ON AN INVERSION FORMULA FOR THE LAPLACE TRANSFORMATION, II

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In an earlier paper (3) we discussed at some length a certain inversion operator for the Laplace transformation. If

$$\text{I} \qquad f(s) = \int_0^\infty e^{-st}\phi(t)\,dt = \mathscr{L}(\phi(t);s)$$

then the inversion operator is given by

$$\text{II} \qquad {}^{\nu}L_{k,t}[f(s)] = \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\int_0^\infty x^{\frac{1}{2}\nu}J_\nu(2kx^{\frac{1}{2}})\,f\!\left(\frac{k(x+1)}{t}\right)dx,$$

and we showed that if $\nu > -1$, then under certain conditions

$$\lim_{k\to\infty} {}^{\nu}L_{k,t}[f(s)] = \phi(t).$$

It is our purpose here to discuss the behaviour of the operator for $\nu \leqslant -1$.

It is clear that if $\nu \leqslant -1$ and ν is not an integer, the operator will not exist. For, by (2, §7.2.1, (2)),

$$x^{\frac{1}{2}\nu}J_\nu(2kx^{\frac{1}{2}}) = (kx)^\nu(1+o(1))/\Gamma(\nu+1), \qquad x\to 0+,$$

if ν is not a negative integer. However, if ν is a negative integer, $\nu = -n$, a different situation appears, for, by (2, §7.2.4, (24)), $J_{-n}(z) = (-1)^n J_n(z)$, and hence

$$x^{-\frac{1}{2}n}J_{-n}(2kx^{\frac{1}{2}}) = (-1)^n k^n(1+o(1))/n! \qquad x\to 0+.$$

Thus there is some prospect of the operator existing in this case.

It will transpire that the operator will exist if $\nu = -n$, under certain hypotheses on $\phi(t)$, and that a suitable modification of the operator will invert the transformation. The theory is contained in the following two theorems.

We make use of the notation

$$\int_0^{\to\infty}\phi(u)\,du$$

to denote the 'Improper' Lebesgue integral. That is,

$$\int_0^{\to\infty}\phi(u)\,du = \lim_{R\to\infty}\int_0^R \phi(u)\,du.$$

Also we define,

$$\text{III} \qquad {}^{\nu}\overrightarrow{L}_{k,t}[f(s)] = \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\int_0^{\to\infty} x^{\frac{1}{2}\nu}J_\nu(2kx^{\frac{1}{2}})\,f\!\left(\frac{k(x+1)}{t}\right)dx.$$

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THEOREM 1. *If* $e^{-\gamma t}\phi(t) \in L(0, \infty)$, $\gamma > 0$, *then*

$$^{-n}\overrightarrow{L}_{k,t}[f(s)]$$

exists for each $t > 0$ *and all* $k > \gamma t$, $(n = 1, 2, \ldots)$, *and*

$$^{-n}\overrightarrow{L}_{k,t}[f(s)] = (-t)^{-n}\ {}^{0}L_{k,t}[f^{(n)}(s)] - \frac{(-1)^{n-1}k^{n+\frac{1}{2}}e^{2k}}{\pi^{\frac{1}{2}}}\sum_{r=1}^{n}\frac{(kt)^{-r}}{(n-r)!}f^{(r-1)}\left(\frac{k}{t}\right).$$

Proof. By (1; ch. 3, §2), $f^{(n)}(s) = L((-t)^n\phi(t); s)$, $s > \gamma$, and thus, by (3, Theorem 2.1), ${}^{0}L_{k,t}[f^{(n)}(s)]$ exists for $k > \gamma t$ $(n = 1, 2, \ldots)$.

Let $k > \gamma t$. Then, since

$$\frac{d}{dz}J_0(z) = -J_1(z) = J_{-1}(z),$$

and since $J_0(z)$ and $f(z) \to 0$ as $z \to \infty$, we obtain on integrating by parts,

$$\begin{aligned}{}^{0}L_{k,t}[f'(s)] &= \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\int_0^{\infty} J_0(2kx^{\frac{1}{2}})f'\left(\frac{k(x+1)}{t}\right)dx\\ &= \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\left\{\frac{t}{k}J_0(2kx^{\frac{1}{2}})f\left(\frac{k(x+1)}{t}\right)\Bigg|_0^{\infty} - t\int_0^{\to\infty}x^{-\frac{1}{2}}J_{-1}(2kx^{\frac{1}{2}})f\left(\frac{k(x+1)}{t}\right)dx\right\}\\ &= -\frac{k^{\frac{1}{2}}e^{2k}}{\pi^{\frac{1}{2}}}f\left(\frac{k}{t}\right) - t\cdot {}^{-1}\overrightarrow{L}_{k,t}[f(s)],\end{aligned}$$

which is the stated result for $n = 1$.

We now proceed by induction. Assuming the result true for n, we have, since $f'(s) = \mathscr{L}(-t\,\phi(t); s)$ for $s > \gamma$, that

$$^{-n}\overrightarrow{L}_{k,t}[f'(s)]$$

exists for $k > \gamma t$ and equals

$$(-t)^{-n}\ {}^{0}L_{k,t}[f^{(n+1)}(s)] - \frac{(-1)^{n-1}k^{n+\frac{1}{2}}}{\pi^{\frac{1}{2}}}\sum_{r=1}^{n}\frac{(kt)^{-r}}{(n-r)!}f^{(r)}\left(\frac{k}{t}\right).$$

Then, for $k > \gamma t$, since, by (2, §7.2.8, (51)),

$$\frac{d}{dz}z^{-n}J_{-n}(z) = z^{-n}J_{-(n+1)}(z)$$

and $J_{-n}(z)$ and $f(z) \to 0$ as $z \to \infty$, we have, on integration by parts, that

$$\begin{aligned}{}^{-n}\overrightarrow{L}_{k,t}[f'(s)] &= \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\int_0^{\to\infty}x^{-\frac{1}{2}n}J_{-n}(2kx^{\frac{1}{2}})f'\left(\frac{k(x+1)}{t}\right)dx\\ &= \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\left\{\frac{t}{k}x^{-\frac{1}{2}n}J_{-n}(2kx^{\frac{1}{2}})f\left(\frac{k(x+1)}{t}\right)\Bigg|_0^{\infty} - t\int_0^{\to\infty}x^{-\frac{1}{2}(n+1)}J_{-(n+1)}(2kx^{\frac{1}{2}})\right.\\ &\qquad\left. f\left(\frac{k(x+1)}{t}\right)dx\right\}\\ &= -\frac{(-1)^n k^{n+\frac{1}{2}}e^{2k}}{\pi^{\frac{1}{2}}n!} - t\cdot {}^{-(n+1)}\overrightarrow{L}_{k,t}[f(s)].\end{aligned}$$

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Hence,

$$ {}^{-(n+1)}\overrightarrow{L}_{k,t}[f(s)] = -t^{-1}\,.\,{}^{-n}\overrightarrow{L}_{k,t}[f'(s)] - \frac{(-1)^n k^{n+\frac{1}{2}} e^{2k}}{t\pi^{\frac{1}{2}} n!} f\left(\frac{k}{t}\right) $$

$$ = (-t)^{-(n+1)}\,{}^{0}L_{k,t}[f^{(n+1)}(s)] + \frac{(-1)^{n-1} k^{n+\frac{1}{2}} e^{2k}}{t\pi^{\frac{1}{2}}} \sum_{r=1}^{n} \frac{(kt)^{-r}}{(n-r)!} f^{(r)}\left(\frac{k}{t}\right) - \frac{(-1)^n k^{n+\frac{1}{2}} e^{2k}}{t\pi^{\frac{1}{2}} n!} f\left(\frac{k}{t}\right) $$

$$ = (-t)^{-(n+1)}\,{}^{0}L_{k,t}[f^{(n+1)}(s)] - \frac{(-1)^n k^{n+3/2} e^{2k}}{\pi^{\frac{1}{2}}} \sum_{r=1}^{n+1} \frac{(kt)^{-r}}{(n+1-r)!} f^{(r-1)}\left(\frac{k}{t}\right). $$

Hence the formula is true for all n.

COROLLARY. *If* $e^{-\gamma t}\phi(t) \in L(0,\infty)$, $\gamma > 0$, *and if* $t^{-\frac{1}{2}}\phi(t) \in L(0,\delta)$, *then* ${}^{-1}L_{k,t}[f(s)]$ *exists and*

$$ {}^{-1}L_{k,t}[f(s)] = -t^{-1}\,{}^{0}L_{k,t}[f'(s)] - \frac{k^{\frac{1}{2}} e^{2k}}{\pi^{\frac{1}{2}} t} f\left(\frac{k}{t}\right). $$

If $e^{-\gamma t}\phi(t) \in L(0,\infty)$, $\gamma > 0$, *then* ${}^{-n}L_{k,t}[f(s)]$ *exists for* $n = 2, 3, \ldots$, *and*

$$ {}^{-n}L_{k,t}[f(s)] = (-t)^{-n}\,{}^{0}L_{k,t}[f^{(n)}(s)] - \frac{(-1)^{n-1} k^{n+\frac{1}{2}} e^{2k}}{\pi^{\frac{1}{2}}} \sum_{r=1}^{n} \frac{(kt)^{-r}}{(n-r)!} f^{(r-1)}\left(\frac{k}{t}\right). $$

Proof. The existence of ${}^{-n}L_{k,t}[f(s)]$ under the various hypotheses follows exactly as in (3, Theorem 2.1). The stated relations now follow from Theorem 1, since

$$ {}^{-n}\overrightarrow{L}_{k,t}[f(s)] = {}^{-n}L_{k,t}[f(s)] $$

when both exist.

THEOREM 2. *If* $e^{-\gamma t}\phi(t) \in L(0,\infty)$, $\gamma > 0$, *then at each point* $t > 0$ *of the Lebesgue set of* ϕ,

$$ \lim_{k\to\infty} \left\{ {}^{-n}\overrightarrow{L}_{k,t}[f(s)] + \frac{(-1)^{n-1} k^{n+\frac{1}{2}} e^{2k}}{\pi^{\frac{1}{2}}} \sum_{r=1}^{n} \frac{(kt)^{-r}}{(n-r)!} f^{(r-1)}\left(\frac{k}{t}\right) \right\} = \phi(t). $$

Proof. This now follows from Theorem 1, and Theorem 3.1 of (3).

COROLLARY. *If* $e^{-\gamma t}\phi(t) \in L(0,\infty)$, $\gamma > 0$, *and if* $t^{-\frac{1}{2}}\phi(t) \in L(0,\delta)$, *for some* $\delta > 0$, *then at each point* $t > 0$ *of the Lebesgue set of* ϕ,

$$ \lim_{k\to\infty} \left\{ {}^{-1}L_{k,t}[f(s)] + \frac{k^{3/2} e^{2k}}{\pi^{\frac{1}{2}} tk} f\left(\frac{k}{t}\right) \right\} = \phi(t). $$

If $e^{-\gamma t}\phi(t) \in L(0,\infty)$, $\gamma > 0$, *then at each point* $t > 0$ *of the Lebesgue set of* ϕ,

$$ \lim_{k\to\infty} \left\{ {}^{-n}L_{k,t}[f(s)] + \frac{(-1)^{n-1} k^{n+\frac{1}{2}} e^{2k}}{\pi^{\frac{1}{2}}} \sum_{r=1}^{n} \frac{(kt)^{-r}}{(n-r)!} f^{(r-1)}\left(\frac{k}{t}\right) \right\} = \phi(t) $$

for $n = 2, 3, \ldots$.

Proof. This now follows from the corollary to Theorem 1, and Theorem 3.1 of **(3)**.

References

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