^2$ for all a there is nothing to prove but if $r_0^2 \geqslant A^2x^2$ for some a's, then $|a|^2 - x^2 \geqslant R_0^2 - x^2 > r_0^2 \geqslant A^2x^2$ or $|a|^2 > |a|^2x^2/\text{Re}^2 a$ or $\text{Re}^2 a > x^2$ and again $X \in I^3$.

3. In order to complete the proof of Theorem 3 we must show that for any fixed $X \in I^3$ the convergence of (6) as $\sigma \to 0$ is uniform on $|\zeta| = 1$, that is, for such X and ζ on $|\zeta| = 1$ the values of u which lie on the half-line $\arg u = \arg a$ are in absolute value less than $|a|$. There are two cases: (i) If X is such that $y^2 + z^2 < A^2x^2$, the equation $y'(\theta) = Ax$ has no solution for θ which means that for any ζ on the unit circle u does not lie on the half-line $\arg u = \arg a$. (ii) If X is such that $y^2 + z^2 \geqslant A^2x^2$, as we have seen the equation $y'(\theta) = Ax$ always has a solution for u on the half-line $\arg u = \arg a$ and

$$|u|^2 = x^2 + y'^2(\theta) = x^2(1 + A^2) = x^2|a|^2/\text{Re}^2 a$$

so that $|u|^2 < |a|^2$ if $x^2 < \text{Re}^2 a$.

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