THE NUMERICAL SOLUTION
OF HAMMERSTEIN EQUATIONS BY A METHOD
BASED ON POLYNOMIAL COLLOCATION

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Abstract

In recent papers we have considered the numerical solution of the Hammerstein
equation

$$y(t) = f(t) + \int_{-1}^{1} k(t,s)g(s,y(s))\,ds, \qquad t \in [-1,1],$$

by a method which first applies the standard collocation procedure to an equivalent
equation for $z(t) := g(t,y(t))$, and then obtains an approximation to y by use of
the equation

$$y(t) = f(t) + \int_{-1}^{1} k(t,s)z(s)\,ds, \qquad t \in [-1,1].$$

In this paper we approximate z by a polynomial z_n of degree $\leq n-1$, with coeffi-
cients determined by collocation at the zeros of the nth degree Chebyshev polyno-
mial of the first kind. We then define the approximation to y to be

$$y_n(t) := f(t) + \int_{-1}^{1} k(t,s)z_n(s)\,ds, \qquad t \in [-1,1],$$

and establish that, under suitable conditions, $\lim_{n\to\infty} y_n(t) = y(t)$, uniformly in t.

1. Introduction

This paper is concerned with the Kumar and Sloan method [10] for the nu-
merical solution of the Hammerstein equation

$$y(t) = f(t) + \int_{-1}^{1} k(t,s)g(s,y(s))\,ds, \qquad t \in [-1,1], \tag{1.1}$$

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where f, k, and g are known functions, with $g(s, v)$ nonlinear in v, and y is the solution to be determined.

The Kumar and Sloan method is a collocation method applied not to equation (1.1), but rather to an equivalent equation (see (1.4)) for the function z defined by

$$z(t) := g(t, y(t)), \qquad t \in [-1, 1]. \tag{1.2}$$

The desired approximation to the solution y of (1.1) is then obtained by the use of the (exact) equation

$$y(t) = f(t) + \int_{-1}^{1} k(t, s) z(s) \, ds, \qquad t \in [-1, 1]. \tag{1.3}$$

Kumar and Sloan established that, under suitable conditions, the approximation to y converges to the exact solution at a rate at least equal to that of the best approximation to z from the space in which the collocation solution is sought. For the particular case where the collocation approximation to z is sought in certain piecewise polynomial function spaces, Kumar [9] showed that the approximation to y may exhibit (global) superconvergence, that is, it may converge to y at a faster rate than the collocation approximation does to z.

In this paper we approximate z by a polynomial z_n of degree $\leq n - 1$, with coefficients determined by collocating the equation

$$z(t) = g\left(t, f(t) + \int_{-1}^{1} k(t, s) z(s) \, ds\right), \qquad t \in [-1, 1], \tag{1.4}$$

at the n zeros $\{\tau_{ni}\}_{i=1}^{n}$ of the nth degree Chebyshev polynomial of the first kind. Thus z_n is that polynomial of degree $\leq n - 1$ which satisfies the n nonlinear equations:

$$z_n(\tau_{ni}) = g\left(\tau_{ni}, f(\tau_{ni}) + \int_{-1}^{1} k(\tau_{ni}, s) z_n(s) \, ds\right), \qquad i = 1, \ldots, n. \tag{1.5}$$

We then take

$$y_n(t) := f(t) + \int_{-1}^{1} k(t, s) z_n(s) \, ds, \qquad t \in [-1, 1], \tag{1.6}$$

as the approximation to y, and establish that, under suitable conditions,

$$\lim_{n \to \infty} y_n(t) = y(t),$$

uniformly in t. We also establish a rate for the convergence of y_n to y that is fast enough to yield, in favourable circumstances, impressive accuracy even for polynomials of low degree (for an example, see Section 6). Obviously, low degree polynomials imply small nonlinear systems, something which is highly

desirable in practical computations. We remark that the above convergence result is the analogue of that proved by Sloan and Burn [15] for the iterated (polynomial) collocation solution of (linear) integral equations of the second kind.

It should be noted that the major difference between the present analysis and that in [10] is the fact that for polynomial interpolation, uniform convergence of the interpolants cannot be guaranteed for *every* continuous function, regardless of the choice of the interpolation points. Hence, the present analysis is based on a mean-convergence property (see (3.5)) of polynomial interpolation.

2. Basic assumptions and operators

The theoretical analysis of (1.1) will be carried out in the Banach space $C = C[-1, 1]$ of continuous, real-valued functions on $[-1, 1]$. This space is equipped with the uniform norm

$$\|x\|_\infty = \sup_{-1 \leq t \leq 1} |x(t)|, \qquad x \in C.$$

The following assumptions are made on the functions f, k, and g in (1.1):
A1: $f \in C$;
A2: the kernel k satisfies

$$\sup_{-1 \leq t \leq 1} \int_{-1}^{1} |k(t, s)|^p \, ds < \infty,$$

and

$$\lim_{t \to t'} \int_{-1}^{1} |k(t, s) - k(t', s)|^p \, ds = 0, \qquad t' \in [-1, 1],$$

for some $p > 1$;
A3: the function $g(t, v)$ is defined and continuous on $[-1, 1] \times \mathbf{R}$;
A4: the partial derivative $g_v(t, v) := \frac{\partial}{\partial v} g(t, v)$ exists and is continuous on $[-1, 1] \times \mathbf{R}$.

With p as in assumption A2, let q be the number given by

$$\frac{1}{q} + \frac{1}{p} = 1. \tag{2.1}$$

Furthermore, let $L_q = L_q(-1, 1)$ denote the Banach space of measurable, real-valued functions on $(-1, 1)$ which have integrable qth power. It then follows from the results of [7] that, under assumption A2, the linear integral operator K, defined by

$$(Kw)(t) := \int_{-1}^{1} k(t, s) w(s) \, ds, \qquad t \in [-1, 1], \ w \in L_q,$$

is a compact operator from L_q to C, and hence also from $C \subset L_q$ to C. Being both compact and linear, it is necessarily (see [8], p. 244) completely continuous. It is also worth mentioning that [7] gives certain tests for deciding whether A2 holds in a given case. In particular, we note that A2 is satisfied for every continuous kernel, and for some weakly singular kernels.

We define another completely continuous operator $T: L_q \to C$ by

$$T(w)(t) := f(t) + (Kw)(t), \qquad t \in [-1, 1], \ w \in L_q,$$

and a continuous, bounded operator $G: C \to C$ by

$$G(x)(t) := g(t, x(t)), \qquad t \in [-1, 1], \ x \in C.$$

With the above notation, the integral equations (1.1) and (1.4) may be written in operator form as

$$y = TG(y), \qquad y \in C, \tag{2.2}$$

and

$$z = GT(z), \quad z \in L_q, \tag{2.3}$$

respectively. The two equations are equivalent in the sense of the following lemma.

LEMMA 1 [10]. *The sets*

$$\Theta_{TG} := \{y \in C : TG(y) = y\}$$

and

$$\Theta_{GT} := \{z \in L_q : GT(z) = z\}$$

are in one-to-one correspondence. Specifically, G is a one-to-one operator from Θ_{TG} onto Θ_{GT}, with inverse T.

For later use we note that, under assumptions A1 to A4, the operator GT is continuously Fréchet differentiable on L_q. Its Fréchet derivative at $z \in L_q$ is the completely continuous linear operator $(GT)'(z)$ given by

$$[(GT)'(z)w](t) = g_v(t, f(t) + (Kz)(t))(Kw)(t), \qquad t \in [-1, 1], \ w \in L_q.$$

3. The polynomial interpolation operator

For $w \in C$, let $P_n w$ denote the unique polynomial of degree $\leq n - 1$ that satisfies

$$(P_n w)(\tau_{nj}) = w(\tau_{nj}), \qquad j = 1, \ldots, n, \tag{3.1}$$

where

$$\tau_{nj} = \cos\frac{2j-1}{2n}\pi, \qquad j = 1, \ldots, n. \tag{3.2}$$

The points $\{\tau_{nj}\}_{j=1}^{n}$ are known as the classical Chebyshev points, and are the zeros of the nth degree Chebyshev polynomial of the first kind T_n, defined by $T_n(\cos\theta) := \cos n\theta$. Recall that polynomial interpolation based on the points (3.2) is relatively well behaved [14, Section 1.3] compared to that based on equally spaced points [5, p. 78].

In the Lagrange form $P_n w$ is

$$(P_n w)(t) = \sum_{j=1}^{n} w(\tau_{nj})l_{nj}(t), \qquad t \in [-1, 1], \tag{3.3}$$

where l_{nj} (a 'fundamental Lagrange polynomial') is the unique polynomial of degree $\leq n - 1$ that satisfies

$$l_{nj}(\tau_{ni}) = \delta_{ij}, \qquad 1 \leq i, j \leq n,$$

where δ_{ij} is the Kronecker delta. Clearly, P_n is a linear operator on C, with the property $P_n^2 = P_n$. It is therefore a projection, whose range is $\mathbf{P}_n$, the set of all polynomials of degree $\leq n - 1$. Furthermore, P_n is a bounded operator on C, with

$$\|P_n\| = \sup_{-1 \leq t \leq 1} \sum_{j=1}^{n} |l_{nj}(t)|.$$

Since P_n is the interpolatory operator defined by (3.3), and the collocation points are given by (3.2), it follows [1, 2, 6, 11, 12] that

$$\lim_{n \to \infty} \int_{-1}^{1} |(P_n w)(s) - w(s)|^q (1 - s^2)^{-1/2} \, ds = 0 \tag{3.4}$$

for every $w \in C$, and for every real number q in the range $0 < q < \infty$. As in Section 2, we choose q to be the number given by (2.1).

Let ρ be the weight function

$$\rho(s) := (1 - s^2)^{-1/2}, \qquad s \in (-1, 1),$$

and let $L_{q,\rho} = L_{q,\rho}(-1, 1)$ be the space of functions $x \in L_q$ for which the weighted L_q norm defined by

$$\|x\|_{q,\rho} := \left(\int_{-1}^{1} |x(s)|^q \rho(s) \, ds \right)^{1/q}$$

is finite. In terms of this norm, (3.4) may be written as

$$\lim_{n \to \infty} \|P_n w - w\|_{q,\rho} = 0, \tag{3.5}$$

from which it follows that

$$\sup_n \|P_n w\|_{q,\rho} < \infty$$

for all $w \in C$. Thus, if P_n is considered as a linear operator from the space C to the space $L_{q,\rho}$ then, from the Banach-Steinhaus theorem [8, p. 203], it follows that there exists a constant $c_1 > 0$, which depends only on q, such that

$$\|P_n x\|_{q,\rho} \le c_1 \|x\|_\infty, \qquad x \in C.$$

REMARK 1. The analysis given in this paper will hold for any set of collocation points and any weight function ρ, provided that (3.5) is satisfied for the q fixed by (2.1), and that $1/\rho$ is bounded. In the case of Jacobi weight functions and interpolation at zeros of the associated Jacobi polynomials, the values of q for which (3.5) holds are easily obtained from [1, 2]. It should be pointed out that while the Jacobi case considered here permits q in (3.5) (ignoring (2.1)) to be any positive number, the other cases are less generous with the values q may assume. For example, $0 < q < 4$ for the Legendre points ($\rho = 1$).

4. Convergence results

The mean-convergence property (3.5) suggests that an appropriate space for the analysis of (1.4) is the Banach space $L_{q,\rho} \subset L_q$ equipped with the norm $\|\cdot\|_{q,\rho}$. Then the theory of [10] goes through in an obvious manner for the equations

$$z = GT(z), \qquad z \in L_{q,\rho}, \tag{4.1}$$

and

$$z_n = P_n GT(z_n), \qquad z_n \in \mathbf{P}_n \subset L_{q,\rho}. \tag{4.2}$$

(Note that (4.2) is the collocation equation (1.5) written in operator form.)

Below we give results analogous to those in [10, Section 4].

THEOREM 1. *Let $y^* \in C$ be a geometrically isolated solution of (2.2), and let z^* be the corresponding solution of (4.1). Suppose A1 to A3 hold.*

(i) If y^* has a nonzero index, then there exists an n_0 such that for $n \ge n_0$, (4.2) has a solution $z_n \in \mathbf{P}_n$ satisfying

$$\|z^* - z_n\|_{q,\rho} \to 0 \quad \text{as } n \to \infty.$$

(ii) Suppose A4 holds, and that 1 is not an eigenvalue of the linear operator $(GT)'(z^*)$. Then there exists a neighbourhood of z^* and an n_1 such that for $n \ge n_1$ a solution z_n of (4.2) is unique in that neighbourhood, and

$$c_2 \|z^* - P_n z^*\|_{q,\rho} \le \|z^* - z_n\|_{q,\rho} \le c_3 \|z^* - P_n z^*\|_{q,\rho},$$

where $c_2, c_3 > 0$ are independent of n.

COROLLARY 1. *Under the conditions in Theorem* 1 (ii) *there exists a constant* $c_4 > 0$ *such that*

$$\|z^* - z_n\|_{q,\rho} \leq c_4 \inf_{\varphi \in \mathbf{P}_n} \|z^* - \varphi\|_\infty.$$

PROOF. For any $\varphi \in \mathbf{P}_n$,

$$
\begin{aligned}
\|z^* - P_n z^*\|_{q,\rho} &= \|(z^* - \varphi) - P_n(z^* - \varphi)\|_{q,\rho} \\
&\leq \|z^* - \varphi\|_{q,\rho} + \|P_n(z^* - \varphi)\|_{q,\rho} \\
&\leq \left[\int_{-1}^1 |z^*(s) - \varphi(s)|^q \rho(s)\, ds \right]^{1/q} + c_1 \|z^* - \varphi\|_\infty \\
&\leq \left(\left[\int_{-1}^1 \rho(s)\, ds \right]^{1/q} + c_1 \right) \|z^* - \varphi\|_\infty,
\end{aligned}
$$

where the second last step follows from the fact that P_n is uniformly bounded as an operator from C to $L_{q,\rho}$.

It should be noted that the quantity

$$\inf_{\varphi \in \mathbf{P}_n} \|z^* - \varphi\|_\infty$$

approaches zero as $n \to \infty$ because of the Weierstrass approximation theorem [5, Theorem 6.1.1], and may be bounded above by use of the Jackson theorems [4, p. 147].

THEOREM 2. *Let* $y^* \in C$ *be a geometrically isolated solution of* (2.2), *and let* z^* *be the corresponding solution of* (4.1). *Suppose* A1 *and* A3 *hold*.
   (i) *If* y^* *has a nonzero index, then with* z_n *as in Theorem* 1(i), *and* $n \geq n_0$, (1.6) *defines an approximation* $y_n \in C$ *satisfying*

$$\|y^* - y_n\|_\infty \to 0 \quad \text{as } n \to \infty.$$

   (ii) *Suppose* A4 *holds, and that* 1 *is not an eigenvalue of the linear operator* $(GT)'(z^*)$. *Then for* $n \geq n_1$ *the approximation* y_n *given by* (1.6), *with* z_n *as in Theorem* 1(ii), *satisfies*

$$\|y^* - y_n\|_\infty \leq c_5 \inf_{\varphi \in \mathbf{P}_n} \|z^* - \varphi\|_\infty,$$

where $c_5 > 0$ *is independent of* n.

PROOF. (i) Since T maps $L_{q,\rho}$ to C, it follows immediately from the definition $y_n := T(z_n)$ that $y_n \in C$. Now from Lemma 1,

$$y^* = T(z^*) = f + Kz^*,$$

and hence

$$|y^*(t) - y_n(t)| = \left| \int_{-1}^{1} k(t,s)[z^*(s) - z_n(s)] \, ds \right|$$

$$\leq \int_{-1}^{1} |k(t,s)/\rho(s)| \, |z^*(s) - z_n(s)| \rho(s) \, ds$$

$$\leq \left[\int_{-1}^{1} |k(t,s)/\rho(s)|^p \rho(s) \, ds \right]^{1/p} \times \left[\int_{-1}^{1} |z^*(s) - z_n(s)|^q \rho(s) \, ds \right]^{1/q},$$

where in the last step we have used the Hölder inequality for the distribution $\rho(s) \, ds$. Now for $p > 1$,

$$\sup_{-1 \leq s \leq 1} [\rho(s)]^{1-p} = 1.$$

Therefore,

$$|y^*(t) - y_n(t)| \leq \left[\int_{-1}^{1} |k(t,s)|^p \, ds \right]^{1/p} \|z^* - z_n\|_{q,\rho},$$

and hence

$$\|y^* - y_n\|_\infty \leq c\|z^* - z_n\|_{q,\rho} \to 0 \quad \text{as } n \to \infty,$$

where

$$c = \sup_{-1 \leq t \leq 1} \left[\int_{-1}^{1} |k(t,s)|^p \, ds \right]^{1/p} < \infty.$$

(ii) This follows similarly, with the aid of Corollary 1.

Thus y_n converges *uniformly* to y^* as $n \to \infty$.

5. Implementation note

In practice, it may be worthwhile expressing z_n as a finite Chebyshev series of the form

$$z_n(t) = \sum_{j=0}^{n-1} a_{nj} T_j(t), \qquad t \in [-1, 1], \tag{5.1}$$

where T_j is the jth degree Chebyshev polynomial of the first kind. Then (1.5) yields a system of n nonlinear equations

$$\sum_{j=0}^{n-1} a_{nj} T_j(\tau_{ni}) = g\tau_{ni}, f(\tau_{ni}) + \sum_{j=0}^{n-1} a_{nj} \int_{-1}^{1} k(\tau_{ni}, s) T_j(s) \, ds,$$

$$i = 1, \ldots, n, \tag{5.2}$$

for the n coefficients $a_{n0}, \dots, a_{n,n-1}$. Once these coefficients have been computed, y_n can be calculated from

$$y_n(t) = f(t) + \sum_{j=0}^{n-1} a_{nj} \int_{-1}^{1} k(t,s) T_j(s)\, ds, \qquad t \in [-1,1]. \qquad (5.3)$$

Clearly, the main practical difficulty lies in calculating the integrals

$$KT_j(t) = \int_{-1}^{1} k(t,s) T_j(s)\, ds, \qquad t \in [-1,1],$$

which are required in both (5.2) and (5.3). It is here that the advantage of expressing z_n as the finite Chebyshev series (5.1) appears: for many kernels of practical interest, the integrals $KT_j(t)$ (where t has any fixed value) may be evaluated recursively [13], hence saving much of the computational work.

6. A numerical example

In this section the proposed method is used to solve an integral equation reformulation of the nonlinear two-point boundary-value problem

$$y''(t) - \exp y(t) = 0, \quad t \in (0,1); \qquad y(0) = y(1) = 0,$$

which apparently is of some interest in magnetohydrodynamics [3, p. 41]. This problem has the unique solution

$$y^*(t) = -\ln 2 + 2\ln\left(c/\cos\left(\tfrac{1}{2}c\left(t - \tfrac{1}{2}\right)\right)\right), \qquad t \in [0,1],$$

where $c = 1.3360556949\ldots$ is the root of $c/\cos(c/4) = \sqrt{2}$. It may be reformulated as the integral equation

$$y(t) = \int_{0}^{1} k(t,s) \exp y(s)\, ds, \qquad t \in [0,1],$$

where

$$k(t,s) = \begin{cases} -s(1-t), & s \le t, \\ -t(1-s), & s > t, \end{cases}$$

is the Green's function for the homogeneous problem

$$y''(t) = 0, \quad t \in (0,1); \qquad y(0) = y(1) = 0.$$

From the integral equation it follows that $y(t)$ is symmetric about $t = 1/2$. Taking advantage of this symmetry, the solution $z^*(t) = \exp(y^*(t))$ of the integral equation

$$z(t) = \exp\left(\int_{0}^{1} k(t,s) z(s)\, ds\right), \qquad t \in [0,1],$$

was approximated by the Chebyshev series

$$z_{2m-1}(t) = \sum_{j=0}^{m-1} a_{2m-1,2j} T_{2j}(2t-1), \qquad t \in [0,1],$$

with the coefficients $a_{2m-1,2j}$ determined by collocation at the points

$$\tau_{2m-1,i} = \frac{1}{2}\left(1 + \cos\frac{2i-1}{4m-2}\pi\right), \qquad i = 1, \ldots, m.$$

Note that these points are just the classical Chebyshev points on the interval [0, 1] shifted linearly to the interval [1/2, 1]. All required integrals were calculated exactly by Gaussian quadrature of appropriate precision.

The results, which are displayed in Table 1, show that very high accuracy is obtained even when the polynomials are of low degree. Note that the degree is $2m-2$, and the size of the corresponding nonlinear system is m. It should be mentioned that the present problem has been solved in [9, 10] by the corresponding piecewise polynomial scheme. There very much larger nonlinear systems had to be solved to obtain accuracy of comparable order.

TABLE 1

m	$\|y^* - y_{2m-1}\|_\infty$
2	5.27 E − 5
3	2.57 E − 7
4	1.47 E − 9
5	8.79 E − 12
6	7.97 E − 14

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