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Spacings Between Integers Having Typically Many Prime Factors

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Abstract. We show that the sequence of integers which have nearly the typical number of distinct prime factors forms a Poisson process. More precisely, for δ arbitrarily small and positive, the nearest neighbor spacings between integers n with $|\omega(n) - loglogn| < (loglogn)^{\delta}$ obey the Poisson distribution law.

1 Introduction

Consider n random variables independently and uniformly taking real values in the interval $[0, n]$. Let $Y_1 < \cdots < Y_n$ denote the order statistics obtained by arranging these random variables in increasing order. Setting $Y_0 = 0$ and $Y_{n+1} = n$, let $D_i = Y_i - Y_{i-1}$ for $1 \leq i \leq n+1$ denote the nearest neighbor spacings of the order statistics. Thus $D_1 + \cdots + D_n = n$, and by symmetry it follows that for $0 < \lambda < n$ a real number,

$$\operatorname{Prob}(D_i > \lambda) = \operatorname{Prob}(D_1 > \lambda) = \Big(\frac{n-\lambda}{n}\Big)^n.$$

Thus $\operatorname{Prob}(D_i > \lambda) \sim e^{-\lambda}$ as $n \to \infty$. For the $(k+1)$-th neighbor spacing it can be shown that

$$\operatorname{Prob}(Y_{i+k+1} - Y_i > \lambda) \sim \sum_{j=0}^{k} \frac{\lambda^j}{j!} e^{-\lambda} \tag{1.1}$$

as $n \to \infty$. The right-hand side of (1.1) is the Poisson distribution function.

We are interested in the spacing distributions of arithmetic sequences. An example of such a sequence is the sequence of prime numbers less than x, which by the prime number theorem form a sparse subset of the integers of density $1/\log x$. This is similar to $Y_1, \ldots, Y_n$ being sparse in the interval $[0, n]$. If p_i denotes the i-th prime less than x, we rescale to consider instead the sequence $\widetilde{p}_i = p_i/\log x$ of "normalized" primes so that that the average spacing between consecutive normalized primes is 1 as $x \to \infty$. This matches the expected value of D_i above as $n \to \infty$. Gallagher [3] showed that assuming the validity of the Hardy–Littlewood prime r-tuple conjectures, we have for $\lambda > 0$ real and $k \geq 0$ integral,

$$\frac{1}{x}\#\{i \leq x : \widetilde{p}_{i+k+1} - \widetilde{p}_i > \lambda\} \sim \sum_{j=0}^{k} \frac{\lambda^j}{j!} e^{-\lambda}$$

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as $x \to \infty$. Thus, conditionally we see that the spacings between primes obey the Poisson distribution law, as in the prototypical situation of randomly dispersed objects mentioned at the start. More recently Kurlberg and Rudnick [8] showed that the spacings between quadratic residues modulo q follow the Poisson distribution as the number of distinct prime divisors of q tends to infinity. There are many other interesting arithmetic sequences that are conjectured to be Poisson processes, but only a few examples exist with proof. For example, it is an open problem to show that the spacings between the fractional parts of $n^2\sqrt{2}$ for $n \leq x$, as $x \to \infty$ are Poisson distributed (see [10]). The reader may find more examples of such work listed in the references [1,4,6]. Of course there are important arithmetic sequences which are not expected to behave like randomly dispersed elements in this sense, such as the non-trivial zeros of the Riemann Zeta function. In this paper we are interested in the spacings between integers with not only one prime factor as in Gallagher's work, but with the typical number of distinct prime factors. We first explain what is meant by "typical".

Let $\omega(n)$ denote the number of distinct prime factors of n. It is easy to see that integers $n \leq x$ have $\log\log x$ distinct prime factors on average:

$$\frac{1}{x}\sum_{n\leq x}\omega(n) = \frac{1}{x}\sum_{n\leq x}\sum_{p\mid n} 1 = \frac{1}{x}\sum_{p\leq x}\sum_{\substack{n\leq x\\ p\mid n}} 1 = \frac{1}{x}\sum_{p\leq x}\left\lfloor \frac{x}{p}\right\rfloor = \log_2 x + O(1), \tag{1.2}$$

where we write $\log_2 x$ for $\log\log x$, and similarly for $\log_j x$. Also, throughout this paper p and q will be used to denote primes. The variance can be shown to be

$$\frac{1}{x}\sum_{n\leq x}(\omega(n) - \log_2 x)^2 \sim \log_2 x. \tag{1.3}$$

Note that (1.2) and (1.3) imply that $\omega(n) \sim \log_2 x$ for all most all $n \leq x$. Erdős and Kac [2] further showed that $\omega(n)$ is normally distributed with mean $\log_2 x$ and standard deviation $\sqrt{\log_2 x}$. Rényi and Turán [9] proved this result with a sharp error term. The following theorem can also be found in Tenenbaum's book [11].

Theorem 1.1 *Given a real number $C > 0$ we have for $0 < c < C$ that the number of integers $n \leq x$ for which $-c < \frac{\omega(n)-\log_2 x}{\sqrt{\log_2 x}} < c$ is*

$$x\frac{1}{\sqrt{2\pi}}\int_{-c}^{c}\exp(-u^2/2)du + O_C(x/\sqrt{\log_2 x}).$$

In [7] we proved a slightly weaker version of the Theorem 1.1 by methods similar to those in this paper.

We conjecture that the spacings between integers $n \leq x$ with $|\omega(n) - \log_2 x| \leq \sqrt{\frac{\pi}{2}}$ (that is, integers with more or less exactly $\log_2 x$ distinct prime factors) obey the Poisson distribution law, but we are unable to prove it. Instead we look at an easier question. For any fixed $0 < \delta < 1/2$, let us say an integer less than x^2 is *δ-normal* if $|\omega(n) - \log_2 x| \leq \sqrt{\frac{\pi}{2}}(\log_2 x)^{\delta}$. We study the sequence of δ-normal numbers.

These are integers having nearly the expected number of prime factors, as $(\log_2 x)^\delta$ is smaller than the standard deviation $\sqrt{\log_2 x}$ of $\omega(n)$. Denote the sequence of δ-normal numbers in increasing order by $N_1, N_2, \dots$. Up to x, there are $x(\log_2 x)^{-1/2+\delta}$ such integers by Theorem 1.1, since an integer is δ-normal if and only if

$$\left|\frac{\omega(n)-\log_2 x}{\sqrt{\log_2 x}}\right| \le \sqrt{\frac{\pi}{2}}(\log_2 x)^{-1/2+\delta}.$$

Thus we should rescale these integers by setting $\widetilde{N}_i = N_i(\log_2 x)^{-1/2+\delta}$. Our main theorem is as follows.

Theorem 1.2 *For any fixed real number $\lambda > 0$ and fixed integer $k \ge 0$ we have*

$$\frac{1}{x}\#\{i \le x : \widetilde{N}_{i+k+1} - \widetilde{N}_i > \lambda\} \sim \sum_{j=0}^{k} \frac{\lambda^j}{j!}e^{-\lambda}$$

Throughout this paper, all implicit constants may depend implicitly on δ, λ and k.

2 Independence Between Additive Shifts of the $\omega(-)$ Function

In this section we show how Theorem 1.2 can be reduced to studying correlations between the additive shifts of the function $\omega(-)$. We will show for example that $\omega(n) - \log_2 x$, $\omega(n+1) - \log_2 x$, and $\omega(n+3) - \log_2 x$ behave independently. Define $\mathcal{N}(x)$ to be the number of integers $n \le x$ for which n is δ-normal. The left-hand side of Theorem 1.2 is asymptotic to

$$(2.1)\quad \frac{1}{x}\#\{i \le x : N_{i+k+1} - N_i > \lambda(\log_2 x)^{1/2-\delta}\}$$
$$\sim \frac{1}{x}\#\{N \le x(\log_2 x)^{1/2-\delta} : \mathcal{N}\big(N + \lambda(\log_2 x)^{1/2-\delta}\big) - \mathcal{N}\big(N\big) \le k\},$$

where N denotes a δ-normal number. Define $\mathcal{N}_{b_1,\dots,b_r}(x)$ to be the number of integers $n \le x$ for which $n + b_i$ is δ-normal for all $1 \le i \le r$ and let $\sigma(m, r)$ denote the number of maps from the set $\{1, \dots, m\}$ onto $\{1, \dots, r\}$. We have the m-th moment of $\mathcal{N}\big(N + \lambda(\log_2 x)^{1/2-\delta}\big) - \mathcal{N}\big(N\big)$:

$$(2.2)\quad \frac{1}{x}\sum_{N \le x(\log_2 x)^{1/2-\delta}} \Big(\mathcal{N}\big(N + \lambda(\log_2 x)^{1/2-\delta}\big) - \mathcal{N}(N)\Big)^m =$$
$$\frac{1}{x}\sum_{r=1}^{m}\sigma(m,r) \sum_{1 \le b_1 < \cdots < b_r \le \lambda(\log_2 x)^{1/2-\delta}} \mathcal{N}_{0,b_1,\dots,b_r}\big(x(\log_2 x)^{1/2-\delta}\big).$$

We will prove the following.

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Theorem 2.1 *For a fixed integer r and any integers*

$$0 \le b_1 < \cdots < b_r \le \lambda(\log_2 x)^{1/2-\delta},$$

we have

$$\frac{1}{x}\mathcal{N}_{b_1,\dots,b_r}(x) \sim (\log_2 x)^{(-1/2+\delta)r}.$$

Throughout this paper all implied constants may depend implicitly on r. Since a randomly chosen integer less than x is δ-normal with probability $(\log_2 x)^{-1/2+\delta}$, the theorem above says that $n+b_1,\dots,n+b_r$ are independently likely to be δ-normal. Theorem 2.1 implies that for fixed m we have that (2.2) is asymptotic to

$$\sim \sum_{r=1}^{m}\sigma(m,r)\frac{\lambda^r}{r!} = \sum_{j=0}^{\infty} j^m \frac{e^{-\lambda}\lambda^j}{j!}, \tag{2.3}$$

the m-th moment of the Poisson distribution (the identity above is known as Dobinski's formula). The Poisson distribution can be recovered from these moments. For example, for $k=0$ we have that (2.1) is

$$\begin{aligned} &\frac{1}{x}\#\{N \le x(\log_2 x)^{1/2-\delta} : \mathcal{N}(N+\lambda(\log_2 x)^{1/2-\delta}) - \mathcal{N}(N) = 0\} \\ &\quad\sim 1 - \frac{1}{x}\sum_{j=1}^{\infty} \sum_{\substack{N \le x(\log_2 x)^{1/2-\delta} \\ \mathcal{N}(N+\lambda(\log_2 x)^{1/2-\delta})-\mathcal{N}(N)=j}} 1 \\ &\quad= 1 - \frac{1}{x}\sum_{j=1}^{\infty}\sum_{m=0}^{\infty}\frac{(2\pi\mathbf{i})^m}{m!} j^m \sum_{\substack{N \le x(\log_2 x)^{1/2-\delta} \\ \mathcal{N}(N+\lambda(\log_2 x)^{1/2-\delta})-\mathcal{N}(N)=j}} 1. \end{aligned} \tag{2.4}$$

Now

$$\frac{1}{x}\sum_{j=1}^{\infty} j^m \sum_{\substack{N \le x(\log_2 x)^{1/2-\delta} \\ \mathcal{N}(N+\lambda(\log_2 x)^{1/2-\delta})-\mathcal{N}(N)=j}} 1$$

is the m-th moment of $\mathcal{N}\big(N+\lambda(\log_2 x)^{1/2-\delta}\big) - \mathcal{N}\big(N\big)$. By (2.3) we get (an explicit dependence on m of the error term is not needed) that (2.4) is asymptotic to

$$\sim 1 - \sum_{j=1}^{\infty}\sum_{m=0}^{\infty}\frac{(2\pi\mathbf{i})^m}{m!} j^m \frac{e^{-\lambda}\lambda^j}{j!} = e^{-\lambda}.$$

Also for $k>0$ the Poisson moments imply that (2.1) $\sim \sum_{j=0}^{k}\frac{\lambda^j}{j!}e^{-\lambda}$. Thus Theorem 1.2 follows from Theorem 2.1. Next we discuss the demonstration of Theorem 2.1.

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The characteristic function of a random variable with a normal distribution is $\exp(-T^2/2)$. We show the independence of

$$\frac{\omega(n+b_i)-\log_2 x}{\sqrt{\log_2 x}}$$

for $1 \le i \le r$ by showing that their joint characteristic function equals essentially $\prod_{i=1}^r \exp(-T_i^2/2)$. Actually it is more convenient to work with $\omega(n; y, z)$ in place of $\omega(n)$, where we set

$$y = y(x) = (\log x)^{3r} \text{ and } z = z(x) = x^{(\log_2 x)^{-3r}},$$

and define

$$\omega(n; y, z) = \sum_{\substack{p|n \\ y<p<z}} 1.$$

Accordingly we work with $\omega(n; y, z) - \sum_{y<p<z} \frac{1}{p}$ in place of $\omega(n) - \log_2 x$. We will soon see that there is not much loss in disregarding the primes less than y or greater than z. In the next section we will prove the following theorem.

Theorem 2.2 *Let $t_i = T_i(\sum_{y<p<z} \frac{1}{p})^{-1/2}$ be real. For $|T_i| \le \frac{1}{1000}(\sum_{y<p<z} \frac{1}{p})^{1/2}$, we have*

$$\frac{1}{x}\sum_{n\le x}\prod_{i=1}^r \exp\Big(\mathbf{i}T_i \frac{\omega(n+b_i; y,z) - \sum_{y<p<z}\frac{1}{p}}{\sqrt{\sum_{y<p<z}\frac{1}{p}}}\Big) = \prod_{i=1}^r \exp\Big(\big(e^{\mathbf{i}t_i} - 1 - \mathbf{i}t_i\big)\sum_{y<p<z}\frac{1}{p}\Big) + O(1/\log x).$$

Observe that for $T_i \le (\log_2 x)^\epsilon$ for small enough $\epsilon > 0$, we have

$$\exp\Big((e^{\mathbf{i}t_i} - 1 - \mathbf{i}t_i)\sum_{y<p<z}\frac{1}{p}\Big) = \exp\Big(\big(\tfrac{-t_i^2}{2} + O(t_i^3)\big)\sum_{y<p<z}\frac{1}{p}\Big) = \exp\big(\tfrac{-T_i^2}{2}\big)\Big(1 + O\Big(\frac{T_i^3}{\sqrt{\log_2 x}}\Big)\Big), \tag{2.5}$$

and for $(\log_2 x)^\epsilon < T_i \le (\log_2 x)^{1/2-\epsilon}$, we have

$$\exp\Big((e^{\mathbf{i}t_i} - 1 - \mathbf{i}t_i)\sum_{y<p<z}\frac{1}{p}\Big) \ll \exp(-T_i^2/4), \tag{2.6}$$

where the implied constants depend on ϵ.

To see how Theorem 2.2 implies Theorem 2.1, we will use the following lemmas.

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Lemma 2.3 *Let $\psi(x)$ be a real function differentiable $\lfloor 4r/\delta \rfloor$ times and satisfying*

$$0 \le \psi(x) \le 1 \text{ for } |x| \in \mathbb{R}, \qquad \psi(x) = 0 \text{ for } |x| \ge 2(\log_2 x)^{-1/2+\delta},$$

$$\int_{-\infty}^{\infty} \psi(x)dx \sim \sqrt{2\pi}(\log_2 x)^{-1/2+\delta}, \text{ and} \tag{2.7}$$

$$|\psi^{(j)}(x)| \ll (\log_2 x)^{j(1-\delta)/2} \quad \text{for any positive integer } j \le \lfloor 4r/\delta \rfloor.$$

We have

$$\frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r}\psi\Big(\frac{\omega(n+b_i;y,z)-\sum_{y<p<z}\frac{1}{p}}{\sqrt{\sum_{y<p<z}\frac{1}{p}}}\Big) \sim (\log_2 x)^{(-1/2+\delta)r}.$$

Proof Let $\hat{\psi}(T) = \int_{-\infty}^{\infty} \psi(u)e^{-\mathbf{i}uT}du$ denote the Fourier transform of ψ. By Fourier inversion we have

$$\frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r}\psi\Big(\frac{\omega(n+b_i;y,z)-\sum_{y<p<z}\frac{1}{p}}{\sqrt{\sum_{y<p<z}\frac{1}{p}}}\Big) = \tag{2.8}$$

$$\frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\psi}(T_i)\exp\Big(\mathbf{i}T_i\frac{\omega(n+b_i;y,z)-\sum_{y<p<z}\frac{1}{p}}{\sqrt{\sum_{y<p<z}\frac{1}{p}}}\Big)dT_i.$$

We have that $|\hat{\psi}(T_i)| \ll (\log_2 x)^{j(1-\delta)/2}|T_i|^{-j}$, by integrating by parts j times and using (2.7). Thus (2.8) equals

$$\frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r}\frac{1}{2\pi}\int_{-(\log_2 x)^{1/2-\delta/4}}^{(\log_2 x)^{1/2-\delta/4}}\hat{\psi}(T_i)\exp\Big(\mathbf{i}T_i\frac{\omega(n+b_i;y,z)-\sum_{y<p<z}\frac{1}{p}}{\sqrt{\sum_{y<p<z}\frac{1}{p}}}\Big)dT_i + O\big((\log_2 x)^{-r}\big).$$

Now by Theorem 2.2 and observations (2.5) and (2.6), the main term above equals

$$\prod_{i=1}^{r}\Big(\frac{1}{2\pi}\int_{-(\log_2 x)^{\epsilon}}^{(\log_2 x)^{\epsilon}}\hat{\psi}(T_i)\exp\Big(\frac{-T_i^2}{2}\Big)dT_i + O\Big(\frac{1}{\sqrt{\log_2 x}}\Big)\Big) = \tag{2.9}$$

$$\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\psi}(T)\exp\Big(\frac{-T^2}{2}\Big)dT + O\Big(\frac{1}{\sqrt{\log_2 x}}\Big)\Big)^r.$$

Recall that the Fourier transform of $\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-u^2}{2}\right)$ is $\exp\left(\frac{-T^2}{2}\right)$. By the Plancherel formula, (2.9) equals

$$\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(u)\exp\Big(\frac{-u^2}{2}\Big)du + O\Big(\frac{1}{\sqrt{\log_2 x}}\Big)\Big)^r \sim (\log_2 x)^{(-1/2+\delta)r}. \qquad \blacksquare$$

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To prove Theorem 2.1 we need to show

$$\frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r}\psi\Big(\frac{\omega(n+b_i)-\log_2 x}{\sqrt{\log_2 x}}\Big)\sim(\log_2 x)^{(-1/2+\delta)r},$$

where ψ is a suitable smooth function approximating the characteristic function of the interval $[-\sqrt{\frac{\pi}{2}}(\log_2 x)^{-1/2+\delta},\sqrt{\frac{\pi}{2}}(\log_2 x)^{-1/2+\delta}]$. This is accomplished by Lemma 2.3, provided that we can show that we may neglect prime factors smaller than y or larger than z without significant loss. This is the purpose of the next lemma.

Lemma 2.4 *Except for $O\big(x(\log_2 x)^{-r}\big)$ integers less than x, we have*

$$\Big|\frac{\omega(n)-\log_2 x}{\sqrt{\log_2 x}}-\frac{\omega(n;y,z)-\sum_{y<p<z}\frac{1}{p}}{\sqrt{\sum_{y<p<z}\frac{1}{p}}}\Big|\ll(\log_2 x)^{-1/2+\delta/2}.$$

Proof Let $\mathcal{E}(x)$ denote the set of integers less than or equal to x with more than $(\log_2 x)^{\delta/2}$ distinct prime factors less than y or more than $(\log_2 x)^{\delta/2}$ distinct prime factors greater than z. The size of this set is

$$\begin{aligned}|\mathcal{E}(x)|&\le\frac{x}{\lfloor(\log_2 x)^{\delta/2}\rfloor!}\Big(\sum_{p\le y}\frac{1}{p}\Big)^{(\log_2 x)^{\delta/2}}+\frac{x}{\lfloor(\log_2 x)^{\delta/2}\rfloor!}\Big(\sum_{z\le p\le x}\frac{1}{p}\Big)^{(\log_2 x)^{\delta/2}}\\&\ll\frac{x}{(\log_2 x)^r},\end{aligned}$$

using $\sum_{p\le x}\frac{1}{p}=\log_2 x+C+O(1/\log x)$ and Stirling's estimate, $n!\sim\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$. For $n\notin\mathcal{E}(x)$ we have $\omega(n)-\omega(n;y,z)\ll(\log_2 x)^{\delta/2}$, and so it follows that

$$\begin{aligned}&\frac{\omega(n)-\log_2 x}{\sqrt{\log_2 x}}-\frac{\omega(n;y,z)-\sum_{y<p<z}\frac{1}{p}}{\sqrt{\sum_{y<p<z}\frac{1}{p}}}\\&\quad=\frac{\omega(n)-\log_2 x-\omega(n;y,z)+\sum_{y<p<z}\frac{1}{p}}{\sqrt{\sum_{y<p<z}\frac{1}{p}}}+O\Big(\frac{|\omega(n)-\log_2 x|\log_3 x}{\log_2 x}\Big)\\&\quad\ll\frac{(\log_2 x)^{\delta/2}}{\sqrt{\log_2 x}}+\frac{|\omega(n)-\log_2 x|\log_3 x}{\log_2 x}.\end{aligned}$$

The proof is complete by noting that except for $O\big(x/(\log_2 x)^r\big)$ integers less than x we have $|\omega(n)-\log_2 x|\ll(\log_2 x)^{1/2+\delta/4}$. This is because $\frac{\omega(n)-\log_2 x}{\sqrt{\log_2 x}}$ has normal moments and in particular

$$\frac{1}{x}\sum_{n\le x}\Big(\frac{\omega(n)-\log_2 x}{\sqrt{\log_2 x}}\Big)^{8\lfloor r/\delta\rfloor}\ll 1,$$

where the implied constant depends on r/δ. ∎

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3 Proof of Theorem 2.2

Define for a prime p

$$f_p(n) = \begin{cases} 1 - \frac{1}{p} & \text{if } p|n \\ -\frac{1}{p} & \text{if } p \nmid n \end{cases},$$

and if $m = \prod_i p_i^{\alpha_i}$, define $f_m(n) = \prod_i f_{p_i}(n)^{\alpha_i}$. (Thus $f_1(n) = 1$.) If we think of a prime p dividing n with probability $1/p$ independently of other primes, then we have $\mathbb{E}(f_m(n)) = 0$ for square-free m. So we have written $\omega(n) - \sum_{p\le x} 1/p = \sum_{p\le x} f_p(n)$ as a sum of independent random variables of mean 0, which already suggests by the Central Limit Theorem that $\omega(n) - \sum_{p\le x} 1/p$ is normally distributed. This simple idea is actually very powerful. It is borrowed from Granville and Soundararajan [5], who use it to efficiently compute very high moments of $\omega(n) - \sum_{p\le x} \frac{1}{p}$ and provide a new proof of the Erdős–Kac theorem.

We have that the left-hand side of Theorem 2.2 equals

$$\begin{aligned}
(3.1)\quad & \frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r} \exp\Big(\mathbf{i}T_i \frac{\omega(n+b_i;y,z) - \prod_{y<p<z}\frac{1}{p}}{\sqrt{\prod_{y<p<z}\frac{1}{p}}}\Big) \\
&= \frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r}\exp\Big(\prod_{y<p<z}\mathbf{i}t_i f_p(n+b_i)\Big) \\
&= \frac{1}{x}\sum_{n\le x}\prod_{y<p<z}\exp\Big(\sum_{i=1}^{r}\mathbf{i}t_i f_p(n+b_i)\Big) \\
&= \frac{1}{x}\sum_{n\le x}\prod_{y<p<z}\Big(1+\Big(\sum_{i=1}^{r}\mathbf{i}t_i f_p(n+b_i)\Big)+\frac{1}{2!}\Big(\sum_{i=1}^{r}\mathbf{i}t_i f_p(n+b_i)\Big)^2+\cdots\Big).
\end{aligned}$$

Now upon expansion of the product, (3.1) equals

$$(3.2)\qquad \frac{1}{x}\sum_{n\le x}\sum_{\substack{a_i\ge 1\\ p|a_i\Rightarrow y<p<z}} K_{a_1,\dots,a_r}\prod_{i=1}^{r} t_i^{\Omega(a_i)} f_{a_i}(n+b_i),$$

for some constants $K_{a_1,\dots,a_r}$ of modulus bounded by 1, where $\Omega(a)$ is the number of prime factors of a counted with multiplicity. Note that when the integers a_i are pairwise coprime we have that

$$(3.3)\qquad K_{a_1,\dots,a_r} = \prod_{i=1}^{r}\prod_{p^\alpha|a_i}\frac{\mathbf{i}^\alpha}{\alpha!}.$$

We will evaluate (3.2) using the following results. The first generalizes a result from [5].

Lemma 3.1 *Let a_i be pairwise coprime integers for $1 \leq i \leq r$. Denote the square-free part of a_i by A_i. We have*

$$\frac{1}{x}\sum_{n\leq x}\prod_{i=1}^{r} f_{a_i}(n+b_i) = \prod_{i=1}^{r}\prod_{p^\alpha \| a_i}\Big(\tfrac{1}{p}\big(1-\tfrac{1}{p}\big)^{\alpha} + \big(\tfrac{-1}{p}\big)^{\alpha}\big(1-\tfrac{1}{p}\big)\Big) + O\Big(\frac{1}{x}\prod_{i=1}^{r}\tau(A_i)^2\Big),$$

where $\tau(A)$ denotes the number of divisors of A and $p^\alpha \parallel a$ means that $p^\alpha | a$ and $p^{\alpha+1} \nmid a$. Note that the main term is zero unless each a_i is square-full (that is, $p|a_i$ implies $p^2|a_i$).

Proof For a fixed integer a, the value $f_a(n)$ depends only on the common prime factors of a and n, so $f_a(n) = f_a((A,n))$. Thus we can group terms this way:

$$\text{(3.4)} \qquad \frac{1}{x}\sum_{n\leq x}\prod_{i=1}^{r} f_{a_i}(n+b_i) = \frac{1}{x}\sum_{d_i|A_i}\sum_{\substack{n\leq x\\ (A_i,n+b_i)=d_i}}\prod_{i=1}^{r} f_{a_i}(d_i)$$

$$= \frac{1}{x}\sum_{d_i|A_i}\sum_{\substack{n\leq x\\ d_i|(n+b_i)}}\sum_{e_i | \frac{(A_i,n+b_i)}{d_i}}\prod_{i=1}^{r}\mu(e_i)f_{a_i}(d_i)$$

$$= \frac{1}{x}\sum_{d_i|A_i}\sum_{e_i|\frac{A_i}{d_i}}\Big(\sum_{\substack{n\leq x\\ e_id_i|n+b_i}} 1\Big)\prod_{i=1}^{r}\mu(e_i)f_{a_i}(d_i).$$

By the Chinese Remainder Theorem, since the integers a_i are pairwise coprime,

$$\sum_{\substack{n\leq x\\ e_id_i|n+b_i}} 1 = \frac{x}{\prod_{i=1}^{r} e_id_i} + O(1).$$

Therefore the above sum is

$$\prod_{i=1}^{r}\sum_{d_i|A_i}\frac{f_{a_i}(d_i)}{d_i}\sum_{e_i|\frac{A_i}{d_i}}\frac{\mu(e_i)}{e_i} + O\Big(\frac{1}{x}\prod_{i=1}^{r}\tau(A_i)^2\Big)$$

$$= \prod_{i=1}^{r}\sum_{d_i|A_i}\frac{f_{a_i}(d_i)}{d_i}\frac{\phi(\frac{A_i}{d_i})}{\frac{A_i}{d_i}} + O\Big(\frac{1}{x}\prod_{i=1}^{r}\tau(A_i)^2\Big)$$

$$= \prod_{i=1}^{r}\sum_{d_i|A_i} f_{a_i}(d_i)\frac{1}{A_i}\phi\Big(\frac{A_i}{d_i}\Big) + O\Big(\frac{1}{x}\prod_{i=1}^{r}\tau(A_i)^2\Big).$$

Now it is easily verified (by multiplicativity in a_i) that the main term of the last line above equals

$$\prod_{i=1}^{r}\prod_{p^\alpha\|a_i}\Big(\tfrac{1}{p}\big(1-\tfrac{1}{p}\big)^{\alpha} + \big(\tfrac{-1}{p}\big)^{\alpha}\big(1-\tfrac{1}{p}\big)\Big). \qquad \blacksquare$$

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In the case that $a_1, \dots, a_r$ are not pairwise coprime, we will need the following.

Lemma 3.2 *Let $r \ge 2$ and $0 \le b_1 < \cdots < b_r \le \lambda(\log_2 x)^{1/2-\delta}$. Suppose that for some prime $y < q < z$, we have that $q|a_1$ and $q|a_2$. Let $q^{\alpha_i} \parallel a_i$ and let $a_i' = a_i q^{-\alpha_i}$. Let A_i denote the square-free part of a_i, and let A_i' denote the square-free part of a_i'. We have*

$$\frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r} f_{a_i}(n+b_i) = O\Big(\frac{1}{q^2}\frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r} f_{a_i'}(n+b_i)\Big) + O\Big(\frac{1}{qx}\prod_{i=1}^{r}\tau(A_i)^2\Big).$$

Proof From (3.4), we have

$$\frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r} f_{a_i}(n+b_i) = \frac{1}{x}\sum_{d_i|A_i}\sum_{e_i|\frac{A_i}{d_i}}\Big(\sum_{\substack{n\le x\\ e_id_i|n+b_i}} 1\Big)\prod_{i=1}^{r}\mu(e_i)f_{a_i}(d_i). \tag{3.5}$$

The sum

$$\sum_{\substack{n\le x\\ e_id_i|n+b_i}} 1$$

is zero for large enough x if q divides more than one integer e_id_i. This is because if $q|n+b_i$ and $q|n+b_j$ then $q|b_i-b_j$ and hence $i=j$ since $q>y$ and the integers b_i are distinct and bounded by $\lambda(\log_2 x)^{1/2-\delta}$. Thus we can suppose that q divides at most one integer e_id_i. First consider the terms of (3.5) with $q\nmid e_id_i$ for all i. These terms contribute

$$\tfrac{1}{x}\sum_{d_i|A_i'}\sum_{e_i|A_i'/d_i}\Big(\sum_{\substack{n\le x\\ e_id_i|n+b_i}} 1\Big)\prod_{i=1}^{r}\mu(e_i)f_{a_i'}(d_i)f_{q^{\alpha_i}}(d_i) = \big(\tfrac{-1}{q}\big)^{\alpha_1+\cdots+\alpha_r}\tfrac{1}{x}\sum_{n\le x}\prod_{i=1}^{r} f_{a_i'}(n+b_i),$$

using the fact that $f_{q^{\alpha_i}}(d_i) = (-1/q)^{\alpha_i}$ and the identity of (3.5). Since $\alpha_1+\alpha_2\ge 2$ we get the desired factor of $1/q^2$. Now say $q|e_1d_1$ and $q\nmid e_id_i$ for $i\ne 1$. The contribution of this case is,

$$\begin{aligned}\frac{1}{x}\sum_{d_i|A_i'}\sum_{e_i|A_i'/d_i}\Bigg(\sum_{\substack{n\le x\\ qe_1d_1|n+b_1\\ e_id_i|n+b_i, i\ne1}} 1\Bigg)\mu(e_1)f_{a_1'}(qd_1)f_{q^{\alpha_1}}(qd_1)\prod_{i=2}^{r}\mu(e_i)f_{a_i'}(d_i)f_{q^{\alpha_i}}(d_i)\\ +\frac{1}{x}\sum_{d_i|A_i'}\sum_{e_i|A_i'/d_i'}\Bigg(\sum_{\substack{n\le x\\ qe_1d_1|n+b_1\\ e_id_i|n+b_i, i\ne1}} 1\Bigg)\mu(qe_1)f_{a_1'}(d_1)f_{q^{\alpha_1}}(d_1)\prod_{i=2}^{r}\mu(e_i)f_{a_i'}(d_i)f_{q^{\alpha_i}}(d_i),\end{aligned} \tag{3.6}$$

where the first line corresponds to $q|d_1$ and the second to $q|e_1$. By the Chinese Remainder Theorem, we have

$$\sum_{\substack{n\le x\\ qe_1d_1|n+b_1\\ e_id_i|n+b_i, i\ne1}} 1 = \frac{1}{q}\sum_{\substack{n\le x\\ e_id_i|n+b_i}} 1 + O(1).$$

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Thus the contribution of the sums of (3.6) is

$$(3.7)\quad \Big(\tfrac{1}{q}\big(1-\tfrac{1}{q}\big)^{\alpha_1}\big(\tfrac{-1}{q}\big)^{\alpha_2+\cdots+\alpha_r}-\tfrac{1}{q}\big(\tfrac{-1}{q}\big)^{\alpha_1+\cdots+\alpha_r}\Big)\tfrac{1}{x}\sum_{n\le x}\prod_{i=1}^{r} f_{a_i'}(n+b_i) + O\Big(\tfrac{1}{x}|f_{q^{\alpha_2}}(d_2)|\prod_{i=1}^{r}\sum_{e_i d_i|A_i'}1\Big).$$

Again we have a factor of $1/q^2$ in the first line above since $\alpha_2 \ge 1$. The second line of (3.7) is $O(\frac{1}{qx}\prod_{i=1}^{r}\tau(A_i)^2)$ since $\alpha_2 \ge 1$ and $q \nmid d_2$. This completes the proof as terms with $q|e_j d_j$ and $q \nmid e_i d_i$ for $i \neq j$ are dealt with similarly. ■

We will also need the following observations.

Lemma 3.3 *We have*

$$\frac{1}{x}\sum_{n\le x}\sum_{\substack{a_i\ge 1\\ p|a_i\Rightarrow y<p<z\\ \omega(a_1)\ge(\log_2 x)^{2r}}}|K_{a_1,\dots,a_r}|\prod_{i=1}^{r}|f_{a_i}(n+b_i)||t_i|^{\Omega(a_i)} \ll \frac{1}{\log x}.$$

Proof We first bound the contribution of terms with $\omega(a_i)=w_i$ for some positive integers w_i. Recall that $|K_{a_1,\dots,a_r}| \le 1$ and note that $|f_{p^\alpha}(n)| \le |f_p(n)|$. Thus

$$(3.8)\quad \frac{1}{x}\sum_{n\le x}\sum_{\substack{a_i\ge 1\\ p|a_i\Rightarrow y<p<z\\ \omega(a_i)=w_i}}|K_{a_1,\dots,a_r}|\prod_{i=1}^{r}|f_{a_i}(n+b_i)||t_i|^{\Omega(a_i)} \ll \frac{1}{x}\sum_{n\le x}\sum_{\substack{a_i\ge 1\\ p|a_i\Rightarrow y<p<z\\ \omega(a_i)=w_i}}\prod_{i=1}^{r}|f_{A_i}(n+b_i)||t_i|^{\Omega(a_i)}.$$

For a fixed square-free integer A_i with $\omega(A_i)=w_i$ we have

$$\sum_{\substack{a_i\ge 1\\ A_i=\text{square-free part of } a_i}}|t_i|^{\Omega(a_i)} \le 1,$$

since $|t_i| \le \frac{1}{1000}$. Thus (3.8) is bounded by

$$(3.9)\quad \ll \frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r}\frac{1}{w_i!}\Big(\sum_{y<p<z}|f_p(n+b_i)|\Big)^{w_i}.$$

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Since $|f_p(n)| \le 1$ if $p|n$ and $|f_p(n)| \le \frac{1}{p}$ if $p \nmid n$, we have that (3.9) is bounded by

$$\ll \frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r}\frac{1}{w_i!}\Big(\omega(n+b_i;y,z)+\log_2 x\Big)^{w_i} \tag{3.10}$$

$$\ll \Big(\prod_{i=1}^{r}\frac{1}{w_i!}\Big)\frac{1}{x}\sum_{n\le x}\sum_{i=1}^{r}2^{rw_i}\Big(\omega(n+b_i;y,z)^{rw_i}+(\log_2 x)^{rw_i}\Big).$$

We have

$$\frac{1}{x}\sum_{n\le x}\omega(n+b_i;y,z)^{rw_i} \ll \frac{1}{x}\sum_{y<p_1,\dots,p_{rw_i}<z}\ \sum_{\substack{n\le x\\ [p_1,\dots,p_{rw_i}]|n+b_i}} 1, \tag{3.11}$$

where $[p_1,\dots,p_{rw_i}]$ denotes the lowest common multiple of $p_1,\dots,p_{rw_i}$. Now (3.11) is bounded by

$$\ll \sum_{y<p_1,\dots,p_{rw_i}<z}\frac{1}{[p_1,\dots,p_{rw_i}]} \ll 2^{rw_i}(\log_2 x)^{rw_i}.$$

Thus we have that (3.10) is bounded by

$$\ll \Big(\prod_{i=1}^{r}\frac{1}{w_i!}\Big)\sum_{i=1}^{r}4^{rw_i}(\log_2 x)^{rw_i}.$$

Summing over integers $w_i \ge 1$ for $i \ge 2$, this is bounded by

$$\ll \frac{(4\log_2 x)^{rw_1}\exp((4\log_2 x)^r)}{w_1!}. \tag{3.12}$$

Finally the sum of (3.12) over integers $w_1 \ge (\log_2 x)^{2r}$ is $\ll 1/\log x$. ∎

Lemma 3.4 *We have*

$$\sum_{\substack{a_i\ge 1\\ p|a_i \Rightarrow y<p<z\\ \omega(a_1)\ge(\log_2 x)^{2r}}}\prod_{i=1}^{r}|t_i|^{\Omega(a_i)}\prod_{p^\alpha\|a_i}\Big|\frac{1}{p}\Big(1-\frac{1}{p}\Big)^{\alpha}+\Big(1-\frac{1}{p}\Big)\Big(\frac{-1}{p}\Big)^{\alpha}\Big| \ll \frac{1}{\log x}.$$

Proof We first bound the contribution of terms with $\omega(a_i) = w_i$ for some positive integers w_i. We have

$$\sum_{\substack{a_i \geq 1 \\ p|a_i \Rightarrow y<p<z \\ \omega(a_i)=w_i}} \prod_{i=1}^{r} |t_i|^{\Omega(a_i)} \prod_{p^\alpha \| a_i} \Big| \frac{1}{p}\Big(1-\frac{1}{p}\Big)^{\alpha} + \Big(1-\frac{1}{p}\Big)\Big(\frac{-1}{p}\Big)^{\alpha} \Big|$$
$$\ll \prod_{i=1}^{r} \frac{1}{w_i!}\Big(\sum_{y<p<z} \frac{1}{p}\Big)^{w_i} \ll \prod_{i=1}^{r} \frac{(\log_2 x)^{w_i}}{w_i!}.$$

Summing over integers $w_i \geq 1$ for $i \geq 2$, this is bounded by

$$\ll \frac{(\log_2 x)^{w_1} \exp(r \log_2 x)}{w_1!}.$$

The sum of (3.12) over integers $w_1 \geq (\log_2 x)^{2r}$ is $\ll 1/\log x$. ■

Back to the Proof

By Lemma 3.3 we see that (3.2) equals, up to an error of $O(1/\log x)$,

$$\frac{1}{x}\sum_{n\leq x} \sum_{\substack{a_i \geq 1 \\ p|a_i \Rightarrow y<p<z \\ \omega(a_i)\leq(\log_2 x)^{2r}}} K_{a_1,\dots,a_r} \prod_{i=1}^{r} f_{a_i}(n+b_i) t_i^{\Omega(a_i)}. \tag{3.13}$$

Let us first treat the terms of (3.13) with $a_1, \dots, a_r$ not pairwise coprime. Applying Lemma 3.2 repeatedly, these terms contribute an amount bounded by

$$\ll \sum_{y<q<z} \frac{1}{q^2} \sum_{\substack{a_i \text{ pairwise coprime} \\ p|a_i \Rightarrow y<p<z \\ \omega(a_i)\leq(\log_2 x)^{2r}}} \Big| \frac{1}{x} \sum_{n\leq x} \prod_{i=1}^{r} f_{a_i}(n+b_i) t_i^{\Omega(a_i)} \Big|$$
$$+ \sum_{y<q<z} \frac{1}{qx} \sum_{\substack{a_i \geq 1 \\ p|a_i \Rightarrow y<p<z \\ \omega(a_i)\leq(\log_2 x)^{2r}}} \prod_{i=1}^{r} |t_i|^{\Omega(a_i)} \tau(A_i)^2. \tag{3.14}$$

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For the second line of (3.14), we have

$$\frac{1}{x}\prod_{i=1}^{r}\sum_{\substack{a_i\ge 1\\ p|a_i\Rightarrow y<p<z\\ \omega(a_i)\le(\log_2 x)^{2r}}}|t_i|^{\Omega(a_i)}\tau(A_i)^2 \ll \frac{1}{x}\prod_{i=1}^{r}\sum_{\substack{a_i\text{square-free}\\ p|a_i\Rightarrow y<p<z\\ \omega(a_i)\le(\log_2 x)^{2r}}}\tau(A_i)^2 \ll \frac{1}{x}\prod_{i=1}^{r} z^{(\log_2 x)^{2r}}4^{(\log_2 x)^{2r}} \ll x^{-1/2}. \tag{3.15}$$

Thus the second line of (3.14) falls into the error term of Theorem 2.2. To bound the first line of (3.14) we use Lemma 3.1 to get that

$$\sum_{y<q<z}\frac{1}{q^2}\sum_{\substack{a_i\text{ pairwise coprime}\\ p|a_i\Rightarrow y<p<z\\ \omega(a_i)\le(\log_2 x)^{2r}}}\Big|\frac{1}{x}\sum_{n\le x}\prod_{i=1}^{r}f_{a_i}(n+b_i)t_i^{\Omega(a_i)}\Big|$$
$$\ll\frac{1}{y}\Big(\sum_{\substack{a_i\ge 1\\ p|a_i\Rightarrow y<p<z\\ \omega(a_i)\le(\log_2 x)^{2r}}}\prod_{i=1}^{r}|t_i|^{\Omega(a_i)}\prod_{p^\alpha\|a_i}\frac{1}{p}+\sum_{\substack{a_i\ge 1\\ p|a_i\Rightarrow y<p<z\\ \omega(a_i)\le(\log_2 x)^{2r}}}\frac{1}{x}\prod_{i=1}^{r}|t_i|^{\Omega(a_i)}\tau(A_i)^2\Big)$$
$$\ll\frac{1}{y}\prod_{i=1}^{r}\prod_{y<p<z}\Big(1+\frac{1}{p}\Big)+x^{-1/2},$$

where we used the bound of (3.15). Now this is less than $\frac{(\log x)^r}{y}+x^{-1/2}\ll 1/\log x$.

Thus only the terms of (3.13) with $a_1,\dots,a_r$ coprime will give a main contribution. Using Lemma 3.1 and (3.3) we get

$$\frac{1}{x}\sum_{n\le x}\sum_{\substack{a_i\text{ pairwise coprime}\\ p|a_i\Rightarrow y<p<z\\ \omega(a_i)\le(\log_2 x)^{2r}}}K_{a_1,\dots,a_r}\prod_{i=1}^{r}f_{a_i}(n+b_i)t_i^{\Omega(a_i)} \tag{3.16}$$
$$=\prod_{i=1}^{r}\sum_{\substack{a_i\text{ pairwise coprime}\\ p|a_i\Rightarrow y<p<z\\ \omega(a_i)\le(\log_2 x)^{2r}}}\prod_{p^\alpha\|a_i}\frac{\mathbf{i}^\alpha}{\alpha!}t_i^\alpha\Big(\frac{1}{p}\Big(1-\frac{1}{p}\Big)^\alpha+\Big(1-\frac{1}{p}\Big)\Big(\frac{-1}{p}\Big)^\alpha\Big)$$
$$+O\Big(\sum_{\substack{a_i\ge 1\\ p|a_i\Rightarrow y<p<z\\ \omega(a_i)\le(\log_2 x)^{2r}}}\frac{1}{x}\prod_{i=1}^{r}|t_i|^{\Omega(a_i)}\tau(A_i)^2\Big).$$

We have already seen in (3.15) that the error term above is negligible. The main term of (3.16) is zero if a_i is not square-full for all i. Thus we may further impose the

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condition $p|a_i \Rightarrow p^2|a_i$. By Lemma 3.4 we may also extend the sum in the main term of (3.16) to *all* pairwise coprime and square-full integers a_i whose prime factors lie between y and z, up to an error of $O(1/\log x)$. Thus (3.16) equals, up to this error,

$$\begin{aligned}(3.17)\quad &\prod_{i=1}^{r}\prod_{y<p<z}\Big(1+\frac{1}{p}\sum_{\alpha\geq 2}\frac{\mathbf{i}^\alpha}{\alpha!}t_i^\alpha\Big(1-\frac{1}{p}\Big)^\alpha+\Big(1-\frac{1}{p}\Big)\sum_{\alpha\geq 2}\frac{\mathbf{i}^\alpha}{\alpha!}t_i^\alpha\Big(-\frac{1}{p}\Big)^\alpha\Big)\\ &=\prod_{i=1}^{r}\prod_{y<p<z}\Big(1+\frac{1}{p}\Big(e^{\mathbf{i}t_i}-1-\mathbf{i}t_i\Big)+O\Big(\frac{1}{p^2}\Big)\Big)\\ &=\prod_{i=1}^{r}\exp\Big(\sum_{y<p<z}\log\Big(1+\frac{1}{p}\Big(e^{\mathbf{i}t_i}-1-\mathbf{i}t_i\Big)+O\Big(\frac{1}{p^2}\Big)\Big)\Big).\end{aligned}$$

Now since

$$\log\Big(1+\frac{1}{p}\Big(e^{\mathbf{i}t_i}-1-\mathbf{i}t_i\Big)+O\Big(\frac{1}{p^2}\Big)\Big)=\frac{1}{p}\Big(e^{\mathbf{i}t_i}-1-\mathbf{i}t_i\Big)+O\Big(\frac{1}{p^2}\Big),$$

we have that (3.17) equals

$$\begin{aligned}&\prod_{i=1}^{r}\exp\Big(\Big(e^{\mathbf{i}t_i}-1-\mathbf{i}t_i\Big)\sum_{y<p<z}\frac{1}{p}+O\Big(\frac{1}{y}\Big)\Big)\\ &\qquad=\prod_{i=1}^{r}\exp\Big(\Big(e^{\mathbf{i}t_i}-1-\mathbf{i}t_i\Big)\sum_{y<p<z}\frac{1}{p}\Big)+O(1/\log x).\end{aligned}$$

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