COMPOSITIO MATHEMATICA

Numerical dimension and a Kawamata–Viehweg–Nadel-type vanishing theorem on compact Kähler manifolds

Junyan Cao

Compositio Math. **150** (2014), 1869–1902.

doi:10.1112/S0010437X14007398

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

Compositio Math. 150 (2014) 1869–1902
doi:10.1112/S0010437X14007398

Numerical dimension and a Kawamata–Viehweg–Nadel-type vanishing theorem on compact Kähler manifolds

Junyan Cao

Abstract

Let X be a compact Kähler manifold and let (L,φ) be a pseudo-effective line bundle on X. We first define a notion of numerical dimension for pseudo-effective line bundles with singular metrics, and then discuss the properties of this numerical dimension. Finally, we prove a very general Kawamata–Viehweg–Nadel-type vanishing theorem on an arbitrary compact Kähler manifold.

1. Introduction

Let X be a compact Kähler manifold and let (L,φ) be a pseudo-effective line bundle[1] on X. Tsuji [Tsu07] defined a notion of numerical dimension for such a pair, using an algebraic approach.

Definition 1.1. Let X be a projective variety and let (L,φ) be a pseudo-effective line bundle. The numerical dimension of (L,φ) is defined to be

$$\nu_{\mathrm{num}}(L,\varphi)=\max\{\dim V \mid V \text{ is a subvariety of } X \text{ such that } \varphi \text{ is well-defined on } V \text{ and } (V,L,\varphi) \text{ is big}\}.$$

Here, (V,L,φ) being 'big' means that there exists a desingularization $\pi:\widetilde{V}\to V$ such that

$$\overline{\lim_{m\to\infty}}\frac{h^0(\widetilde{V},m\pi^*(L)\otimes\mathcal{I}(m\varphi\circ\pi))}{m^n}>0,$$

where n is the dimension of V.[2]

Since Tsuji's definition depends on the existence of subvarieties, it would be convenient to find a more analytic definition in the case where the base manifold is not projective. Following a suggestion of J.-P. Demailly, we first define a notion of numerical dimension, $\mathrm{nd}(L,\varphi)$ (see Definition 3.1), for a pseudo-effective line bundle (L,φ) on a manifold X which is just assumed to be compact Kähler. The definition involves a certain cohomological intersection product of positive currents, introduced in § 2. We discuss the properties of $\mathrm{nd}(L,\varphi)$ in §§ 3 and 4. The main properties are as follows.

Received 20 October 2012, accepted in final form 21 January 2014, published online 27 August 2014.
2010 Mathematics Subject Classification 32L20, 14F17, 14Exx (primary).
Keywords: vanishing theorem, compact Kähler manifolds, Nadel vanishing theorem, pseudo-effective line bundle.

[1] We refer to Definition 2.1 for the definition of a pseudo-effective pair (L,φ).

[2] Tsuji proved in [Tsu07] that bigness does not depend on the choice of desingularization.

Proposition 1.2 (Proposition 3.7). *Let (L,φ) be a pseudo-effective line bundle on a projective variety X of dimension n. If $\mathrm{nd}(L,\varphi)=n$, then*

$$\varliminf_{m\to\infty}\frac{h^0(X,mL\otimes\mathcal{I}(m\varphi))}{m^n}>0.$$

Proposition 1.3 (Proposition 4.2). *Let (L,φ) be a pseudo-effective line bundle on a projective variety X. Then*

$$\nu_{\mathrm{num}}(L,\varphi)=\mathrm{nd}(L,\varphi).$$

Our main goal in this article is to prove a general Kawamata–Viehweg–Nadel vanishing theorem on an arbitrary compact Kähler manifold. Our result is as follows.

Theorem 1.4 (Theorem 5.12). *Let (L,φ) be a pseudo-effective line bundle on a compact Kähler manifold X of dimension n. Then*

$$H^p(X,\mathcal{O}(K_X+L)\otimes\mathcal{I}_+(\varphi))=0\quad\textit{for every } p\geqslant n-\mathrm{nd}(L,\varphi)+1,$$

where $\mathcal{I}_+(\varphi)$ is the upper semicontinuous variant of the multiplier ideal sheaf associated to φ (cf. (2.1) or [FJ05]).

The organization of the article is as follows. In § 2, we recall some elementary results about the analytic multiplier ideal sheaves and define our cohomological product of positive currents by quasi-equisingular approximation. In § 3, using the product defined in § 2, we give our definition of the numerical dimension $\mathrm{nd}(L,\varphi)$ for a pseudo-effective line bundle L equipped with a singular metric φ. The main goal of this section is to obtain an asymptotic estimate of sections when $\mathrm{nd}(L,\varphi)=\dim X$. In § 4, we prove that our numerical dimension coincides with Definition 1.1 when X is projective; we also give a numerical criterion for the numerical dimension and discuss a relationship between the numerical dimension without multiplier ideal sheaves and the numerical dimension defined here. In § 5, we first give a quick proof of our Kawamata–Viehweg–Nadel vanishing theorem on projective varieties; finally, we generalize the vanishing theorem to arbitrary compact Kähler manifolds by the methods developed in [DP03], [Eno93] and [Mou95].

2. Cohomological product of positive currents

We first recall some basic definitions and results about quasi-psh functions (see [Dem12] for details). Let X be a complex manifold. We say that φ is a psh function (respectively, a quasi-psh function) on X if $\varphi: X\to[-\infty,+\infty[$ is upper semicontinuous and

$$i\partial\overline{\partial}\varphi\geqslant 0\quad(\text{respectively, } i\partial\overline{\partial}\varphi\geqslant -c\cdot\omega_X),$$

where c is a positive constant and ω_X is a smooth hermitian metric on X. We say that a quasi-psh function φ has analytic singularities if φ is locally of the form

$$\varphi(z)=c\cdot\ln\Big(\sum|g_i|^2\Big)+O(1),$$

where $c>0$ and the g_i are holomorphic functions. Let φ and ψ be two quasi-psh functions. We say that φ is less singular than ψ if

$$\psi\leqslant\varphi+C$$

for some constant C. We write this as $\varphi\preccurlyeq\psi$.

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

We now recall the analytic definition of multiplier ideal sheaves. Let φ be a quasi-psh function. We define the multiplier ideal sheaves associated to the quasi-psh function φ as

$$\mathcal{I}(\varphi)_x = \left\{ f \in \mathcal{O}_X : \int_{U_x} |f|^2 e^{-2\varphi} < +\infty \right\}$$

where U_x is some open neighborhood of x in X. It is well known that $\mathcal{I}(\varphi)$ is a coherent sheaf (cf. [Dem12] for a more detailed introduction to the concept of multiplier ideal sheaf). When φ does not possess analytic singularities, one needs to introduce the 'upper semicontinuous regularization' of $\mathcal{I}(\varphi)$, namely the ideal sheaf

$$\mathcal{I}_+(\varphi) = \lim_{\epsilon \to 0^+} \mathcal{I}((1+\epsilon)\varphi). \tag{2.1}$$

By the Noetherian property of coherent ideal sheaves, there exists an $\epsilon > 0$ such that

$$\mathcal{I}_+(\varphi) = \mathcal{I}((1+\epsilon')\varphi) \quad \text{for every } 0 < \epsilon' < \epsilon.$$

When φ has analytic singularities, it is easy to see that

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi). \tag{2.2}$$

Conjecturally, the equality (2.2) holds for all quasi-psh functions. Recently, Berndtsson [Ber13] proved that (2.2) holds for quasi-psh functions φ such that $\mathcal{I}(\varphi) = \mathcal{O}_X$. However, it is unknown whether his method can be generalized to arbitrary quasi-psh functions.[3]

Important convention. When we talk about a line bundle L on X, we always implicitly fix a smooth metric h_0 on L. Given a quasi-psh function φ on X, we can therefore construct a new metric (which may be singular) on L by setting $h_0 \cdot e^{-\varphi}$. In a similar fashion, when we prescribe a 'singular metric' φ on L, we actually mean that the metric on L is given by $h_0 \cdot e^{-\varphi}$. Recall that the curvature form for the metric φ is

$$\frac{i}{2\pi}\Theta_\varphi(L) = \frac{i}{2\pi}\Theta_{h_0}(L) + dd^c\varphi$$

by the Poincaré–Lelong formula.

Definition 2.1. Let L be a pseudo-effective line bundle on a compact Kähler manifold X equipped with a metric φ. We say that (L, φ) is a pseudo-effective pair (or sometimes a pseudo-effective line bundle) if the curvature form $(i/2\pi)\Theta_\varphi(L)$ is positive as a current, i.e. $(i/2\pi)\Theta_\varphi(L) \geqslant 0$.

Let $\pi : \widetilde{X} \to X$ be a modification of a smooth variety X, and let φ and ψ be two quasi-psh fuctions on X such that $\mathcal{I}(\varphi) \subset \mathcal{I}(\psi)$. In general, this inclusion does not imply that $\mathcal{I}(\varphi \circ \pi) \subset \mathcal{I}(\psi \circ \pi)$. In order to compare $\mathcal{I}(\varphi \circ \pi)$ and $\mathcal{I}(\psi \circ \pi)$, we need the following lemma.

Lemma 2.2. *Let $E = \pi^* K_X - K_{\widetilde{X}}$. If $\mathcal{I}(\varphi) \subset \mathcal{I}(\psi)$, then*

$$\mathcal{I}(\varphi \circ \pi) \otimes \mathcal{O}(-E) \subset \mathcal{I}(\psi \circ \pi),$$

where the sheaf $\mathcal{O}(-E)$ consists of the germs of holomorphic functions f such that $\operatorname{div}(f) \geqslant E$.

[3] The equality (2.2) is well known in dimension 1 and was proved to be true in dimension 2 by Favre and Jonsson [FJ05]. See [DP03] for more details about $\mathcal{I}_+(\varphi)$.

Proof. It is known that $\mathcal{I}(\varphi\circ\pi)\subset\pi^*\mathcal{I}(\varphi)$ (cf. [Dem12, Proposition 14.3]). Then, for any $f\in\mathcal{I}(\varphi\circ\pi)_x$ we have

$$\int_{\pi(U_x)}|\pi_*(f)|^2e^{-2\varphi}<+\infty, \tag{2.3}$$

where U_x is some open neighborhood of x (though its image $\pi(U_x)$ is not necessarily open). Combining (2.3) with the condition $\mathcal{I}(\varphi)\subset\mathcal{I}(\psi)$, we get

$$\int_{\pi(U_x)}|\pi_*f|^2e^{-2\psi}<+\infty. \tag{2.4}$$

The inequality (2.4) implies that

$$\int_{U_x}|f|^2|J|^2e^{-2\psi\circ\pi}<+\infty, \tag{2.5}$$

where J is the Jacobian of π. Since $\mathcal{O}(-E)=J\cdot\mathcal{O}_X$, (2.5) implies the lemma. $\square$

Let X be a compact Kähler manifold and let T be a closed positive $(1,1)$-current. It is well known that T can be written as

$$T=\theta+dd^c\varphi,$$

where θ is a smooth $(1,1)$-closed form representing $[T]\in H^{1,1}(X,\mathbb{R})$ and φ is a quasi-psh function. Demailly's famous regularization theorem states that φ can be approximated by a sequence of quasi-psh functions with analytic singularities. This type of approximation is called an analytic approximation of φ. Among all such analytic approximations, we want to deal with those which somehow preserve the information concerning the singularities of T. More precisely, we introduce the following definition.

Definition 2.3. Let $\theta+dd^c\varphi$ be a positive current on a compact Kähler manifold (X,ω), where θ is a smooth form and φ is a quasi-psh function on X. We say that $\{\varphi_k\}_{k=1}^\infty$ is a quasi-equisingular approximation of φ for the current $\theta+dd^c\varphi$ if it satisfies the following conditions:

(i) the sequence $\{\varphi_k\}_{k=1}^\infty$ converges to φ in L^1 topology and

$$\theta+dd^c\varphi_k\geqslant-\tau_k\cdot\omega$$

for some constants $\tau_k\to0$ as $k\to+\infty$;

(ii) all the φ_k have analytic singularities, and $\varphi_k\preccurlyeq\varphi_{k+1}$ for all k;

(iii) for any $\delta>0$ and $m\in\mathbb{N}$, there exists $k_0(\delta,m)\in\mathbb{N}$ such that

$$\mathcal{I}(m(1+\delta)\varphi_k)\subset\mathcal{I}(m\varphi)\quad\text{for every } k\geqslant k_0(\delta,m).$$

Remark 1. By condition (i), the concept of a quasi-equisingular approximation depends not on φ only but on the current $\theta+dd^c\varphi$.

The existence of quasi-equisingular approximations was essentially proved in [DPS01, Theorem 2.2.1] by a Bergman kernel method. Such approximations are in some sense the most singular ones asymptotically. The following proposition makes this assertion more precise.

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Proposition 2.4. *Let $\theta + dd^c\varphi_1$ and $\theta + dd^c\varphi_2$ be two positive currents on a compact Kähler manifold X. We assume that the quasi-psh function φ_2 is more singular than φ_1. Let $\{\varphi_{i,1}\}_{i=1}^{\infty}$ be an analytic approximation of φ_1, and let $\{\varphi_{i,2}\}_{i=1}^{\infty}$ be a quasi-equisingular approximation of φ_2. For any closed smooth $(n-1,n-1)$-semipositive form u, we have*

$$\underline{\lim}_{i\to\infty}\int_X (dd^c\varphi_{i,1})_{\mathrm{ac}}\wedge u \geqslant \overline{\lim}_{i\to\infty}\int_X (dd^c\varphi_{i,2})_{\mathrm{ac}}\wedge u, \tag{2.6}$$

where $(dd^c\varphi_{i,1})_{\mathrm{ac}}$ denotes the absolutely continuous part of the current $dd^c\varphi_{i,1}$.

Proof. The idea of the proof is rather standard (cf. [Bou02] or [Dem12, Theorem 18.12]). To prove (2.6), it is enough to show that

$$\int_X (dd^c\varphi_{s,1})_{\mathrm{ac}}\wedge u \geqslant \overline{\lim}_{i\to\infty}\int_X (dd^c\varphi_{i,2})_{\mathrm{ac}}\wedge u \tag{2.7}$$

for every $s\in\mathbb{N}$ fixed. Since $\{\varphi_{i,2}\}_{i=1}^{\infty}$ is a quasi-equisingular approximation of φ_2, for any $\delta>0$ and $m\in\mathbb{N}$, there exists a $k_0(\delta,m)\in\mathbb{N}$ such that

$$\mathcal{I}(m(1+\delta)\varphi_{k,2})\subset\mathcal{I}(m\varphi_2)\quad\text{for every } k\geqslant k_0(\delta,m). \tag{2.8}$$

Since $\varphi_{s,1}\preccurlyeq\varphi_1\preccurlyeq\varphi_2$ by assumption, (2.8) implies that

$$\mathcal{I}(m(1+\delta)\varphi_{k,2})\subset\mathcal{I}(m\varphi_{s,1}) \tag{2.9}$$

for any $s\in\mathbb{N}$ and $k\geqslant k_0(\delta,m)$.

Using (2.9), we can proceed to prove (2.7). Let $\pi:\widehat{X}\to X$ be a log resolution of $\varphi_{s,1}$, i.e. $dd^c(\varphi_{s,1}\circ\pi)$ is locally of the form

$$dd^c(\varphi_{s,1}\circ\pi)=[F]+C^{\infty},$$

where F is a $\mathbb{R}$-normal crossing divisor. By Lemma 2.2, (2.9) implies that

$$\mathcal{I}(m(1+\delta)\varphi_{k,2}\circ\pi)\otimes\mathcal{O}(-J)\subset\mathcal{I}(m\varphi_{s,1}\circ\pi)=\mathcal{O}(-\lfloor mF\rfloor) \tag{2.10}$$

for $k\geqslant k_0(\delta,m)$, where J is the Jacobian of the blow-up π. Since F is a normal crossing divisor, (2.10) implies that $m(1+\delta)dd^c\varphi_{k,2}\circ\pi+[J]-\lfloor mF\rfloor$ is a positive current. Then

$$\int_{\widehat{X}}(m(1+\delta)\cdot dd^c\varphi_{k,2}\circ\pi)_{\mathrm{ac}}\wedge u\leqslant C+\int_{\widehat{X}}(m\cdot dd^c\varphi_{s,1}\circ\pi)_{\mathrm{ac}}\wedge u$$

for $k\geqslant k_0(\delta,m)$, where C is a constant independent of m and k. Letting $m\to+\infty$, we get

$$\int_{\widehat{X}}(dd^c\varphi_{k,2}\circ\pi)_{\mathrm{ac}}\wedge u\leqslant O\left(\frac{1}{m}\right)+C_1\delta+\int_{\widehat{X}}(dd^c\varphi_{s,1}\circ\pi)_{\mathrm{ac}}\wedge u \tag{2.11}$$

for $k\geqslant k_0(\delta,m)$, where C_1 is a constant independent of m and k. Then

$$\int_X(dd^c\varphi_{k,2})_{\mathrm{ac}}\wedge u\leqslant O\left(\frac{1}{m}\right)+C_1\delta+\int_X(dd^c\varphi_{s,1})_{\mathrm{ac}}\wedge u\quad\text{for } k\geqslant k_0(\delta,m).$$

Letting $m\to+\infty$ and $\delta\to 0$, we get

$$\overline{\lim}_{k\to\infty}\int_X(dd^c\varphi_{k,2})_{\mathrm{ac}}\wedge u\leqslant\int_X(dd^c\varphi_{s,1})_{\mathrm{ac}}\wedge u,$$

and so (2.7) is proved. ∎

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Remark 2. By taking $\varphi_1 = \varphi_2$ and $\varphi_{i,1} = \varphi_{i,2}$ in Proposition 2.4, we obtain that the sequence $\{\int_X (dd^c\varphi_{i,2})_{\mathrm{ac}} \wedge u\}_{i=1}^{\infty}$ is in fact convergent. Moreover, if $\{\varphi_{i,1}\}$ and $\{\varphi_{i,2}\}$ are two quasi-equisingular approximations of φ, Proposition 2.4 implies that

$$\lim_{i\to\infty}\int_X (dd^c\varphi_{i,1})_{\mathrm{ac}} \wedge u = \lim_{i\to\infty}\int_X (dd^c\varphi_{i,2})_{\mathrm{ac}} \wedge u. \tag{2.12}$$

Thanks to Proposition 2.4 and (2.12), we can define a related cohomological product of closed positive $(1,1)$-currents.

Definition 2.5. Let $T_1,\ldots,T_k$ be closed positive $(1,1)$-currents on a compact Kähler manifold X. We write them in the potential forms $T_i = \theta_i + dd^c\varphi_i$ as usual. Let $\{\varphi_{i,j}\}_{j=1}^{\infty}$ be a quasi-equisingular approximation of φ_i. Then we can define a product

$$\langle T_1, T_2, \ldots, T_k\rangle$$

as an element in $H^{k,k}_{\geqslant 0}(X)$ (cf. [Bou02] or [Dem12, Theorem 18.12]) such that for all $u \in H^{n-k,n-k}(X)$,

$$\langle T_1, T_2, \ldots, T_k\rangle \wedge u = \lim_{j\to\infty}\int_X (\theta_1 + dd^c\varphi_{1,j})_{\mathrm{ac}} \wedge \cdots \wedge (\theta_k + dd^c\varphi_{k,j})_{\mathrm{ac}} \wedge u,$$

where $\wedge$ is the usual wedge product in cohomology.

Remark 3. Let $\{\psi_{i,j}\}_{j=1}^{\infty}$ be an analytic approximation (not necessarily quasi-equisingular) of φ_i. Thanks to Proposition 2.4 and some standard arguments (cf. [Dem12, Theorem 18.12]), we have

$$\begin{aligned}&\varliminf_{j\to\infty}\int_X (\theta_1 + dd^c\psi_{1,j})_{\mathrm{ac}} \wedge \cdots \wedge (\theta_k + dd^c\psi_{k,j})_{\mathrm{ac}} \wedge u\\ &\quad\geqslant \lim_{j\to\infty}\int_X (\theta_1 + dd^c\varphi_{1,j})_{\mathrm{ac}} \wedge \cdots \wedge (\theta_k + dd^c\varphi_{k,j})_{\mathrm{ac}} \wedge u.\end{aligned}$$

This means that the product defined in Definition 2.5 is smaller than the product defined by any other analytic approximation. In particular, the product defined in Definition 2.5 does not depend on the choice of the quasi-equisingular approximation.

3. Numerical dimension

Using Definition 2.5, we can give our definition of the numerical dimension.

Definition 3.1. Let (L,φ) be a pseudo-effective line bundle on a compact Kähler manifold X. We define the numerical dimension $\mathrm{nd}(L,\varphi)$ to be the largest $v \in \mathbb{N}$ such that $\langle (i\Theta_\varphi(L))^v\rangle \neq 0$, where the cohomological product $\langle (i\Theta_\varphi(L))^v\rangle$ is the v-fold product of $i\Theta_\varphi(L)$ given in Definition 2.5.

Let (L,φ) be a pseudo-effective line bundle on X of dimension n such that $\mathrm{nd}(L,\varphi) = n$. If the quasi-psh function φ has analytic singularities, it is not difficult to see that

$$\frac{h^0(X, mL \otimes \mathcal{I}(m\varphi))}{m^n}$$

admits a strictly positive limit by using the Riemann–Roch formula. When φ is just a quasi-psh function, Tsuji conjectured in [Tsu07] that

$$\frac{h^0(X, mL \otimes \mathcal{I}(m\varphi))}{m^n}$$

also admits a strictly positive limit. The main goal of this section is to prove Proposition 1.2, i.e. if $\mathrm{nd}(L,\varphi)=n$, then

$$\varliminf_{m\to\infty}\frac{h^0(X,mL\otimes\mathcal{I}(m\varphi))}{m^n}>0.$$

To begin with, we explain the construction of quasi-equisingular approximations by a Bergman kernel method. Before doing so, we first give a useful estimate that uses the comparison-of-integrals method in [DPS01, Theorem 2.2.1, Step 2]. Although the proof is almost the same, we present it here for the sake of completeness.

Lemma 3.2. *Let A be a very ample line bundle on a projective manifold X, and let (L,φ) be a pseudo-effective line bundle. Let φ_m be the metric on L constructed by the Bergman kernel of $H^0(X,\mathcal{O}(A+mL)\otimes\mathcal{O}(m\varphi))$ with respect to the metric $m\varphi$. Then*

$$\mathcal{I}\left(\frac{sm}{m-s}\varphi_m\right)\subset\mathcal{I}(s\varphi)\quad\textit{for any } m,s\in\mathbb{N}.$$

Proof. First of all, we have the following estimate on X:

$$\begin{aligned}\int_{s\cdot\varphi(x)\leqslant(sm/(m-s))\cdot\varphi_m(x)}e^{-2s\cdot\varphi(x)}&=\int_{s\cdot\varphi(x)\leqslant sm/(m-s)\cdot\varphi_m(x)}e^{2(m-s)\cdot\varphi(x)-2m\cdot\varphi(x)}\\&\leqslant\int_X e^{2m\cdot\varphi_m}e^{-2m\cdot\varphi}\\&=h^0(X,\mathcal{O}(A+mL)\otimes\mathcal{I}(m\varphi))<+\infty.\end{aligned}$$

Using the finiteness obtained above, for any $f\in\mathcal{I}((sm/(m-s))\varphi_m)_x$ we have

$$\begin{aligned}\int_{U_x}|f|^2e^{-2s\varphi}&\leqslant\int_{s\varphi(x)\leqslant(sm/(m-s))\varphi_m(x)}|f|^2e^{-2s\varphi}+\int_{U_x}|f|^2e^{-(2sm/(m-s))\varphi_m}\\&\leqslant\sup|f|^2\cdot\int_{s\varphi(x)\leqslant(sm/(m-s))\varphi_m(x)}e^{-2s\varphi}+\int_{U_x}|f|^2e^{-2(sm/(m-s))\varphi_m}<+\infty.\end{aligned}$$

Then $f\in\mathcal{I}(s\varphi)$, so the lemma is proved. □

We are going to construct a quasi-equisingular approximation to φ. Although such approximations were implicitly constructed in [DPS01, Theorem 2.2.1] for the local case, we can easily adapt that construction to a global situation by using the same techniques.

Proposition 3.3. *Let X be a projective variety of dimension n and let ω be a Kähler metric in $H^{1,1}(X,\mathbb{Q})$. Let (L,φ) be a pseudo-effective line bundle on X (see Definition 2.1) such that $\mathrm{nd}(L,\varphi)=n$.*

Let (G,h_G) be an ample line bundle on X equipped with a smooth metric h_G, such that the curvature form $i\Theta_{h_G}(G)$ is positive and sufficiently large (e.g. G is very ample and $G-K_X$ is ample). Let $\{\tau_{p,q,i}\}_i$ be an orthonormal basis of

$$H^0(X,\mathcal{O}(2^pG+2^qL)\otimes\mathcal{I}(2^q\varphi))$$

with respect to the singular metric $h_G^{2^p}\cdot h_0^{2^q}\cdot e^{-2^q\varphi}$. We define

$$\varphi_{p,q}=\frac{1}{2^q}\ln\sum_i|\tau_{p,q,i}|^2_{h_G^{2^p}\cdot h_0^{2^q}}.$$

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Then there exist two increasing integral sequences $p_m \to +\infty$ and $q_m \to +\infty$ with

$$\lim_{m\to+\infty}(q_m/p_m) = +\infty$$

and

$$q_m - q_{m-1} \geqslant p_m - p_{m-1} \quad \text{for all } m \in \mathbb{N}$$

such that $\{\varphi_{p_m,q_m}\}_{m=1}^{+\infty}$ is a quasi-equisingular approximation of φ for the current $(i/(2\pi))\Theta_{h_0}(L)+ dd^c\varphi$. Set $\varphi_m := \varphi_{p_m,q_m}$ for simplicity.

Moreover, $\{\varphi_m\}$ satisfies the following two properties.

(i) *$H^0(X,\mathcal{O}(2^{p_m}G + 2^{q_m}L)\otimes\mathcal{I}(2^{q_m}\varphi_m)) = H^0(X,\mathcal{O}(2^{p_m}G + 2^{q_m}L)\otimes\mathcal{I}(2^{q_m}\varphi))$ for every $m\in\mathbb{N}^+$.*

(ii) *There exists a constant $C > 0$ independent of G and m such that*

$$\int_X \left(\frac{i}{2\pi}\Theta_{\varphi_m}(L) + \epsilon\omega\right)^n_{\mathrm{ac}} > C$$

for all $\epsilon > 0$ and $m \geqslant m_0(\epsilon)$ (i.e. m is larger than a constant depending on ϵ).

Proof. By [Dem12, Theorems 13.21 and 13.23], there exist two sequences $p_m \to +\infty$ and $q_m \to +\infty$ with

$$\lim_m q_m/p_m = +\infty$$

and

$$q_m - q_{m-1} \geqslant p_m - p_{m-1} \quad \text{for all } m \in \mathbb{N}$$

such that $\{\varphi_m\}$ is an analytic approximation of φ for the current $(i/2\pi)\Theta_\varphi(L)$. Since φ_m is constructed using the Bergman kernel, by Lemma 3.2, $\{\varphi_m\}$ satisfies condition (iii) in Definition 2.3. To prove that $\{\varphi_m\}$ is a quasi-equisingular approximation, it remains to verify condition (ii) in Definition 2.3.

We first prove that

$$\varphi_{p-1,q-1} \preccurlyeq \varphi_{p,q} \quad \text{and} \quad \varphi_{p,q-1} \preccurlyeq \varphi_{p-1,q-1} \tag{3.1}$$

by using the standard diagonal trick (cf. [DEL00] or [DPS01, Theorem 2.2.1, Step 3]). Let Δ be the diagonal of $X\times X$, and let π_1 and π_2 be two projections from $X\times X$ to X. Let

$$F := 2^{p-1}\pi_1^*G + 2^{p-1}\pi_2^*G + 2^{q-1}\pi_1^*L + 2^{q-1}\pi_2^*L$$

be a new bundle on $X\times X$ equipped with a singular metric $2^{q-1}\pi_1^*(\varphi) + 2^{q-1}\pi_2^*(\varphi)$. Since $2^{p-1}G - K_X$ is ample enough, we can apply the Ohsawa–Takegoshi extension theorem from Δ to $X\times X$ for the line bundle F. Then the following map is surjective:

$$(H^0(X,\mathcal{O}(2^{p-1}G + 2^{q-1}L)\otimes\mathcal{I}(2^{q-1}\varphi)))^2 \to H^0(X,\mathcal{O}(2^pG + 2^qL)\otimes\mathcal{I}(2^q\varphi)). \tag{3.2}$$

Let $\{f_{p-1,q-1,i}\}_{i=1}^N$ be an orthonormal basis of

$$H^0(X,\mathcal{O}(2^{p-1}G + 2^{q-1}L)\otimes\mathcal{I}(2^{q-1}\varphi))$$

with respect to the singular metric $h_G^{2^{p-1}}\cdot h_0^{2^{q-1}}\cdot e^{-2^{q-1}\varphi}$. For any

$$g \in H^0(X,\mathcal{O}(2^pG + 2^qL)\otimes\mathcal{I}(2^q\varphi)),$$

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

by applying the effective version of the Ohsawa–Takegoshi extension theorem to (3.2), there exist constants $\{c_{i,j}\}$ such that

$$g(z) = \left(\sum_{i,j} c_{i,j} f_{p-1,q-1,i}(z) f_{p-1,q-1,j}(w)\right)\Bigg|_{z=w}$$

and

$$\sum_{i,j} |c_{i,j}|^2 \leqslant C_1 \|g\|^2,$$

where C_1 depends only on X and $\|g\|$ is the L^2 norm with respect to the singular metric $h_G^{2p} \cdot h_0^{2^q} e^{-2^q \varphi}$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |g(z)|^2_{h_G^{2p} \cdot h_0^{2^q}} &\leqslant \left(\sum_{i,j} |c_{i,j}|^2\right)\left(\sum_{i,j} |f_{p-1,q-1,i}(z) f_{p-1,q-1,j}(z)|^2_{h_G^{2p} \cdot h_0^{2^q}}\right) \\ &\leqslant C_1 \|g\|^2 \left(\sum_i |f_{p-1,q-1,i}(z)|^2_{h_G^{2p-1} \cdot h_0^{2^{q-1}}}\right)^2. \end{aligned}$$

Assuming $\|g\| = 1$, we get

$$\begin{aligned} \frac{1}{2^q} \ln |g(z)|^2_{h_G^{2p} \cdot h_0^{2^q}} &\leqslant \frac{\ln C_1}{2^q} + \frac{1}{2^{q-1}} \ln\left(\sum_i |f_{p-1,q-1,i}(z)|^2_{h_G^{2p-1} \cdot h_0^{2^{q-1}}}\right) \\ &= \frac{\ln C_1}{2^q} + \varphi_{p-1,q-1}(z). \end{aligned}$$

By the extremal property of the Bergman kernel, we finally obtain that

$$\varphi_{p-1,q-1} \preccurlyeq \varphi_{p,q}.$$

Thus the first inequality in (3.1) is proved. The second inequality in (3.1) is evident by observing that G is very ample. Thanks to the construction of p_m and q_m, (3.1) implies that $\varphi_{m-1} \preccurlyeq \varphi_m$. Therefore φ_m is a quasi-equisingular approximation of φ for the current $(i/2\pi)\Theta_\varphi(L)$.

It remains to check properties (i) and (ii) of the proposition. Property (i) comes directly from the construction of φ_m. Property (ii) follows from the fact that $\mathrm{nd}(L, \varphi) = n$ and φ_m is a quasi-equisingular approximation. □

The rest of this section is devoted to the proof of Proposition 1.2. The strategy is as follows. Thanks to property (ii) of Proposition 3.3, we can construct a new metric on L with strictly positive curvature that is more singular than φ in an asymptotic sense (cf. (3.11)). Then Proposition 1.2 follows from a standard estimate for this new metric. Before giving the construction of the new metric, we need the following two preparatory propositions.

Proposition 3.4. *Let φ_m be the quasi-psh function constructed in Proposition 3.3. Then there exists another quasi-psh function $\widetilde{\varphi}_m$ such that:*

(i) $\sup_{x \in X} \widetilde{\varphi}_m(x) = 0$;
(ii) $(i/2\pi)\Theta_{\widetilde{\varphi}_m}(L) \geqslant (\delta/2) \cdot \omega$, *where δ is a strictly positive number independent of m;*
(iii) $\varphi_m \preccurlyeq \widetilde{\varphi}_m$.

Proof. Let $\pi : X_m \to X$ be a log resolution of φ_m. We can therefore assume that

$$\frac{i}{2\pi}\Theta_{\varphi_m\circ\pi}(\pi^* L) = [E] + \beta,$$

where $[E]$ is a normal crossing divisor and $\beta \in C^\infty$. Keeping the notation used in Proposition 3.3, since $\omega \in H^{1,1}(X, \mathbb{Q})$, we can find a $\mathbb{Q}$-ample line bundle A on X such that $c_1(A) = \omega$. Let ϵ be a positive rational number. By property (ii) of Proposition 3.3, we have

$$\int_X \left(\frac{i}{2\pi}\Theta_{\varphi_m}(L) + \epsilon\omega\right)^n_{\mathrm{ac}} > C.$$

Thanks to Proposition 3.3,

$$\left(\frac{i}{2\pi}\Theta_{\varphi_m\circ\pi}(\pi^* L) + \epsilon\pi^*\omega\right)_{\mathrm{ac}}$$

is a $\mathbb{Q}$-nef class for m large enough. We can thus choose a $\mathbb{Q}$-nef line bundle F_m on X_m such that

$$c_1(F_m) = \left(\frac{i}{2\pi}\Theta_{\varphi_m\circ\pi}(\pi^* L) + \epsilon\pi^*\omega\right)_{\mathrm{ac}}. \tag{3.3}$$

We now prove that

$$F_m - \delta\pi^*\omega \tag{3.4}$$

is pseudo-effective for a uniform constant $\delta > 0$ independent of ϵ and m. In order to prove (3.4), we first give a uniform upper bound on $F_m^{n-1}\cdot\pi^* A$. Let C_1 be a constant such that $C_1 \cdot A - L$ is effective. Using the nefness of F_m and $\pi^* A$, (3.3) implies that

$$\begin{aligned} F_m^{n-1}\cdot\pi^* A &\leqslant F_m^{n-2}\cdot(\pi^* L + \epsilon\pi^*\omega)\cdot\pi^* A \leqslant F_m^{n-2}\cdot(C_1+\epsilon)\pi^* A\cdot\pi^* A \\ &\leqslant F_m^{n-3}\cdot((C_1+\epsilon)\pi^* A)^2\cdot\pi^* A \leqslant \cdots \leqslant ((C_1+\epsilon)\pi^* A)^{n-1}\cdot\pi^* A. \end{aligned}$$

Therefore $\{F_m^{n-1}\cdot\pi^* A\}_m$ is uniformly bounded (for $\epsilon < 1$). Combining this with property (ii) of Proposition 3.3, we can thus choose a rational constant $\delta > 0$ independent of ϵ and m such that

$$F_m^n > n\delta F_m^{n-1}\cdot\pi^* A. \tag{3.5}$$

Using the holomorphic Morse inequality (cf. [Dem12, ch. 8] or [Tra11]) for the $\mathbb{Q}$-bundle $F_m - \delta\cdot\pi^*(A)$ on X_m, we have

$$h^0(X_m, kF_m - k\delta\cdot\pi^* A) \geqslant C\,\frac{k^n}{n!}(F_m^n - n\delta F_m^{n-1}\cdot\pi^* A) + O(k^{n-1}). \tag{3.6}$$

Combining (3.5) and (3.6), we obtain that $F_m - \delta\pi^*\omega$ is pseudo-effective.

By taking $\epsilon \leqslant \delta/2$, the pseudo-effectiveness (3.4) implies that $(i/2\pi)\Theta_{\varphi_m\circ\pi}(\pi^* L)_{\mathrm{ac}} - (\delta/2)\pi^*\omega$ is pseudo-effective. In other words, there exists a quasi-psh function ψ_m on X_m such that

$$\frac{i}{2\pi}\Theta_{\varphi_m\circ\pi}(\pi^* L) + dd^c\psi_m \geqslant \frac{\delta}{2}\pi^*\omega. \tag{3.7}$$

Let C_1 be a constant such that

$$\sup_{x\in X_m}(\varphi_m\circ\pi + \psi_m + C_1)(x) = 0.$$

Then (3.7) implies that $\varphi_m\circ\pi(x) + \psi_m(x) + C_1$ induces a quasi-psh function on X, which we denote by $\widetilde{\varphi}_m$. It is easy to check that $\widetilde{\varphi}_m$ satisfies all the requirements in the proposition. $\square$

Remark 4. In the proof of Proposition 3.4 we assumed that ϵ is rational. The reason is that we want to use the holomorphic Morse inequality (3.6). However, by using the techniques in [DP04], one can get the same results without assuming that ϵ is rational.

Thanks to Proposition 3.4, we are able to construct a singular metric on L which is a type of limit of $\widetilde{\varphi}_m$. We first recall the notion of upper semicontinuous regularization. Let $\Omega \subset \mathbb{R}^n$ and let $(u_\alpha)_{\alpha\in I}$ be a family of upper semicontinuous functions $\Omega \to [-\infty, +\infty[$. Assume that (u_α) is locally uniformly bounded from above. Since the upper envelope

$$u = \sup_{\alpha\in I} u_\alpha$$

need not be upper semicontinuous, we consider its upper semicontinuous regularization,

$$u^*(z) = \lim_{\epsilon\to 0} \sup_{B(z,\epsilon)} u.$$

We denote this upper semicontinuous regularization by $\widetilde{\sup}_\alpha(u_\alpha)$. It is easy to prove that if $\{u_\alpha\}_{\alpha\in I}$ are psh functions which are locally uniformly bounded from above, then $\widetilde{\sup}_\alpha(u_\alpha)$ is also a psh function (see [Dem12] for details).

We need the following lemma.

Lemma 3.5. *Let φ be a quasi-psh function with normal crossing singularities, i.e. φ is locally of the form*

$$\varphi = \sum_i a_i \ln|f_i| + O(1)$$

where the f_i are holomorphic functions and $\sum_i \operatorname{div}(f_i)$ is a normal crossing divisor. Let $\{\psi_i\}$ be quasi-psh functions such that

$$\sup_{z\in X} \psi_i(z) \leqslant 0 \quad \text{and} \quad dd^c\psi_i \geqslant -C\omega$$

for some uniform constant C independent of i. If $\varphi \preccurlyeq \psi_i$ for all i, then

$$\varphi \preccurlyeq \widetilde{\sup_i}(\psi_i).$$

Proof. Since φ has normal crossing singularities and φ is less singular than φ_i, the differences $\psi_i - \varphi$ are quasi-psh functions and

$$dd^c(\psi_i - \varphi) \geqslant -C_1\omega \tag{3.8}$$

for some uniform constant C_1 independent of i. Since $\sup_{z\in X} \psi_i(z) \leqslant 0$ and $dd^c\psi_i \geqslant -C\omega$ for a uniform constant C, standard potential theory implies that there exists a constant M such that

$$\int_X \psi_i \leqslant M \quad \text{for all } i.$$

Therefore

$$\int_X (\psi_i - \varphi) \leqslant M' \tag{3.9}$$

for a uniform constant M'.

Combining (3.8) and (3.9), there exists a uniform constant C_2 such that

$$\sup_{z\in X}(\psi_i(z) - \varphi(z)) \leqslant C_2 \quad \text{for all } i.$$

Therefore $\varphi \preccurlyeq \widetilde{\sup}_i(\psi_i)$ and the lemma is proved. □

Thanks to Propositions 3.4 and 3.5, we can construct the following crucial metric mentioned in the paragraph before Proposition 3.4.

Proposition 3.6. *In the situation of Proposition 3.4, set*

$$\widetilde{\varphi}(z) := \lim_{m\to\infty} \widetilde{\sup_{s\geqslant 0}}((\widetilde{\varphi}_{m+s}(z))).$$

Then the new metric $\widetilde{\varphi}$ satisfies

$$\frac{i}{2\pi}\Theta_{\widetilde{\varphi}}(L) \geqslant \frac{\delta}{2}\omega \tag{3.10}$$

and

$$\varphi_m \preccurlyeq \widetilde{\varphi} \quad \textit{for every } m \geqslant 1. \tag{3.11}$$

Proof. By Proposition 3.4, we have

$$\frac{i}{2\pi}\Theta_{\widetilde{\varphi}_m}(L) \geqslant \frac{\delta}{2}\omega \quad \text{for } m \geqslant 1.$$

By letting $m \to +\infty$, (3.10) is proved. To check (3.11), since $\widetilde{\varphi} \leqslant \widetilde{\sup}_{s\geqslant 0}(\widetilde{\varphi}_{m+s})$ by construction, it is enough to show that

$$\varphi_m \preccurlyeq \widetilde{\sup_{s\geqslant 0}}(\widetilde{\varphi}_{m+s}). \tag{3.12}$$

Combining Propositions 3.3 and 3.4, we obtain that

$$\varphi_m \preccurlyeq \varphi_{m+s} \preccurlyeq \widetilde{\varphi}_{m+s} \quad \text{for every } m \text{ and } s. \tag{3.13}$$

Let $\pi : \widehat{X} \to X$ be a log resolution of φ_m. By (3.13), we have

$$\varphi_m \circ \pi \preccurlyeq \varphi_{m+s} \circ \pi \preccurlyeq \widetilde{\varphi}_{m+s} \circ \pi. \tag{3.14}$$

Since $\varphi_m \circ \pi$ has normal crossing singularities, by Lemma 3.5, (3.14) implies that

$$\varphi_m \circ \pi \preccurlyeq \widetilde{\sup_{s\geqslant 0}}(\widetilde{\varphi}_{m+s} \circ \pi).$$

Upon passing to π_*, (3.12) is proved. □

Using the new metric $\widetilde{\varphi}$, we can give the following asymptotic estimate.

Proposition 3.7 (Proposition 1.2). *Let X be a projective variety of dimension n and let (L, φ) be a pseudo-effective line bundle on X such that $\mathrm{nd}(L, \varphi) = n$. Then*

$$\varliminf_{m\to\infty} \frac{h^0(X,\, mL \otimes \mathcal{I}(m\varphi))}{m^n} > 0.$$

Proof. Let $\{\varphi_m\}$ be the quasi-equisingular approximation of φ constructed in Proposition 3.3. By Lemma 3.2, for every $m \in \mathbb{N}$ we have

$$h^0(X,\, mL \otimes \mathcal{I}(m\varphi)) \geqslant h^0\bigg(X,\, mL \otimes \mathcal{I}\bigg(\frac{m\cdot 2^{q_k}}{2^{q_k} - m}\varphi_k\bigg)\bigg). \tag{3.15}$$

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Let $\widetilde{\varphi}$ be the metric constructed in Proposition 3.6. By (3.11) in Proposition 3.6, we have

$$h^0\left(X,\, mL \otimes \mathcal{I}\left(\frac{m \cdot 2^{q_k}}{2^{q_k} - m}\varphi_k\right)\right) \geqslant h^0\left(X,\, mL \otimes \mathcal{I}\left(\frac{m \cdot 2^{q_k}}{2^{q_k} - m}\widetilde{\varphi}\right)\right) \tag{3.16}$$

for every k and m. Combining (3.15) with (3.16), we obtain that

$$h^0(X,\, mL \otimes \mathcal{I}(m\varphi)) \geqslant h^0\left(X,\, mL \otimes \mathcal{I}\left(\frac{m \cdot 2^{q_k}}{2^{q_k} - m}\widetilde{\varphi}\right)\right). \tag{3.17}$$

Since (3.17) is true for every m and k, we can take k so large that $2^{q_k} \gg m$. By applying (3.10) to (3.17), we have

$$\varliminf_{m \to \infty} \frac{h^0(X,\, mL \otimes \mathcal{I}(m\varphi))}{m^n} > 0,$$

so the proposition is proved. $\square$

Remark 5. Proposition 3.7 implies that if $\mathrm{nd}(L, \varphi) = \dim X$, then $\nu_{\mathrm{num}}(L, \varphi) = \dim X$ (cf. Definition 1.1). In the next section, we will study the relation between $\mathrm{nd}(L, \varphi)$ and $\nu_{\mathrm{num}}(L, \varphi)$ in more detail.

4. A numerical criterion

Up to now, we have two concepts of numerical dimension for a pseudo-effective pair (L, φ): the 'algebraic' concept $\nu_{\mathrm{num}}(L, \varphi)$ and the more analytic concept $\mathrm{nd}(L, \varphi)$ (see Definitions 1.1 and 3.1). We prove in this section that these two definitions coincide when X is projective. Before giving the proof, we first list some properties of multiplier ideal sheaves which will be useful in our context. The essential tool here is the Ohsawa–Takegoshi extension theorem (cf. [Dem12, ch. 12]).

Lemma 4.1. *Let (L, φ) be a pseudo-effective line bundle on a projective variety X of dimension n, and let $\{\varphi_k\}$ be a quasi-equisingular approximation of φ. Let s_1 be a positive number such that*

$$\mathcal{I}_+(\varphi) = \mathcal{I}((1+\epsilon')\varphi) \quad \textit{for every } 0 < \epsilon' \leqslant s_1. \tag{4.1}$$

Assume that A is a very ample line bundle and S is the zero divisor of a very general global section of $H^0(X, A)$. Then the following properties hold.

(i) *The restrictions*

$$\mathcal{I}(m\varphi_k) \to \mathcal{I}(m\varphi_k|_S), \quad \mathcal{I}_+(m\varphi_k) \to \mathcal{I}_+(m\varphi_k|_S), \tag{4.2}$$
$$\mathcal{I}(m\varphi) \to \mathcal{I}(m\varphi|_S), \quad \mathcal{I}_+(m\varphi) \to \mathcal{I}_+(m\varphi|_S) \tag{4.3}$$

are well-defined for all $m \in \mathbb{N}$, where $\varphi|_S$ denotes the restriction of φ on S and $\mathcal{I}(\varphi|_S)$ is the multiplier ideal sheaf associated to $\varphi|_S$ on S.[4] *Moreover, we have*

$$\mathcal{I}((1+\epsilon')\varphi|_S) = \mathcal{I}((1+s_1)\varphi|_S) \quad \textit{for every } 0 < \epsilon' \leqslant s_1.$$

(ii) *$\{\varphi_k|_S\}$ is a quasi-equisingular approximation of $\varphi|_S$.*

[4] Note that $\varphi|_S$ is also quasi-psh if it is well-defined.

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

(iii) *If the restrictions are well-defined, we have an exact sequence*

$$0 \to \mathcal{I}_+(\varphi) \otimes \mathcal{O}(-S) \to \operatorname{adj}_S^{\epsilon}(\varphi) \to \mathcal{I}_+(\varphi|_S) \to 0$$

for every $0 < \epsilon \leqslant s_1$, *where*

$$\operatorname{adj}_S^{\epsilon}(\varphi)_x = \left\{ f \in \mathcal{O}_x : \int_{U_x} \frac{|f|^2}{|s|^{2(1-\epsilon/2)}} e^{-2(1+\epsilon)\varphi} < +\infty \right\}.$$

(iv) $\operatorname{adj}_S^{\epsilon}(\varphi) = \mathcal{I}_+(\varphi)$ *for every* $0 < \epsilon \leqslant s_1$.

Proof. (i) First of all, since S is very general, φ_k and φ are well-defined on S. Since the multiplier ideal sheaves here are coherent and the restrictions (4.2) and (4.3) contain only countable morphisms, by Fubini's theorem it is easy to see that the restrictions (4.2) and (4.3) are well-defined.

To show the second part of (i), since S is very general, we can suppose that

$$\mathcal{I}((1+s_1)\varphi) \to \mathcal{I}((1+s_1)\varphi|_S) \tag{4.4}$$

is well-defined. Combining this with (4.1), we obtain that

$$\mathcal{I}((1+\epsilon')\varphi) \to \mathcal{I}((1+\epsilon')\varphi|_S) \tag{4.5}$$

is well-defined for every $0 < \epsilon' < s_1$. Let $f \in \mathcal{I}(S, (1+s_1)\varphi|_S)_x$. Applying the Ohsawa–Takegoshi extension theorem to (4.4), there exists a function $\widetilde{f} \in \mathcal{I}((1+s_1)\varphi)$ such that $\widetilde{f}|_S = f$. Thanks to (4.1) and (4.5), $\widetilde{f}|_S \in \mathcal{I}((1+\epsilon')\varphi|_S)$ for every $0 < \epsilon' < s_1$, so (i) is proved.

(ii) Since $\{\varphi_k\}$ is a quasi-equisingular approximation of φ, we have that

$$\mathcal{I}(m(1+\delta)\varphi_k) \subset \mathcal{I}(m\varphi) \quad \text{for every } k \geqslant k_0(\delta, m). \tag{4.6}$$

To prove (ii), it is enough to show that

$$\mathcal{I}(m(1+\delta)\varphi_k|_S) \subset \mathcal{I}(m\varphi|_S) \quad \text{for every } k \geqslant k_0(\delta, m). \tag{4.7}$$

Let $f \in \mathcal{I}(m(1+\delta)\varphi_k|_S)_x$. By the Ohsawa–Takegoshi extension theorem, there exists a $\widetilde{f} \in \mathcal{I}(X, m(1+\delta)\varphi_k)_x$ such that $\widetilde{f}|_S = f$. By (4.6), $\widetilde{f} \in \mathcal{I}(m\varphi)$. Thanks to (4.3), we have $\widetilde{f}|_S \in \mathcal{I}(S, m\varphi\,|_S)$. Hence (4.7) is proved.

(iii) First of all, the Ohsawa–Takegoshi extension theorem implies the surjectivity of the sequence. It remains to prove the exactness of the middle term, i.e. for any $f \in \mathcal{O}_x$ satisfying the conditions

$$\frac{f}{s} \in \mathcal{O}_x \quad \text{and} \quad \int_{U_x} \frac{|f|^2}{|s|^{2(1-\epsilon/2)}} e^{-2(1+\epsilon)\varphi} < +\infty, \tag{4.8}$$

we should prove the existence of some $\epsilon' > 0$ such that

$$\int_{U_x} \frac{|f|^2}{|s|^2} e^{-2(1+\epsilon')\varphi} < +\infty, \tag{4.9}$$

where s is a local defining function for S. In fact, if $f/s \in \mathcal{O}_x$, then

$$\int_{U_x} \frac{|f|^2}{|s|^{4-\delta}} < +\infty \quad \text{for every } \delta > 0. \tag{4.10}$$

By taking $\epsilon' = \epsilon/4$ in (4.9), we have

$$\int_{U_x} \frac{|f|^2}{|s|^2} e^{-2(1+\epsilon/4)\varphi} \leqslant \left(\int_{U_x} \frac{|f|^2}{|s|^{2(1-\epsilon/2)}} e^{-2(1+\epsilon)\varphi}\right)^{(1+\epsilon/4)/(1+\epsilon)} \left(\int_{U_x} \frac{|f|^2}{|s|^{\alpha}}\right)^{(3\epsilon/4)/(1+\epsilon)} \tag{4.11}$$

by Hölder's inequality, where

$$\alpha = \left(2 - 2\left(1 - \frac{\epsilon}{2}\right)\frac{1+\epsilon/4}{1+\epsilon}\right) \cdot (1+\epsilon) \cdot \frac{4}{3\epsilon} = \frac{10\epsilon + \epsilon^2}{3\epsilon} < 4.$$

Thanks to (4.8) and (4.10), the right-hand side of (4.11) is finite. Thus (4.9) is proved.

(iv) By the definition of $\mathcal{I}_+(\varphi)$, we have an obvious inclusion

$$\mathrm{adj}_S^{\epsilon}(\varphi) \subset \mathcal{I}_+(\varphi).$$

In order to prove the equality, it is enough to show that for any $f \in \mathcal{I}((1+\epsilon)\varphi)_x$ we have

$$\int_{U_x} \frac{|f|^2}{|s|^{2(1-\epsilon/2)}} e^{-2(1+\epsilon)\varphi}\, dV < +\infty, \tag{4.12}$$

where s is a general global section of $H^0(X, A)$ independent of the choice of f and x.

The bound (4.12) comes from Fubini's theorem. In fact, let $\{s_0, \ldots, s_N\}$ be a basis for $H^0(X, A)$. Then

$$\sum_{i=0}^{N} |s_i(x)|^2 \neq 0 \quad \text{for every } x \in X.$$

Taking $\{\tau_0, \ldots, \tau_N\} \in \mathbb{C}^{N+1}$, we have

$$\begin{aligned}
&\int_{\sum_{i=0}^N |\tau_i|^2 = 1} d\tau \int_{U_x} \frac{|f|^2}{\left|\sum_{i=0}^N \tau_i s_i\right|^{2(1-\epsilon/2)}} e^{-2(1+\epsilon)\varphi}\, dV \\
&= \int_{U_x} \frac{|f|^2}{\left|\sum_{i=0}^N |s_i(x)|^2\right|^{(1-\epsilon/2)}} e^{-2(1+\epsilon)\varphi}\, dV \int_{\sum_{i=0}^N |\tau_i|^2 = 1} \frac{1}{\left(\sum_{i=0}^N \tau_i (s_i / \sum_{i=0}^N |s_i(x)|^2)\right)^{2(1-\epsilon/2)}}\, d\tau \\
&= \int_{U_x} \frac{|f|^2}{\left|\sum_{i=0}^N |s_i(x)|^2\right|^{(1-\epsilon/2)}} e^{-2(1+\epsilon)\varphi}\, dV \int_{\sum_{i=0}^N |\tau_i|^2 = 1} \frac{1}{|\tau_0|^{2(1-\epsilon/2)}}\, d\tau < +\infty.
\end{aligned} \tag{4.13}$$

For any $f \in \mathcal{I}((1+\epsilon)\varphi)_x$ fixed, by applying Fubini's theorem to (4.13) we obtain that

$$\int_{U_x} \frac{|f|^2}{|s|^{2(1-\epsilon/2)}} e^{-2(1+\epsilon)\varphi} < +\infty \tag{4.14}$$

for a general element $s \in H^0(X, A)$. Observing that $\mathcal{I}((1+\epsilon)\varphi)$ is finitely generated on X, we can therefore choose a general section s such that (4.14) is true for any $f \in \mathcal{I}((1+\epsilon)\varphi)$. Thus (4.12) is proved. □

The next proposition confirms that our definition of the numerical dimension coincides with Tsuji's definition.

Proposition 4.2. *If (L, φ) is a pseudo-effective line bundle on a projective variety X of dimension n, then*

$$\nu_{\mathrm{num}}(L, \varphi) = \mathrm{nd}(L, \varphi).$$

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Proof. We first prove that

$$\nu_{\mathrm{num}}(L,\varphi) \geqslant \mathrm{nd}(L,\varphi) \tag{4.15}$$

by induction on the dimension. If $\mathrm{nd}(L,\varphi)=n$, (4.15) comes from Proposition 3.7. Assume that $\mathrm{nd}(L,\varphi)<n$. Let A be a general hypersurface given by a very ample line bundle, and let $\{\varphi_k\}$ be a quasi-equisingular approximation of φ. By Lemma 4.1, $\varphi_k|_A$ is a quasi-equisingular approximation of $\varphi|_A$. Since A is a general section and $\mathrm{nd}(L,\varphi)<n$, we have

$$\lim_{k\to\infty}\int_A\bigg(\bigg(\frac{i}{2\pi}\Theta_{\varphi_k}(L)\bigg)_{\mathrm{ac}}\bigg)^s\wedge\omega^{n-s-1}>0,$$

where $s=\mathrm{nd}(L,\varphi)$. By Definition 2.3, we have

$$\mathrm{nd}(L,\varphi|_A)\geqslant s=\mathrm{nd}(L,\varphi), \tag{4.16}$$

where $\mathrm{nd}(L,\varphi|_A)$ is the numerical dimension of $(L,\varphi|_A)$ on A. Note, moreover, that by the definition of ν_{num},

$$\nu_{\mathrm{num}}(L,\varphi)\geqslant\nu_{\mathrm{num}}(L,\varphi|_A). \tag{4.17}$$

Thanks to (4.16) and (4.17), we get (4.15) by induction on the dimension.

We now prove that

$$\nu_{\mathrm{num}}(L,\varphi)\leqslant\mathrm{nd}(L,\varphi). \tag{4.18}$$

Assume that $\nu_{\mathrm{num}}(L,\varphi)=s$. By Definition 1.1, there exists a subvariety V of dimension s such that

$$\overline{\lim_{m\to\infty}}\frac{h^0(V,\ mL\otimes\mathcal{I}(m\varphi))}{m^s}>0. \tag{4.19}$$

Let $\{\varphi_k\}$ be a quasi-equisingular approximation of φ. To prove (4.18), by Definition 3.1, it is sufficient to show that

$$\lim_{k\to+\infty}(i\Theta_{\varphi_k}(L))_{\mathrm{ac}}^s\wedge[V]>0. \tag{4.20}$$

We prove (4.20) by using the holomorphic Morse inequality for line bundles equipped with singular metrics (cf. [Bon98]). Let $\pi:\widetilde{X}\to X$ be a desingularization of V in X, and let $\widetilde{V}$ be the strict transform of V. Thanks to (4.19), we have

$$\overline{\lim_{m\to\infty}}\frac{h^0(\widetilde{V},\ m\pi^*(L)\otimes\mathcal{I}(m\varphi_k\circ\pi))}{m^s}>0\quad\text{for every }k. \tag{4.21}$$

Let A be an ample line bundle on X and let ω be a Kähler metric such that $c_1(A)=\omega$. By Definition 3.1, we can find a positive sequence $\epsilon_k\to0$ such that $(i\Theta_{\varphi_k}(L))_{\mathrm{ac}}+\epsilon_k\omega>0$. Using [Bon98, Theorem 1.1], we have

$$\int_V(i\Theta_{\varphi_k}(L)+\epsilon_k\omega)_{\mathrm{ac}}^s\geqslant\overline{\lim_{m\to\infty}}\frac{h^0(\widetilde{V},\ m\pi^*(L)\otimes\mathcal{I}(m\varphi_k\circ\pi))}{m^s}.$$

Combining this with (4.21), we obtain that

$$(i\Theta_{\varphi_k}(L)+\epsilon_k\omega)_{\mathrm{ac}}^s\wedge[V]>0.$$

Upon letting $k\to+\infty$, (4.20) is proved. □

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Remark 6. From the proof, it is easy to conclude that if $S_1, S_2, \dots, S_k$ are divisors of general global sections of a very ample line bundle, then

$$\mathrm{nd}(L, \varphi|_{S_1 \cap S_2 \cap \cdots \cap S_k}) = \max(\mathrm{nd}(L, \varphi), n - k). \tag{4.22}$$

In fact, if $\mathrm{nd}(L, \varphi) \leqslant n - k$, then by the same argument as above, $\varphi_m|_{S_1 \cap S_2 \cap \cdots \cap S_k}$ is also a quasi-equisingular approximation of $\varphi|_{S_1 \cap S_2 \cap \cdots \cap S_k}$. Then (4.22) is proved by a simple calculation.

Before giving a numerical criterion to calculate the numerical dimension, we mention the following interesting example, [Tsu07, Example 3.6]. The example tells us that we cannot expect an equality of the form

$$\sup_A \overline{\lim_{m \to \infty}} \frac{\ln h^0(X, \mathcal{O}(A + mL) \otimes \mathcal{I}(m\varphi))}{\ln m} = \mathrm{nd}(L, \varphi) \tag{4.23}$$

where A runs over all the ample bundles on X. In fact, Tsuji defined a closed positive $(1,1)$-current T on $\mathbb{P}^1$,

$$T = \sum_{i=1}^{+\infty} \sum_{j=1}^{3^{i-1}} \frac{1}{4^i} P_{i,j}$$

where the $P_{i,j}$ are distinct points on $\mathbb{P}^1$. Thus, there exists a singular metric φ on $L = \mathcal{O}(1)$ with $(i/2\pi)\Theta_\varphi(L) = T$. It is easy to construct a quasi-equisingular approximation $\{\varphi_k\}$ of φ such that

$$\frac{i}{2\pi}\Theta_{\varphi_k}(L) = \sum_{i=1}^{k} \sum_{j=1}^{3^{i-1}} \frac{1}{4^i} P_{i,j} + C^\infty.$$

Then $\mathrm{nd}(L, \varphi) = 0$.

On the other hand, owing to the construction of φ, we have

$$\overline{\lim_{m \to \infty}} \frac{h^0(\mathbb{P}^1, \mathcal{O}(s + m) \otimes \mathcal{I}(m\varphi))}{m} = \overline{\lim_{k \to \infty}} \frac{h^0(\mathbb{P}^1, \mathcal{O}(s + 4^k - 1) \otimes \mathcal{I}((4^k - 1)\varphi))}{4^k - 1}$$

for every $s \geqslant 1$. By construction,

$$\mathcal{I}((4^k - 1)\varphi)_x = \mathcal{O}_x$$

for $x \notin \{P_{i,j}\}_{i \leqslant k-1}$, and $\mathcal{I}((4^k - 1)\varphi)$ has multiplicity $\lfloor (4^k - 1)/4^i \rfloor = 4^{k-i} - 1$ on 3^{i-1} points $\{P_{i,1}, \dots, P_{i,3^{i-1}}\}$. Therefore

$$\begin{aligned} h^0(\mathbb{P}^1, \mathcal{O}(s + 4^k - 1) \otimes \mathcal{I}((4^k - 1)\varphi)) &= s + 4^k - \sum_{i=1}^{k-1} 3^{i-1}(4^{k-i} - 1) \\ &= \frac{9}{2} 3^{k-1} + s - \frac{1}{2}. \end{aligned}$$

Then

$$\sup_A \overline{\lim_{m \to \infty}} \frac{\ln h^0(\mathbb{P}^1, \mathcal{O}(A + m) \otimes \mathcal{I}(m\varphi))}{\ln m} = \frac{\ln 3}{\ln 4}.$$

Therefore

$$\mathrm{nd}(L, \varphi) \neq \sup_A \overline{\lim_{m \to \infty}} \frac{\ln h^0(\mathbb{P}^1, \mathcal{O}(A + m) \otimes \mathcal{I}(m\varphi))}{\ln m}.$$

In view of the above example, we propose the following modified formula for calculating the numerical dimension.

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Proposition 4.3. *Let (L,φ) be a pseudo-effective line bundle on a projective variety X, and let A be a very ample line bundle. Then $\mathrm{nd}(L,\varphi)=d$ if and only if*

$$\lim_{\epsilon\to 0}\frac{\ln\Big(\overline{\lim}_{m\to\infty}\,(h^0(X,\ m\epsilon A+mL\otimes\mathcal{I}(m\varphi))/m^n)\Big)}{\ln\epsilon}=n-d.$$

Proof. First of all, the inclusion

$$\begin{aligned}H^0(X,\ m\epsilon A+mL\otimes\mathcal{I}(m\varphi))&\supset H^0(X,\ m\epsilon A+mL\otimes\mathcal{I}_+(m\varphi))\\&\supset H^0(X,\ m\epsilon A+mL\otimes\mathcal{I}((m+1)\varphi))\end{aligned}$$

implies that $h^0(X,m\epsilon A+mL\otimes\mathcal{I}_+(m\varphi))$ has the same asymptotic comportment as $h^0(X,m\epsilon A+mL\otimes\mathcal{I}(m\varphi))$. Since we have constructed the exact sequence for $\mathcal{I}_+$ in Lemma 4.1, we prefer to calculate $h^0(X,m\epsilon A+mL\otimes\mathcal{I}_+(m\varphi))$ instead of $h^0(X,m\epsilon A+mL\otimes\mathcal{I}(m\varphi))$.

If $\mathrm{nd}(L,\varphi)=n$, the proposition follows directly from Proposition 4.2. Assume that $\mathrm{nd}(L,\varphi)=d<n$. Let $\{Y_i\}_{i=1}^n$ be the zero divisors of n very general global sections of $H^0(X,A)$. By the remark after Proposition 4.2, there exists a uniform constant $C>0$ such that for all m and ϵ,

$$h^0(Y_1\cap\cdots\cap Y_{n-d},\ m\epsilon A+mL\otimes\mathcal{I}_+(m\varphi))=C(\epsilon,m)\cdot m^d \tag{4.24}$$

with $C(\epsilon,m)\geqslant C$. Our aim is to prove by induction on s that

$$\begin{aligned}&\frac{1}{m^{n-s}}h^0(Y_1\cap\cdots\cap Y_s,\ m\epsilon A+mL\otimes\mathcal{I}_+(m\varphi))\\&\quad=C(\epsilon,m)\epsilon^{n-s-d}\frac{1}{(n-d-s)!}+O(\epsilon^{n-s-d+1})+O\left(\frac{1}{m}\right)\end{aligned} \tag{4.25}$$

for $0\leqslant s\leqslant n-d$. If $s=n-d$, (4.25) comes from (4.24). Assume that (4.25) is true for $s_0\leqslant s\leqslant n-d$. We now prove (4.25) for $s=s_0-1$.

Let Y be the intersection of zero divisors of s_0-1 general global sections of $H^0(X,A)$, and let

$$e_1^{0,q}(\epsilon,m)=\binom{m\epsilon}{q}h^0(Y\cap Y_1\cap\cdots\cap Y_q,\ m\epsilon A\otimes mL\otimes\mathcal{I}_+(m\varphi)). \tag{4.26}$$

We claim that

$$\frac{1}{m^{n-s_0+1}}h^0(Y,m\epsilon A+mL\otimes\mathcal{I}_+(m\varphi))=-\frac{1}{m^{n-s_0+1}}\left(\sum_{q\geqslant 1}(-1)^q e_1^{0,q}(\epsilon,m)\right)+O\left(\frac{1}{m}\right). \tag{4.27}$$

We postpone the proof of (4.27) to Lemma 4.4 and conclude the proof of (4.25) first. If $q>n-d-s_0+1$, we have, by definition,

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}e_1^{0,q}(\epsilon)=O(\epsilon^q)\leqslant O(\epsilon^{n-d-s_0+2}). \tag{4.28}$$

Then (4.27) and the induction hypothesis of (4.25) imply that

$$\begin{aligned}&\frac{1}{m^{n-s_0+1}}\,h^0(Y,\ m\epsilon A+mL\otimes\mathcal{I}_+(m\varphi))\\&\quad=-\left(\sum_{q=1}^{n-d-s_0+1}(-1)^q\frac{\epsilon^{n-d-s_0+1}C(\epsilon,m)}{q!(n-q-s_0+1-d)!}\right)+O(\epsilon^{n-d-s_0+2})+O\left(\frac{1}{m}\right)\end{aligned}$$

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

$$= -\left(\sum_{q=1}^{n-d-s_0+1}(-1)^q \frac{\epsilon^{n-d-s_0+1}C(\epsilon,m)}{(n-s_0+1-d)!}\binom{n-s_0+1-d}{q}\right) + O(\epsilon^{n-d-s_0+2}) + O\left(\frac{1}{m}\right)$$
$$= -\frac{\epsilon^{n-d-s_0+1}C(\epsilon,m)}{(n-s_0+1-d)!}\left(\sum_{q=1}^{n-d-s_0+1}(-1)^q\binom{n-s_0+1-d}{q}\right) + O(\epsilon^{n-d-s_0+2}) + O\left(\frac{1}{m}\right)$$
$$= C(\epsilon,m)\epsilon^{n-d-s_0+1}\frac{1}{(n-d-s_0+1)!} + O(\epsilon^{n-d-s_0+2}) + O\left(\frac{1}{m}\right).$$

Therefore (4.25) is proved for $s = s_0 - 1$.

In particular, taking $s = 0$ in (4.25), we have

$$\lim_{\epsilon\to 0}\overline{\lim_{m\to\infty}}\frac{1}{m^n\epsilon^{n-d}}h^0(X, m\epsilon A + mL\otimes\mathcal{I}_+(m\varphi)) > 0.$$

So the proposition is proved. □

We now prove the claim (4.27) in Proposition 4.3.

Lemma 4.4. *In the situation of Proposition 4.3, we have*

$$\frac{1}{m^{n-s_0+1}}h^0(Y,\ m\epsilon A + mL\otimes\mathcal{I}_+(m\varphi)) = \frac{1}{m^{n-s_0+1}}e_1^{0,0}(\epsilon,m)$$
$$= -\frac{1}{m^{n-s_0+1}}\left(\sum_{q\geqslant 1}(-1)^q e_1^{0,q}(\epsilon,m)\right) + O\left(\frac{1}{m}\right).$$

Proof. Thanks to properties (iii) and (iv) of Lemma 4.1 and [Kür06, § 4], $\mathcal{O}_Y(mL\otimes\mathcal{I}_+(m\varphi))$ is resolved by a complex of sheaves

$$\begin{aligned}\mathcal{O}_Y(m\epsilon A + mL\otimes\mathcal{I}_+(m\varphi)) &\to \bigoplus_{1\leqslant i\leqslant m\epsilon}\mathcal{O}_{Y\cap Y_i}(m\epsilon A + mL\otimes\mathcal{I}_+(m\varphi))\\ &\to \bigoplus_{1\leqslant i_1<i_2\leqslant m\epsilon}\mathcal{O}_{Y\cap Y_{i_1}\cap Y_{i_2}}(m\epsilon A + mL\otimes\mathcal{I}_+(m\varphi))\\ &\to\cdots\end{aligned}\tag{$*$}$$

and then

$$H^k(Y,\ mL\otimes\mathcal{I}_+(m\varphi)) = \mathbb{H}^k(\epsilon,m),\tag{4.29}$$

where $\mathbb{H}^k(\epsilon,m)$ represents the hypercohomology of $(*)$.

We now calculate the asymptotic behavior of both sides of (4.29). The Nadel vanishing theorem implies that

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}h^k(Y,\ mL\otimes\mathcal{I}_+(m\varphi)) = 0 \quad \text{for every } k\geqslant 1.\tag{4.30}$$

Moreover, since we assume that $\mathrm{nd}(L,h) = d < \dim Y$, we have

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}h^0(Y,\ mL\otimes\mathcal{I}_+(m\varphi)) = 0.\tag{4.31}$$

By calculating the asymptotic cohomology on both sides of (4.29), (4.30) and (4.31) imply in particular that

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}\sum_k(-1)^k h^k(\epsilon,m)=0, \tag{4.32}$$

where $h^k(\epsilon,m)$ denotes the dimension of $\mathbb{H}^k(\epsilon,m)$.

We now prove the lemma by using (4.32). By the Nadel vanishing theorem, we have

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}\binom{m\epsilon}{q}h^p(Y\cap Y_1\cap\cdots\cap Y_q,\ m\epsilon A\otimes mL\otimes\mathcal{I}_+(m\varphi))=0$$

for every $p\geqslant 1$. If $p=0$, we have

$$\binom{m\epsilon}{q}h^0(Y\cap Y_1\cap\cdots\cap Y_q,\ m\epsilon A\otimes mL\otimes\mathcal{I}_+(m\varphi))=e_1^{0,q}(\epsilon,m)$$

by (4.26). Then (4.32) implies that

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}\left(\sum_{q\geqslant 0}(-1)^q e_1^{0,q}(\epsilon,m)\right)=0\quad\text{for every }\epsilon>0,$$

which is equivalent to saying that

$$\begin{aligned}\frac{1}{m^{n-s_0+1}}h^0(Y,\ m\epsilon A+mL\otimes\mathcal{I}_+(m\varphi))&=\frac{1}{m^{n-s_0+1}}e_1^{0,0}(\epsilon,m)\\&=-\frac{1}{m^{n-s_0+1}}\left(\sum_{q\geqslant 1}(-1)^q e_1^{0,q}(\epsilon,m)\right)+O\left(\frac{1}{m}\right).\end{aligned}$$

Thus the lemma is proved. □

Remark 7. On a compact Kähler manifold, Boucksom defined in [Bou02] a concept of numerical dimension, $\mathrm{nd}(L)$, for a pseudo-effective line bundle L without any specified metric. Let $\varphi_{\min}$ be a positive metric of L with minimal singularities. Proposition 4.3 implies, in particular, that

$$\mathrm{nd}(L)\geqslant\mathrm{nd}(L,\varphi_{\min}). \tag{4.33}$$

Example 1.7 from [DPS94] tells us that we cannot hope for an equality

$$\mathrm{nd}(L)=\mathrm{nd}(L,\varphi_{\min}).$$

In that example, the line bundle L is nef and $\mathrm{nd}(L)=1$. On the other hand, [DPS94, Example 1.7] shows that there exists a unique singular metric h on L such that the curvature form $(i/2\pi)\Theta_h(L)$ is positive. Moreover,

$$\frac{i}{2\pi}\Theta_h(L)=[C]$$

for a curve C on X. Therefore $\varphi_{\min}=h$ and $\mathrm{nd}(L,\varphi_{\min})=0$. Hence

$$\mathrm{nd}(L)>\mathrm{nd}(L,\varphi_{\min})$$

in this example.

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

5. A Kawamata–Viehweg–Nadel vanishing theorem

The classic Nadel vanishing theorem states the following.

Theorem 5.1 [Nad90, Dem93]. *Let (X, ω) be a projective manifold and let (L, φ) be a pseudo-effective line bundle on X. If $i\Theta_\varphi(L) \geqslant c \cdot \omega$ for some constant $c > 0$, then*

$$H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}(\varphi)) = 0 \quad \textit{for every } q \geqslant 1.$$

One of the limitations of Theorem 5.1 is that the curvature $i\Theta_\varphi(L)$ should be strictly positive. Various attempts have been made to overcome this limitation. For example, the following more classic Kawamata–Viehweg vanishing theorem has found many applications in complex algebraic geometry (cf. [Dem12, ch. 6.D])

Theorem 5.2 [Dem12]. *Let X be a projective manifold and let F be a line bundle over X such that some positive multiple mF can be written as $mF = L + D$ where L is a nef line bundle and D is an effective divisor. Then*

$$H^q(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(m^{-1}D)) = 0 \quad \textit{for every } q > n - \mathrm{nd}(L).$$

The classic proof of Theorem 5.2 uses an ample line bundle on X and a hyperplane section argument to perform an induction on the dimension. Therefore the hypothesis that X is projective is crucial in Theorem 5.2. However, we believe that it would be useful to find a Kawamata–Viehweg-type vanishing theorem for arbitrary Kähler manifolds. In this direction, Demailly and Peternell have proved the following result.

Theorem 5.3 [DP03]. *Let (L, h) be a line bundle over a compact Kähler n-fold X. Assume that L is nef. Then the natural morphism*

$$H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}_+(h)) \to H^q(X, \mathcal{O}(K_X + L))$$

vanishes for $q \geqslant n - \mathrm{nd}(L) + 1$.

Following several ideas and techniques from [DP03], we will prove in this section our main theorem, Theorem 1.4, which says that given a pseudo-effective line bundle (L, φ) over a compact Kähler manifold X of dimension n, one has

$$H^p(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}_+(\varphi)) = 0 \quad \text{for } p \geqslant n - \mathrm{nd}(L, \varphi) + 1.$$

By (4.33), our vanishing theorem can be viewed as a generalization of Theorem 5.3. The main advantage of this version of the Kawamata–Viehweg–Nadel vanishing theorem is that we do not need strict positivity of the line bundle; but as compensation, we have to use the multiplier ideal sheaf $\mathcal{I}_+(\varphi)$ instead of $\mathcal{I}(\varphi)$. When X is projective, the proof of our vanishing theorem is much easier. We first give a quick proof of Theorem 1.4 in the projective case using the tools developed in the previous sections. To begin with, we prove Theorem 1.4 in the case where $\mathrm{nd}(L, \varphi) = \dim X$.

Proposition 5.4. *Let X be a smooth projective variety of dimension n. Let (L, φ) be a pseudo-effective line bundle over X with $\mathrm{nd}(L, \varphi) = n$. Then*

$$H^i(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}_+(\varphi)) = 0 \quad \textit{for every } i > 0.$$

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Proof. Recall that we first fix a smooth metric h_0 on L. The quasi-psh function φ gives a metric $h_0 e^{-\varphi}$ on L. That (L,φ) is pseudo-effective means that

$$\frac{i}{2\pi}\Theta_{\varphi}(L) = \frac{i}{2\pi}\Theta_{h_0}(L) + dd^c\varphi \geqslant 0.$$

Since $(i/2\pi)\Theta_{\varphi}(L)$ is not strictly positive, we cannot apply Theorem 5.1 directly. The idea is to add a portion of the metric $\widetilde{\varphi}$ constructed in Proposition 3.6 to make the curvature form for the new metric strictly positive. We will see that this operation preserves the multiplier ideal sheaves $\mathcal{I}_+(\varphi)$.

First of all, by the definition of $\mathcal{I}_+$ (see § 2), there exists a $\delta > 0$ such that

$$\mathcal{I}_+(\varphi) = \mathcal{I}((1+\delta)\varphi). \tag{5.1}$$

Let $\widetilde{\varphi}$ be the psh function constructed in Proposition 3.6. Set $\varphi_1 := (1+\sigma(\epsilon)-\epsilon)\varphi + \epsilon\widetilde{\varphi}$, where $0 < \epsilon < 1$ and $0 < \sigma(\epsilon) \ll \epsilon$. Since $dd^c\varphi \geqslant -c\omega$ for some constant c,[5] the condition $\sigma(\epsilon) \ll \epsilon$ implies that

$$\frac{i}{2\pi}\Theta_{\varphi_1}(L) = (1+\sigma(\epsilon)-\epsilon)\frac{i}{2\pi}\Theta_{\varphi}(L) + \epsilon\frac{i}{2\pi}\Theta_{\widetilde{\varphi}}(L) + \sigma(\epsilon)dd^c\varphi > 0.$$

Applying the standard Nadel vanishing theorem (cf. Theorem 5.1) to $(X, L, \mathcal{I}(\varphi_1))$, we get

$$H^i(X, \mathcal{O}(K_X + L)\otimes\mathcal{I}(\varphi_1)) = 0 \quad \text{for } i > 0. \tag{5.2}$$

On the other hand, it not hard to prove that

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi_1) \quad \text{for } \epsilon \ll 1. \tag{5.3}$$

We postpone the proof of (5.3) to Lemma 5.5 and conclude the proof of Proposition 5.4 first. By taking ϵ small enough, (5.2) and (5.3) imply the proposition. $\square$

Lemma 5.5. *In the situation of Proposition 5.4, if ϵ is small enough, then*

$$\mathcal{I}(\varphi_1) = \mathcal{I}_+(\varphi). \tag{5.4}$$

Proof. By (3.11) of Proposition 3.6, we have

$$(1+\sigma(\epsilon))\varphi_m = (1+\sigma(\epsilon)-\epsilon)\varphi_m + \epsilon\varphi_m \preccurlyeq (1+\sigma(\epsilon)-\epsilon)\varphi + \epsilon\widetilde{\varphi}.$$

Therefore

$$\mathcal{I}(\varphi_1) \subset \mathcal{I}((1+\sigma(\epsilon))\varphi_m). \tag{5.5}$$

Note that, by Lemma 3.2, we have

$$\mathcal{I}((1+\sigma(\epsilon))\varphi_m) \subset \mathcal{I}_+(\varphi) \tag{5.6}$$

for m large enough with respect to $\sigma(\epsilon)$. Combining (5.5) with (5.6), we obtain that

$$\mathcal{I}(\varphi_1) \subset \mathcal{I}_+(\varphi).$$

[5] In our context, since φ is a function on X, we have $(i/2\pi)\Theta_{\varphi}(L) = (i/2\pi)\Theta_{h_0}(L) + dd^c\varphi \geqslant 0$. Therefore $dd^c\varphi \geqslant -c\omega$.

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

For the reverse inclusion of (5.4), we need to prove that if $f \in \mathcal{I}_+(\varphi)_x$, then

$$f \in \mathcal{I}(\varphi_1)_x.$$

By (5.1), we have

$$\int_{U_x} |f|^2 e^{-2(1+\delta)\varphi} < +\infty. \tag{5.7}$$

Since $\widetilde{\varphi}$ is a quasi-psh function, by taking ϵ small enough, we have

$$\int_{U_x} e^{-2(\epsilon/\delta)\widetilde{\varphi}} < +\infty. \tag{5.8}$$

Therefore (5.7) and (5.8) imply that

$$\int_{U_x} |f|^2 e^{-2(1+\sigma(\epsilon)-\epsilon)\varphi - 2\epsilon\widetilde{\varphi}} \leqslant \int_{U_x} |f|^2 e^{-2(1+\delta)\varphi} \int_{U_x} e^{-2(\epsilon/\delta)\widetilde{\varphi}} < +\infty$$

by Hölder's inequality. Since $\varphi_1 = (1+\sigma(\epsilon)-\epsilon)\varphi + \epsilon\widetilde{\varphi}$ by construction, we have $f \in \mathcal{I}(\varphi_1)$. The lemma is proved. □

Using Proposition 5.4, we can prove the following Kawamata–Viehweg–Nadel vanishing theorem by induction on the dimension.

Proposition 5.6. *Let (L, φ) be a pseudo-effective line bundle on a projective variety X of dimension n. Then*

$$H^p(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}_+(\varphi)) = 0 \quad \text{for } p \geqslant n - \mathrm{nd}(L, \varphi) + 1.$$

Proof. If $\mathrm{nd}(L, \varphi) = n$, the result has been proved in Proposition 5.4. Assume that $\mathrm{nd}(L, \varphi) < n$. Let A be an ample line bundle that is large enough with respect to L, and let S be the zero divisor of a very general global section of $H^0(X, A)$. Let $\epsilon > 0$ be small enough that property (iv) of Lemma 4.1 is satisfied (by Lemma 4.1, ϵ is independent of A). By Lemma 4.1, we have an exact sequence

$$0 \to \mathcal{I}_+(\varphi) \otimes \mathcal{O}(-S) \to \mathcal{I}_+(\varphi) \to \mathcal{I}_+(S, \varphi_S) \to 0. \tag{5.9}$$

Therefore we get an exact sequence

$$\begin{aligned} H^q(S, \mathcal{O}(K_S + L) \otimes \mathcal{I}_+(\varphi|_S)) &\to H^{q+1}(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}_+(\varphi)) \\ &\to H^{q+1}(X, \mathcal{O}(K_X + A + L) \otimes \mathcal{I}_+(\varphi)) \end{aligned}$$

for every $q \geqslant 0$. Since A is ample enough with respect to L, we have

$$H^{q+1}(X, \mathcal{O}(K_X + A + L) \otimes \mathcal{I}_+(\varphi)) = 0$$

by the Nadel vanishing theorem. Thus the above exact sequence implies that

$$H^q(S, \mathcal{O}(K_S + L) \otimes \mathcal{I}_+(\varphi|_S)) \to H^{q+1}(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}_+(\varphi))$$

is surjective for every q. The proposition is proved by induction on the dimension. □

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

The main goal of this section is to prove Theorem 1.4 for arbitrary Kähler manifolds. To achieve this, we use the methods developed in [DP03], [Eno93] and [Mou95]. To clarify the idea of the proof, we first consider the following easy case. Assume that (X, ω) is a compact Kähler manifold and (L, φ) is a pseudo-effective line bundle with analytic singularities. Let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ be the eigenvalues of $i\Theta_{\varphi}(L)$ with respect to ω. Let f be a smooth (n,p)-form representing an element in $H^p(X, K_X \otimes L \otimes \mathcal{I}(\varphi))$ for some $p \geqslant n - \mathrm{nd}(L, \varphi) + 1$. Then $\int_X |f|^2 e^{-2\varphi} \omega^n < +\infty$. By using a L^2 estimate (cf. [DP03] or Proposition A.1 in the Appendix), f can be written as

$$f = \overline{\partial} u_k + v_k, \tag{5.10}$$

with the following estimate:

$$\int_X |u_k|^2 e^{-2\varphi} + \frac{1}{2p\epsilon_k} \int_X |v_k|^2 e^{-2\varphi} \leqslant \int_X \frac{1}{2p\epsilon_k + \lambda_1 + \lambda_2 + \cdots + \lambda_p} |f|^2 e^{-2\varphi}, \tag{5.11}$$

where $\{\epsilon_k\}$ is a positive sequence tending to 0. Since $p \geqslant n - \mathrm{nd}(L, \varphi) + 1$, we have

$$\int_X \left(\sum_{i \geqslant p} \lambda_i(z) \right) \omega^n > 0. \tag{5.12}$$

If $\lambda_p(z)$ is generically strictly positive, (5.11) implies that

$$\lim_{k \to +\infty} \int_X |v_k|^2 e^{-2\varphi} = 0.$$

By some standard results from functional analysis (cf. Lemma 5.8), we obtain that

$$f = 0 \in H^p(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}(\varphi)).$$

The situation becomes more complicated when $\lambda_p(z)$ is not necessarily generically strictly positive. In this case, thanks to the condition (5.12) and the fact that φ has analytic singularities, we can use Monge–Ampère equations to construct a sequence of new metrics $\widehat{\varphi}_k$ on L such that $\int_X |f|^2 e^{-2\widehat{\varphi}_k} \omega^n$ can be controlled by $\int_X |f|^2 e^{-2\varphi} \omega^n$ and, more importantly, the place where the pth eigenvalue of $i\Theta_{\widehat{\varphi}_k}(L)$ is strictly positive tends to cover the whole X. Letting $k \to +\infty$, we can thus prove that

$$f = 0 \in H^p(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}(\varphi)).$$

In the general case, since φ does not necessarily possess analytic singularities, we are in trouble when using L^2 estimates. Therefore we replace φ by a quasi-equisingular approximation $\{\varphi_k\}$ and get estimates similar to (5.10) and (5.11) with φ replaced by φ_k. We can use a Monge–Ampère equation to construct other metrics $\widehat{\varphi}_k$ for which we can control the eigenvalues. Therefore we can use L^2 estimates for every $\widehat{\varphi}_k$. By a delicate analysis, we then prove the theorem. Such ideas were already used in [DP03], [Eno93] and [Mou95]. We will construct the key metric $\widehat{\varphi}_k$ in Lemma 5.9 and prove some important properties of $\widehat{\varphi}_k$ in Lemmas 5.10 and 5.11. Finally, we prove the vanishing theorem in Theorem 5.12.

To begin with, we show that $\mathcal{I}_+$ has analytic singularities. More precisely, we prove the following result.

LEMMA 5.7. *Let (L, φ) be a pseudo-effective line bundle over a compact Kähler manifold X. Then there exists a quasi-equisingular approximation $\{\varphi_k\}$ of φ such that*

$$\mathcal{I}\left(\left(1 + \frac{2}{k}\right) \varphi_k \right) = \mathcal{I}_+(\varphi) \quad \text{for } k \gg 1. \tag{5.13}$$

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Proof. By [DPS01, Theorem 2.2.1], there exists a quasi-equisingular approximation $\{\varphi_k\}$ of φ. The technique of comparing integrals discussed in [DPS01] implies that we can choose a subsequence $\{\varphi_{f(k)}\} \subset \{\varphi_k\}$ such that

$$\mathcal{I}\left(\left(1+\frac{2}{k}\right)\varphi_{f(k)}\right) \subset \mathcal{I}_+(\varphi). \tag{5.14}$$

In fact, if X is projective, we can take $s = 1+\epsilon$ and $f(k) \gg k$ in Lemma 3.2. By Lemma 3.2, we get (5.14). If X is an arbitrary compact Kähler manifold, we can get the inclusion (5.14) on any Stein open set of X. Using standard gluing techniques, we also obtain the global inclusion (5.14) (see [DPS01, Theorem 2.2.1] for details).

For the opposite inclusion, we observe that $\varphi_{f(k)}$ is less singular than φ, and the definition of $\mathcal{I}_+(\varphi)$ implies that

$$\mathcal{I}\left(\left(1+\frac{2}{k}\right)\varphi_{f(k)}\right) \supset \mathcal{I}_+(\varphi) \quad \text{for } k \gg 1.$$

Thus the lemma is proved. □

The following lemma will be important in the proof of our Kawamata–Viehweg–Nadel vanishing theorem. The main substance of the lemma is that to prove the convergence in higher-degree cohomology with multiplier ideal sheaves, we just need to check the convergence for some smooth metric. Although this technique is well known (see, for example, [DPS01, Part 2.4.2]), we will present the proof here for the reader's convenience.

First we fix some notation. Let (L,φ) be a pseudo-effective line bundle over a compact Kähler manifold X and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be a Stein covering of X. Set $U_{\alpha_0\alpha_1\cdots\alpha_q} := U_{\alpha_0}\cap\cdots\cap U_{\alpha_q}$. Let $\check{C}^q(\mathcal{U}, K_X\otimes L\otimes\mathcal{I}_+(\varphi))$ be the Čech q-cochain associated to $K_X\otimes L\otimes\mathcal{I}_+(\varphi)$. For an element $c\in\check{C}^q(\mathcal{U}, K_X\otimes L\otimes\mathcal{I}_+(\varphi))$, we denote its component on $U_{\alpha_0\alpha_1\cdots\alpha_q}$ by $c_{\alpha_0\alpha_1\cdots\alpha_q}$. Let

$$\delta_p : \check{C}^{p-1}(\mathcal{U},\mathcal{I}_+(\varphi)) \to \check{C}^p(\mathcal{U},\mathcal{I}_+(\varphi)) \tag{5.15}$$

be the Čech operator, and let $\check{Z}^p(\mathcal{U},\mathcal{I}_+(\varphi)) = \operatorname{Ker}\delta_{p+1}$.

Lemma 5.8. *Let L be a line bundle over a compact Kähler manifold X and let φ be a singular metric on L. Let $\{U_\alpha\}_{\alpha\in I}$ be a Stein covering of X. Let u be an element in $\check{H}^p(X,\mathcal{O}(K_X+L)\otimes\mathcal{I}_+(\varphi))$. If there exists a sequence $\{v_k\}_{k=1}^\infty\subset\check{C}^p(\mathcal{U},K_X\otimes L\otimes\mathcal{I}_+(\varphi))$ in the same cohomology class as u satisfying the L^2 convergence condition*

$$\lim_{k\to\infty}\int_{U_{\alpha_0\cdots\alpha_p}} |v_{k,\alpha_0\cdots\alpha_p}|^2 \to 0, \tag{5.16}$$

where the L^2 norm $|v|^2$ in (5.16) is taken for some fixed smooth metric on L, then $u=0$ in $\check{H}^p(X,\mathcal{O}(K_X+L)\otimes\mathcal{I}_+(\varphi))$.

Proof. On the p-cochain space $\check{C}^p(\mathcal{U},\mathcal{I}_+(\varphi))$, we first define a family of natural semi-norms: for $f\in\check{C}^p(\mathcal{U},\mathcal{I}_+(\varphi))$, define a family of semi-norms by

$$\sum_{\alpha_0\cdots\alpha_p}\int_{V_{\alpha_0\cdots\alpha_p}} |f|^2\omega^n \quad \text{for any open set } V_{\alpha_0\cdots\alpha_p}\Subset U_{\alpha_0\cdots\alpha_p}. \tag{5.17}$$

Claim. *$\check{C}^p(\mathcal{U},\mathcal{I}_+(\varphi))$ is a Fréchet space with respect to the family of semi-norms (5.17).*

Proof of the claim. We need to prove that if $f_i \in \mathcal{I}_+(\varphi)$ and $f_i \to f_0$ with respect to the semi-norms (5.17), then $f_0 \in \mathcal{I}_+(\varphi)$. First of all, by (5.17), f_0 is holomorphic. By Lemma 5.7, we can choose a quasi-psh function ψ with analytic singularities such that

$$\mathcal{I}(\psi) = \mathcal{I}_+(\varphi).$$

Let $\pi : \widehat{X} \to X$ be a log resolution of ψ. Then the current $E = \lfloor dd^c(\psi \circ \pi)\rfloor$ has normal crossing singularities. Since $f_i \in \mathcal{I}_+(\varphi) = \mathcal{I}(\psi)$, we have

$$(f_i \circ \pi) \cdot J \in \mathcal{O}(-E), \tag{5.18}$$

where J is the Jacobian of π. Since $f_i \circ \pi \rightharpoonup f_0 \circ \pi$ in the sense of weak convergence and E has normal crossing singularities, (5.18) implies that

$$(f_0 \circ \pi) \cdot J \in \mathcal{O}(-E).$$

Therefore $f_0 \in \mathcal{I}_+(\varphi)$ and the claim is proved. □

As a consequence of the claim, the Čech operator (5.15) is continuous and its kernel $\check{Z}^{p-1}(\mathcal{U}, \mathcal{I}_+(\varphi))$ is also a Fréchet space. Therefore we have a continuous boundary morphism between Fréchet spaces,

$$\delta_p : \check{C}^{p-1}(\mathcal{U}, \mathcal{I}_+(\varphi)) \to \check{Z}^p(\mathcal{U}, \mathcal{I}_+(\varphi)). \tag{5.19}$$

Since the cokernel of δ_p in (5.19) is $\check{H}^p(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}_+(\varphi))$, which is of finite dimension, by the open mapping theorem from functional analysis, the image of δ_p in (5.19) is closed. Therefore the quotient morphism

$$\mathrm{pr} : \check{Z}^p(\mathcal{U}, \mathcal{I}_+(\varphi)) \to \frac{\check{Z}^p(\mathcal{U}, \mathcal{I}_+(\varphi))}{\mathrm{Im}(\delta_p)} = \check{H}^p(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}_+(\varphi)) \tag{5.20}$$

is continuous. Thanks to the claim, the condition (5.16) implies that $\{v_k\}_{k=1}^{\infty}$ tends to 0 in the Fréchet space $\check{Z}^p(\mathcal{U}, \mathcal{I}_+(\varphi))$. By the continuity of (5.20), we have

$$\lim_{k \to +\infty} \mathrm{pr}(v_k) = 0 \in \check{H}^p(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}_+(\varphi)). \tag{5.21}$$

Since, by construction, the $\mathrm{pr}(v_k)$ are in the same class as $[u]$, we conclude by (5.21) that $u = 0$ in $\check{H}^p(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}_+(\varphi))$. □

Remark 8. Recently, Matsumura proved in [Mat13] that the above lemma is also true for the space $\check{H}^p(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}(\varphi))$.

We proceed to construct the new singular metrics mentioned in the paragraphs before Lemma 5.7.

Lemma 5.9. *Let (L, φ) be a pseudo-effective line bundle over a compact Kähler manifold (X, ω) of dimension n, and let $p \geqslant n - \mathrm{nd}(L, \varphi) + 1$. Then there exists a sequence of metrics $\{\widehat{\varphi}_k\}_{k=1}^{\infty}$ with analytic singularities on L which satisfy the following properties.*

(i) *$\mathcal{I}(\widehat{\varphi}_k) = \mathcal{I}_+(\varphi)$ for all k.*

(ii) *Let $\lambda_{1,k} \leqslant \lambda_{2,k} \leqslant \cdots \leqslant \lambda_{n,k}$ be the eigenvalues of $(i/2\pi)\Theta_{\widehat{\varphi}_k}(L)$ with respect to the base metric ω. Then there exist two sequences $\tau_k \to 0$ and $\epsilon_k \to 0$ such that*

$$\epsilon_k \gg \tau_k + \frac{1}{k} \quad \text{and} \quad \lambda_{1,k}(x) \geqslant -\epsilon_k - \frac{C}{k} - \tau_k$$

for all $x \in X$ and k, where C is a constant independent of k.

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

(iii) *We can choose $\beta > 0$ and $0 < \alpha < 1$ independent of k such that for every k, there exists an open subset U_k of X satisfying*

$$\mathrm{vol}(U_k) \leqslant \epsilon_k^{\beta} \quad \textit{and} \quad \lambda_p + 2\epsilon_k \geqslant \epsilon_k^{\alpha} \textit{ on } X \backslash U_k.$$

Proof. Recall that we first fix a smooth metric h_0 on L. Taking φ as a weight, we just mean that the hermitian metric on L is $h_0 \cdot e^{-\varphi}$.

By definition, there exists $s_1 > 0$ such that

$$\mathcal{I}_+(\varphi) = \mathcal{I}((1+s_1)\varphi). \tag{5.22}$$

Let $\{\varphi_k\}$ be the quasi-equisingular approximation of φ in Lemma 5.7. Then there is a positive sequence $\tau_k \to 0$ such that

$$\frac{i}{2\pi}\Theta_{\varphi_k}(L) \geqslant -\tau_k\omega \quad \text{and} \quad \mathcal{I}\left(\left(1+\frac{2}{k}\right)\varphi_k\right) = \mathcal{I}_+(\varphi) \tag{5.23}$$

for every k. We can choose a positive sequence $\epsilon_k \to 0$ such that $\epsilon_k \gg \tau_k + 1/k$.

Fix a positive sequence $\{\delta_k\}$ tending to 0. We begin to construct new metrics by solving a Monge–Ampère equation. Let $\pi : X_k \to X$ be a log resolution of φ_k. Then $dd^c(\varphi_k \circ \pi)$ is of the form $[E_k] + C^{\infty}$ where $[E_k]$ is a normal crossing $\mathbb{Q}$-divisor. Let $Z_k = \pi_*(E_k)$. By [Bou02], there exists a smooth metric h_k on $[E_k]$ such that for all $\delta > 0$ small enough,

$$\pi^*(\omega) + \delta\frac{i}{2\pi}\Theta_{h_k}(-E_k)$$

is a Kähler form on X_k. Then we can solve a Monge–Ampère equation on X_k,

$$\begin{aligned}\left(\left(\frac{i}{2\pi}\pi^*\Theta_{\varphi_k}(L)\right)_{\mathrm{ac}} + \epsilon_k\pi^*\omega + \delta_k\frac{i}{2\pi}\Theta_{h_k}(-E_k) + dd^c\psi_{k,\epsilon,\delta_k}\right)^n \\ = C(k,\delta,\epsilon)\cdot\epsilon_k^{n-d}\left(\omega + \delta_k\frac{i}{2\pi}\Theta_{h_k}(-E_k)\right)^n,\end{aligned} \tag{5.24}$$

with the normalization condition

$$\sup_{z\in X_k}(\varphi_k\circ\pi + \psi_{k,\epsilon,\delta_k} + \delta_k\ln|E_k|_{h_k})(z) = 0 \tag{5.25}$$

where $d = \mathrm{nd}(L,\varphi)$. Thanks to the definition of numerical dimension, there exists a uniform constant $C > 0$ such that $C(k,\delta,\epsilon) \geqslant C$. By observing, moreover, that

$$i\partial\overline{\partial}\ln|E_k|_{h_k} = [E_k] + \frac{i}{2\pi}\Theta_{h_k}(-E_k),$$

(5.24) implies that

$$\frac{i}{2\pi}\Theta_{\varphi_k+\psi_{k,\epsilon,\delta_k}+\delta_k\ln|E_k|_{h_k}}(\pi^*L) \geqslant -\epsilon_k\omega. \tag{5.26}$$

Set

$$\widehat{\varphi}_k := \left(1+\frac{2}{k}-s\right)\varphi_k\circ\pi + s(\varphi_k\circ\pi + \psi_{k,\epsilon,\delta} + \delta\ln|E_k|_{h_k}), \tag{5.27}$$

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

where $0 < s \ll s_1$ will be made precise[6] in Lemma 5.10. Now we have a new metric $\widehat{\varphi}_k$ on $(X_k, \pi^* L)$ (i.e. $h_0 e^{-\widehat{\varphi}_k}$ as the actual hermitian metric on $\pi^* L$). We prove that $\widehat{\varphi}_k$ induces a natural metric on (X, L). In fact, by (5.27) we have

$$\frac{i}{2\pi}\Theta_{\widehat{\varphi}_k}(\pi^* L) = (1-s)\frac{i}{2\pi}\Theta_{\varphi_k}(\pi^* L) + s\frac{i}{2\pi}\Theta_{\varphi_k+\psi_{k,\epsilon,\delta_k}+\delta_k \ln|E_k|_{h_k}}(\pi^* L) + \frac{2}{k}dd^c\varphi_k. \quad (5.28)$$

Inequality (5.26) gives the estimate for the second term in the right-hand side of (5.28). For the last term in the right-hand side of (5.28), we observe that φ_k is a function on X satisfying

$$\frac{i}{2\pi}\Theta_{\varphi_k}(L) = \frac{i}{2\pi}\Theta_{h_0}(L) + dd^c\varphi_k \geqslant -c\omega,$$

and thus

$$dd^c\varphi_k \geqslant -C\omega$$

for some uniform constant C, and

$$\frac{i}{2\pi}\Theta_{\widehat{\varphi}_k}(\pi^* L) \geqslant -\epsilon_k\omega - \tau_k\omega - \frac{C}{k}\omega. \quad (5.29)$$

Thus $\widehat{\varphi}_k$ induces a quasi-psh function on X by extending it from $X \backslash Z_k$ to the whole X. This is the metric that we wanted to construct. We denote it also by $\widehat{\varphi}_k$ for simplicity. We will prove properties (i) to (iii) of Lemmas 5.10 and 5.11. □

Lemma 5.10. *If we take s in (5.27) small enough with respect to s_1 in (5.22) of Lemma 5.9, then*

$$\int_U |f|^2 e^{-2\widehat{\varphi}_k} \leqslant C_{|f|_{L^\infty}} \left(\int_U |f|^2 e^{-2(1+s_1)\varphi}\right)^{1/(1+s_1)} \quad (5.30)$$

for all U in X and $k \gg 1$, where $C_{|f|_{L^\infty}}$ is a constant depending only on $|f|_{L^\infty}$ (in particular, it is independent of the open subset U and of k). As a consequence, we have

$$\mathcal{I}(\widehat{\varphi}_k) = \mathcal{I}_+(\varphi) \quad \textit{for every } k. \quad (5.31)$$

Proof. Thanks to (5.26), $\varphi_k + \psi_{k,\epsilon,\delta_k} + \delta \ln|E_k|_{h_k}$ induces a quasi-psh function on X. We denote it also by $\varphi_k + \psi_{k,\epsilon,\delta_k} + \delta \ln|E_k|_{h_k}$ for simplicity. Then (5.25) and (5.26) in Lemma 5.9 imply the existence of a constant $a > 0$ such that

$$\int_X e^{-2a(\varphi_k+\psi_{k,\epsilon,\delta}+\delta_k \ln|E_k|_{h_k})}$$

is uniformly bounded for all k.

By Hölder's inequality and the construction (5.27), we have

$$\int_U |f|^2 e^{-2\widehat{\varphi}_k} \leqslant \left(\int_U |f|^2 e^{-2(1+s_1)\varphi_k}\right)^{1/(1+s_1)} \left(\int_U |f|^2 e^{-(2s(1+s_1)/s_1)(\varphi_k+\psi_{k,\epsilon,\delta_k}+\delta_k \ln|E_k|_{h_k})}\right)^{s_1/(1+s_1)} \quad (5.32)$$

for $k \gg 1$, where U is any open subset of X. If we take an $s > 0$ satisfying $s(1+s_1)/s_1 \leqslant a$, then the uniform boundedness of $\int_X e^{-2a(\varphi_k+\psi_{k,\epsilon,\delta_k}+\delta_k \ln|E_k|_{h_k})}$ implies that

$$\int_U |f|^2 e^{-(2s(1+s_1)/s_1)(\varphi_k+\psi_{k,\epsilon,\delta_k}+\delta_k \ln|E_k|_{h_k})} \leqslant C \cdot |f|_{L^\infty} \quad (5.33)$$

[6] Note that s_1 is the constant in (5.22).

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

for any $U \subset X$ and $k \gg 1$. Combining (5.32) with (5.33), we obtain that

$$\int_U |f|^2 e^{-2\widehat{\varphi}_k} \leqslant C_{|f|_{L^\infty}} \bigg(\int_U |f|^2 e^{-2(1+s_1)\varphi_k} \bigg)^{1/(1+s_1)} \leqslant C_{|f|_{L^\infty}} \bigg(\int_U |f|^2 e^{-2(1+s_1)\varphi} \bigg)^{1/(1+s_1)} \quad (5.34)$$

for some constant $C_{|f|_{L^\infty}}$ independent of the open subset U and of $k \gg 1$.

It remains to prove (5.31). The inclusion $\mathcal{I}(\widehat{\varphi}_k) \supset \mathcal{I}_+(\varphi)$ comes directly from (5.34). By construction, $\widehat{\varphi}_k$ is more singular than $(1+2/k)\varphi_k$. Then (5.23) implies that $\mathcal{I}(\widehat{\varphi}_k) \subset \mathcal{I}_+(\varphi)$, and so the equality (5.31) is proved. □

The following lemma was essentially proved in [Mou95].

Lemma 5.11. *In the situation of Lemma 5.9, the new metrics $\{\widehat{\varphi}_k\}_{k=1}^{\infty}$ satisfy properties (ii) and (iii) of Lemma 5.9.*

Proof. Let $\lambda_1(z) \leqslant \lambda_2(z) \leqslant \cdots \leqslant \lambda_n(z)$ be the eigenvalues of $i\Theta_{\widehat{\varphi}_k}(L)$ with respect to the base metric ω. Note that λ_i is equal to $\lambda_{i,k}$ in Lemma 5.9. Since the proof here is for a fixed k, this simplification should not lead to misunderstanding. By (5.29), we have

$$\lambda_i(z) \geqslant -\epsilon_k - \frac{C}{k} - \tau_k,$$

so property (ii) of Lemma 5.9 is proved.

Set $\widehat{\lambda}_i := \lambda_i + 2\epsilon_k$. Since s is a fixed positive constant, the Monge–Ampère equation (5.24) implies that

$$\prod_{i=1}^{n} \widehat{\lambda}_i(z) \geqslant C(s)\epsilon_k^{n-d}, \quad (5.35)$$

where $C(s) > 0$ does not depend on k. Since $p > n-d$, we can take α such that $0 < \alpha < 1$ and $n - d < \alpha p$. Set $U_k := \{z \in X \mid \widehat{\lambda}_p(z) < \epsilon_k^{\alpha}\}$.

We now check that U_k satisfies property (iii) of Lemma 5.9. Since $\epsilon_k \gg \tau_k + 1/k$, we have $\widehat{\lambda}_i(z) = \lambda_i(z) + 2\epsilon_k \geqslant 0$ for any z and i. Thus the cohomological condition

$$\int_X (\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_n)\omega^n \leqslant M$$

implies that

$$\int_{U_k} (\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_n)\omega^n \leqslant M. \quad (5.36)$$

Observing that (5.35) and the definition of U_k imply that

$$\prod_{p+1 \leqslant i \leqslant n} \widehat{\lambda}_i(z) \geqslant C(s)\frac{\epsilon_k^{n-d}}{\epsilon_k^{\alpha p}} \quad \text{for } z \in U_k,$$

we have

$$\sum_{p+1 \leqslant i \leqslant n} \widehat{\lambda}_i(z) \geqslant C\bigg(\frac{\epsilon_k^{n-d}}{\epsilon_k^{\alpha p}}\bigg)^{1/(n-p)} \quad \text{for } z \in U_k \quad (5.37)$$

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

by the inequality between arithmetic and geometric means. Applying (5.37) to (5.36), we have

$$\int_{U_k}\left(\frac{\epsilon_k^{n-d}}{\epsilon_k^{\alpha p}}\right)^{1/(n-p)}\omega^n \leqslant M'. \tag{5.38}$$

Since $n-d<\alpha p$, (5.38) implies that

$$\mathrm{vol}(U_k)\leqslant \epsilon_k^{\beta}$$

for some $\beta>0$. Thus property (iii) of Lemma 5.9 is proved. □

We now reach the final conclusion.

Theorem 5.12 (Theorem 1.4). *Let (L,φ) be a pseudo-effective line bundle on a compact Kähler manifold (X,ω). Then*

$$H^p(X,\mathcal{O}(K_X+L)\otimes\mathcal{I}_+(\varphi))=0 \quad \textit{for } p\geqslant n-\mathrm{nd}(L,\varphi)+1.$$

Remark 9. One reason to use $\mathcal{I}_+(\varphi)$ instead of $\mathcal{I}(\varphi)$ is that it does not seem to be easy to prove that

$$H^p(X,\mathcal{O}(K_X+L)\otimes\mathcal{I}(\varphi))=0 \quad \text{for } p\geqslant n-\mathrm{nd}(L,\varphi)+1,$$

even when X is projective. (However, see [Mat13] for some recent progress.)

Proof. We prove the theorem in two steps.

Step 1. L^2 estimates.

Let $\{\widehat{\varphi}_k\}_{k=1}^{\infty}$ be the metrics constructed in Lemma 5.9, and let $[u]$ be an element in $H^p(X, K_X\otimes L\otimes\mathcal{I}_+(\varphi))$. Let f be a smooth (n,p)-form representing $[u]$. Then

$$\int_X |f|^2 e^{-2(1+s_1)\varphi}<+\infty,$$

where s_1 is the constant in (5.22) of Lemma 5.9. By Lemma 5.10, we have

$$\int_U |f|^2e^{-2\widehat{\varphi}_k}\leqslant C\left(\int_U |f|^2 e^{-2(1+s_1)\varphi}\right)^{1/(1+s_1)} \quad \text{for every } k\gg 1 \tag{5.39}$$

for any open subset U of X, where C is a constant independent of U and k (but which certainly depends on $|f|_{L^\infty}$). We now use the L^2 method from [DP03] to get a key estimate, namely that f can be written as

$$f=\overline{\partial}u_k+v_k \tag{5.40}$$

with the bound

$$\int_X |u_k|^2e^{-2\widehat{\varphi}_k}+\frac{1}{2p\epsilon_k}\int_X |v_k|^2e^{-2\widehat{\varphi}_k}\leqslant \int_X \frac{1}{\widehat{\lambda}_{1,k}+\widehat{\lambda}_{2,k}+\cdots+\widehat{\lambda}_{p,k}}|f|^2e^{-2\widehat{\varphi}_k}, \tag{5.41}$$

where $\widehat{\lambda}_{i,k}=\lambda_{i,k}+2\epsilon_k$. The estimate (5.41) comes from the Bochner inequality

$$\|\overline{\partial}u\|^2_{\widehat{\varphi}_k}+\|\overline{\partial}^*u\|^2_{\widehat{\varphi}_k}\geqslant \int_{X-Z_k}(\widehat{\lambda}_{1,k}+\widehat{\lambda}_{2,k}+\cdots+\widehat{\lambda}_{p,k}-C\epsilon_k)|u|^2_{\widehat{\varphi}_k}\,dV,$$

where Z_k is the singular locus of φ_k in X (see [DP03, Theorem 3.3] or Proposition A.1 in the Appendix for details).

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Using (5.41), we claim that

$$\lim_{k\to\infty}\int_X |v_k|^2 e^{-2\widehat{\varphi}_k} \to 0. \tag{5.42}$$

To prove the claim, observe that properties (ii) and (iii) of Lemma 5.9 and (5.41) imply that

$$\int_X |u_k|^2 e^{-2\widehat{\varphi}_k} + \frac{1}{2p\epsilon_k}\int_X |v_k|^2 e^{-2\widehat{\varphi}_k} \leqslant \int_X \frac{C_1}{\epsilon_k^\alpha}|f|^2 e^{-2\widehat{\varphi}_k} + \int_{U_k}\frac{1}{C_2\epsilon_k}|f|^2 e^{-2\widehat{\varphi}_k}.$$

Then

$$\int_X |v_k|^2 e^{-2\widehat{\varphi}_k} \leqslant C_3\epsilon_k^{1-\alpha}\int_X |f|^2 e^{-2\widehat{\varphi}_k} + C_4\int_{U_k}|f|^2 e^{-2\widehat{\varphi}_k}. \tag{5.43}$$

Since $\operatorname{vol}(U_k)\to 0$ by property (iii) of Lemma 5.9, (5.39) implies that the second term of the right-hand side of (5.43) tends to 0. Since $0<\alpha<1$ and $\epsilon_k\to 0$ as $k\to\infty$, (5.39) therefore implies that the first term of the right-hand side of (5.43) also tends to 0. Thus (5.42) is proved.

Step 2. Final stage.

We use Lemma 5.8 to obtain the final result. Let $\mathcal{U}=\{U_\alpha\}_{\alpha\in I}$ be a Stein covering of X. Thanks to (5.42), we get a p-cocycle representing v_k by solving $\overline{\partial}$-equations, i.e. v_k can be written as

$$v_k=\{v_{k,\alpha_0\cdots\alpha_p}\}\in \check{C}^p(\mathcal{U},\mathcal{O}(K_X+L)\otimes\mathcal{I}(\widehat{\varphi}_k)),$$

where the components satisfy the L^2 conditions

$$\int_{U_{\alpha_0\cdots\alpha_p}} |v_{k,\alpha_0\cdots\alpha_p}|^2 e^{-2\widehat{\varphi}_k} \leqslant C\int_X |v_k|^2 e^{-2\widehat{\varphi}_k}, \tag{5.44}$$

with C not depending on k. Inequality (5.44) and property (i) of Lemma 5.9 imply that $\{v_k\}$ is in $\check{C}^p(\mathcal{U},\mathcal{O}(K_X+L)\otimes\mathcal{I}_+(\varphi))$ for every k.

Since $\widehat{\varphi}_k\leqslant 0$ by construction, (5.42) and (5.44) imply that

$$\lim_{k\to\infty}\int_{U_{i_0\cdots i_p}} |v_{k,i_0\cdots i_p}|^2 = 0. \tag{5.45}$$

By (5.40), $\{v_k\}_{k=1}^\infty$ are in the same cohomology class as u in $H^p(X,\mathcal{O}(K_X+L)\otimes\mathcal{I}_+(\varphi))$. By Lemma 5.8, (5.45) implies that $[u]=0$. So the theorem is proved. □

Acknowledgements

I would like to thank Professor J.-P. Demailly for numerous ideas and suggestions for this article, and also for his availability and patience. I also thank the referees for their valuable suggestions.

Appendix. An L^2 estimate

For the reader's convenience, we give the proof of estimate (5.41) in Theorem 5.12. For the most part, the proof is just extracted from [DP03].

Proposition A.1. *Let (X,ω) be a compact Kähler manifold and let $(L,h_0e^{-\varphi})$ be a line bundle on X, where h_0 is a smooth metric on L and the quasi-psh function φ has analytic singularities and is smooth outside a subvariety Z. Assume that*

$$\frac{i}{2\pi}\Theta_\varphi(L)\geqslant -\epsilon\omega$$

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

on $X\backslash Z$ and that f is a smooth L-valued (n,p)-form satisfying

$$\int_X |f|^2 e^{-2\varphi}\, dV < \infty. \tag{A1}$$

Let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ be the eigenvalues of $(i/2\pi)\Theta_\varphi(L)$ and let $\widehat{\lambda}_i = \lambda_i + 2\epsilon \geqslant \epsilon$. Then there exist u and v such that $f = \overline{\partial} u + v$ and the following estimate is satisfied:

$$\int_X |u|^2 e^{-2\varphi}\, dV + \frac{1}{2p\epsilon}\int_X |v|^2 e^{-2\varphi}\, dV \leqslant \int_X \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p} |f|^2 e^{-2\varphi}\, dV.$$

Proof. Let ω_1 be a complete Kähler metric on $X\backslash Z$ and let $\omega_\delta = \omega + \delta\omega_1$ for some $\delta > 0$. We now do the standard L^2 estimate on $(X\backslash Z, \omega_\delta, L, \varphi)$.

If s is an L-valued (n,p)-form in $C_c^\infty(X\backslash Z)$, then the Bochner inequality implies that

$$\|\overline{\partial} s\|_\delta^2 + \|\overline{\partial}^* s\|_\delta^2 \geqslant \int_{X\backslash Z} (\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p - 2p\epsilon)|s|^2 e^{-2\varphi}\omega_\delta^n, \tag{A2}$$

where $\|s\|_\delta^2 = \int_X |s|^2 e^{-2\varphi}\omega_\delta^n$. Note that there is an abuse of notation here: we calculate the norm $|u|^2$ by the metric (or volume form) written in the equations. For example, if the volume form is ω_δ^n, then we calculate the norm of u by means of the metrics ω_δ and h_0.

Since f is an (n,p)-form, (A1) implies that

$$f \in L^2(X\backslash Z, L, \varphi, \omega_\delta) \quad \text{for } \delta > 0.$$

We write every form s in the domain of the L^2 extension of $\overline{\partial}^*$ as $s = s_1 + s_2$ with

$$s_1 \in \operatorname{Ker} \overline{\partial} \quad \text{and} \quad s_2 \in (\operatorname{Ker} \overline{\partial})^\perp \subset \operatorname{Ker} \overline{\partial}^*.$$

Since $f \in \operatorname{Ker} \overline{\partial}$, by (A2) we obtain

$$\begin{aligned}
|\langle f, s\rangle|_{\varphi,\delta}^2 &= |\langle f, s_1\rangle|_{\varphi,\delta}^2 \\
&\leqslant \int_{X\backslash Z} \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p}|f|^2 e^{-2\varphi}\, dV_\delta \int_{X\backslash Z} (\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p)|s_1|^2 e^{-2\varphi}\, dV_\delta \\
&\leqslant \int_{X\backslash Z} \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p}|f|^2 e^{-2\varphi}\, dV_\delta (\|\overline{\partial}^* s_1\|_\delta^2 + 2p\epsilon\|\overline{\partial} s_1\|_\delta^2) \\
&\leqslant \int_{X\backslash Z} \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p}|f|^2 e^{-2\varphi}\, dV_\delta (\|\overline{\partial}^* s\|_\delta^2 + 2p\epsilon\|\overline{\partial} s\|_\delta^2).
\end{aligned}$$

By the Hahn–Banach theorem, we can find v_δ and u_δ such that

$$\langle f, s\rangle_\delta = \langle u_\delta, \overline{\partial}^* s\rangle_\delta + \langle v_\delta, s\rangle_\delta \quad \text{for every } s$$

and which satisfy the estimate

$$\|u_\delta\|_\delta^2 + \frac{1}{2p\epsilon}\|v_\delta\|_\delta^2 \leqslant C\int_X \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p}|f|^2 e^{-2\varphi}\omega_\delta^n.$$

Therefore

$$f = \overline{\partial} u_\delta + v_\delta. \tag{A3}$$

Since the norm $\|\cdot\|_\delta$ of (n,p)-forms is increasing as δ decreases to 0, we obtain limits

$$u=\lim_{\delta\to 0}u_\delta \quad\text{and}\quad v=\lim_{\delta\to 0}v_\delta \tag{A4}$$

satisfying

$$\|u\|_\delta^2+\frac{1}{2p\epsilon}\|v\|_\delta^2\leqslant C\int_X\frac{1}{\widehat{\lambda}_1+\widehat{\lambda}_2+\cdots+\widehat{\lambda}_p}|f|^2e^{-2\varphi}\omega_\delta^n\leqslant C\int_X\frac{1}{\widehat{\lambda}_1+\widehat{\lambda}_2+\cdots+\widehat{\lambda}_p}|f|^2e^{-2\varphi}\omega^n \tag{A5}$$

for every $\delta>0$. Formulas (A3) and (A4) imply that $f=\overline{\partial}u+v$. Letting $\delta\to 0$ in (A5), we obtain the estimate in the proposition. □

References

Ber13 B. Berndtsson, *The openness conjecture for plurisubharmonic functions*, Preprint (2013), arXiv:1305.5781.

Bon98 L. Bonavero, *Inégalités de morse holomorphes singulières*, J. Geom. Anal. **8** (1998), 409–425; MR 1707735 (2000k:32020).

Bou02 S. Boucksom, *Cones positifs des variétés complexes compactes*, PhD thesis, Université Joseph Fourier, Grenoble (2002).

Dem93 J.-P. Demailly, *A numerical criterion for very ample line bundles*, J. Differential Geom. **37** (1993), 323–374; MR 1205448 (94d:14007).

Dem12 J.-P. Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1 (International Press, Somerville, MA, 2012); MR 2978333.

DEL00 J.-P. Demailly, L. Ein and R. Lazarsfeld, *A subadditivity property of multiplier ideals*, Michigan Math. J. **48** (2000), 137–156; dedicated to William Fulton on the occasion of his 60th birthday; MR 1786484 (2002a:14016).

DP04 J.-P. Demailly and M. Păun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. (2) **159** (2004), 1247–1274; MR 2113021 (2005i:32020).

DP03 J.-P. Demailly and T. Peternell, *A Kawamata-Viehweg vanishing theorem on compact Kähler manifolds*, J. Differential Geom. **63** (2003), 231–277; MR 2015548 (2004m:32040).

DPS94 J.-P. Demailly, T. Peternell and M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom. **3** (1994), 295–345; MR 1257325 (95f:32037).

DPS01 J.-P. Demailly, T. Peternell and M. Schneider, *Pseudo-effective line bundles on compact Kähler manifolds*, Internat. J. Math. **12** (2001), 689–741; MR 1875649 (2003a:32032).

Eno93 I. Enoki, *Kawamata-Viehweg vanishing theorem for compact Kähler manifolds*, in *Einstein Metrics and Yang-Mills Connections (Sanda, 1990)*, Lecture Notes in Pure and Applied Mathematics, vol. 145 (Marcel Dekker, New York, 1993), 59–68; MR 1215279 (94f:32063).

FJ05 C. Favre and M. Jonsson, *Valuations and multiplier ideals*, J. Amer. Math. Soc. **18** (2005), 655–684; MR 2138140 (2007b:14004).

Kür06 A. Küronya, *Asymptotic cohomological functions on projective varieties*, Amer. J. Math. **128** (2006), 1475–1519; MR 2275909 (2007k:14009).

Mat13 S.-i. Matsumura, *An injectivity theorem with multiplier ideal sheaves of singular metrics with transcendental singularities*, Preprint (2013), arXiv:1308.2033.

Mou95 C. Mourougane, *Versions kählériennes du théorème d'annulation de Bogomolov-Sommese*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), 1459–1462; MR 1366101 (96k:32066).

Nad90 A. Nadel, *Multiplier ideal sheaves and Kähler–Einstein metrics of positive scalar curvature*, Ann. of Math. (2) **132** (1990), 549–596; MR 1078269 (92d:32038).

https://doi.org/10.1112/S0010437X14007398 Published online by Cambridge University Press

Tra11 S. Trapani, *Divisorial Zariski decomposition and algebraic Morse inequalities*, Proc. Amer. Math. Soc. **139** (2011), 177–183; MR 2729081 (2012a:32019).

Tsu07 H. Tsuji, *Extension of log pluricanonical forms from subvarieties*, Preprint (2007), arXiv:0709.2710.

Junyan Cao junyan@kias.re.kr, jycao136@yahoo.com
KIAS, 85 Hoegiro, Dongdaemun-gu, Seoul 130-722, Republic of Korea