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COCHAIN SEQUENCES AND THE QUILLEN CATEGORY OF A COCLASS FAMILY

BETTINA EICK and DAVID J. GREEN✉

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Abstract

We introduce the concept of infinite cochain sequences and initiate a theory of homological algebra for them. We show how these sequences simplify and improve the construction of infinite coclass families (as introduced by Eick and Leedham-Green) and also how they can be applied to prove that almost all groups in such a family have equivalent Quillen categories. We also include some examples of infinite families of p-groups from different coclass families that have equivalent Quillen categories.

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1. Introduction

Coclass theory was initiated by Leedham-Green and Newman [14]. The fundamental aim of this theory is to classify and investigate finite p-groups using the coclass as primary invariant. The infinite coclass families of finite p-groups of fixed coclass, as defined by Eick and Leedham-Green [10], are considered to be a step towards these aims. Their definition is based on a splitting theorem for a certain type of second cohomology group.

Various interesting properties of the infinite coclass families have been determined. For example, the automorphism groups and the Schur multiplicators of the groups in one family can be described, simultaneously, for all groups in the family (see [5–7]). It is conjectured that almost all groups in an infinite coclass family have isomorphic mod-p cohomology rings. This conjecture is still open, but it is underlined by computational evidence obtained by Eick and King [9] and by our earlier result [8], which says that the Quillen categories of almost all groups in an infinite coclass family are equivalent. The proof of the latter theorem uses a splitting theorem for cohomology groups.

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In this paper, we derive a generalization of the splitting theorems obtained and used in [10] and [8], and describe the splitting at the cocycle level. Based on this, we introduce the concept of an infinite cochain sequence and take the first steps towards the development of a theory of homological algebra for them.

We then show that the infinite coclass families of [10] can be defined using the infinite cochain sequence. This way of defining the families is more explicit than the definition in [10], since it is based on cocycles rather than just cohomology classes. This difference is significant: for example, it is useful for the investigation of the Quillen categories of the groups in a coclass family. Further, we use the infinite cochain sequences to give a new, more conceptual proof of our main theorem in [8] on the Quillen categories of the groups in an infinite coclass family.

In the final part of this paper, we give some examples of groups from different coclass families with equivalent Quillen categories. Let $q = p^\ell$ for a prime p, let $\mathbb{Z}_p$ denote the p-adic integers and consider the irreducible action of C_q on $T = \mathbb{Z}_p^{(p-1)p^{\ell-1}}$ (this is unique up to equivalence). Then $G_q = T \rtimes C_q$ is an infinite pro-p-group of coclass ℓ. For $\ell = 1$, it is the unique infinite pro-p-group of coclass one (or of maximal class) and, for $\ell > 1$, it is an interesting example for an infinite pro-p-group of coclass ℓ.

The main line groups associated with an infinite pro-p-group G of coclass r are the infinite number of lower central series quotients $G/\gamma_i(G)$ that have coclass r; this infinite sequence is not necessarily a coclass family itself, but it consists of a finite number of different coclass families. The skeleton groups associated with an infinite pro-p-group G of coclass r form a significantly larger family of groups containing the main line groups and they play an important role in coclass theory; we refer to [13, Section 8.4] or [11, Section 3], for details.

Theorem 1.1.

(1) *For an arbitrary, fixed prime p, the Quillen categories of almost all main line groups associated with G_p are pairwise equivalent.*

(2) *The Quillen categories of almost all skeleton groups associated with G_9 are pairwise equivalent.*

Proof. (1) See Section 8.1 for odd p; for $p = 2$ the main line groups are the dihedral 2-groups, and the result is well known (see e.g. [8, Section 9]).

(2) See Section 8.2. □

Remark 1.2. Theorem 1.1(1) can be made more explicit: the Quillen categories of the quotients $G_p/\gamma_i(G_p)$ are equivalent for all $i \geq p + 1$. Note that $G_3/\gamma_i(G_3)$ have isomorphic mod-3 cohomology rings for $i \in \{5, 6, 7\}$, but the cohomology ring for $i = 4$ is different, see [9].

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2. Infinite cochain sequences

2.1. Preliminaries. We work, throughout, with the normalized standard resolution (see [12, page 8]): that is, a cochain $f \in C^n(G, M)$ is a map $f : G^n \to M$ with the additional property that $f(g_1, \ldots, g_n)$ is zero if any g_i is the identity element.

We denote the coboundary operator by $\Delta : C^n(G, M) \to C^{n+1}(G, M)$. Recall that the coboundary of a 2-cochain is given by

$$\Delta f(g_1, g_2, g_3) = {}^{g_1}f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2)$$

and, more generally, the coboundary of an n-cochain is

$$\begin{aligned}\Delta f(g_1, \ldots, g_{n+1}) = {}^{g_1}f(g_2, \ldots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \ldots, g_n).\end{aligned}$$

The n-cocycles are the elements of

$$Z^n(G, M) = \ker(C^n(G, M) \xrightarrow{\Delta} C^{n+1}(G, M)),$$

and the n-coboundaries are the elements of

$$B^n(G, M) = \mathrm{Im}(C^{n-1}(G, M) \xrightarrow{\Delta} C^n(G, M)).$$

Since $\Delta^2 = 0$, it follows that $B^n(G, M) \subseteq Z^n(G, M)$, and we set $H^n(G, M) = Z^n(G, M)/B^n(G, M)$. Elements of $H^n(G, M)$ are called cohomology classes; if f is an n-cocycle, then its cohomology class is $f + B^n(G, M) \in H^n(G, M)$.

Remark 2.1. By transfer theory, $|G| \cdot H^n(G, M) = 0$ for all $n \geq 1$ (see, for example, [3, Proposition 3.6.17]).

2.2. Splitting theorems. For the remainder of this section, we fix the following notation.

Notation 2.2. Let G be a finite p-group with $m = \log_p(|G|)$, let R be a commutative ring, let M an RG-module and let N be a submodule of M with $\mathrm{Ann}_N(p) = \{0\}$.

We need the following generalization of [8, Theorem 7], which is itself a generalization of [10, Theorem 18].

Theorem 2.3. *We use Notation 2.2 and let $n \geq 1$ and $r \geq 2m$. Then there is a splitting*

$$H^n(G, M/p^rN) \cong H^n(G, M) \oplus H^{n+1}(G, N),$$

which is natural with respect to restriction to subgroups of G.

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Notation 2.4. Projection $M \twoheadrightarrow M/p^rN$ induces maps

$$C^n(G, M) \twoheadrightarrow C^n(G, M/p^rN)$$

of cochain modules and $H^n(G, M) \to H^n(G, M/p^rN)$ of cohomology modules. We shall denote all three maps by pro_r.

Proof. Recall that $|G| = p^m$ and define $i_r : N \to M : x \mapsto p^r x$. Consider the long exact sequence in group cohomology induced by the following short exact sequence of coefficient modules

$$0 \longrightarrow N \xrightarrow{i_r} M \xrightarrow{\mathrm{pro}_r} M/p^rN \longrightarrow 0.$$

The proof of [8, Theorem 7] readily generalizes, showing that, if $n \geq 0$ and $r \geq 2m$, then

$$0 \longrightarrow H^n(G, M) \xrightarrow{\mathrm{pro}_r} H^n(G, M/p^rN) \xrightarrow{\mathrm{con}_r} H^{n+1}(G, N) \longrightarrow 0$$

is a split short exact sequence, where con_r is the connecting homomorphism. □

We now describe how Theorem 2.3 works at the cocycle level.

Proposition 2.5. *Use Notation 2.2. Let $n \geq 1$, and pick $\rho \in Z^n(G, M)$ and $\eta \in Z^{n+1}(G, N)$.*

(1) *There is a (not necessarily unique) n-cochain $\sigma \in C^n(G, N)$ such that $\Delta(\sigma) = p^m\eta$.*
(2) *For every $r \geq m$ and for every choice of σ in (1), the induced cochain $\mathrm{pro}_r(\rho + p^{r-m}\sigma)$ lies in $Z^n(G, M/p^rN)$.*
(3) *For every $r \geq 2m$ and for every choice of σ in (1), the cohomology class*

$$\mathrm{pro}_r(\rho + p^{r-m}\sigma) + B^n(G, M/p^rN) \in H^n(G, M/p^rN)$$

is the unique class corresponding, via the isomorphism of Theorem 2.3, to

$$(\rho + B^n(G, M), \eta + B^{n+1}(G, N)) \in H^n(G, M) \oplus H^{n+1}(G, N).$$

Proof. (1) $p^m\eta$ is a coboundary, since $p^mH^{n+1}(G, N) = 0$, by Remark 2.1.

(2) pro_r and Δ commute, and $\Delta(\rho + p^{r-m}\sigma) = p^r\eta$ lies in the kernel of pro_r.

(3) The proof of [8, Theorem 7] says that the component maps of the isomorphism $H^n(G, M/p^rN) \to H^n(G, M) \oplus H^{n+1}(G, N)$ are the connecting homomorphism $\mathrm{con}_r : H^n(G, M/p^rN) \to H^{n+1}(G, N)$ and the map

$$H^n(G, M/p^rN) \xrightarrow{\pi_*} H^n(G, M/p^{r-m}N) \xrightarrow{(\mathrm{pro}_{r-m})^{-1}} H^n(G, M),$$

where $\pi : M/p^rN \to M/p^{r-m}N$, is the projection map $x + p^rN \mapsto x + p^{r-m}N$. As

$$\pi_* \,\mathrm{pro}_r(\rho + p^{r-m}\sigma) = \mathrm{pro}_{r-m}(\rho + p^{r-m}\sigma) = \mathrm{pro}_{r-m}(\rho),$$

the image in $H^n(G, M)$ is $\rho + B^n(G, M)$.

Recall from, for example, the proof of [13, Theorem 9.1.5], that con_r is constructed as the composition

$$Z^n(G, M/p^rN) \xrightarrow{(\mathrm{pro}_r)^{-1}} C^n(G, M) \xrightarrow{\Delta} Z^{n+1}(G, M) \xrightarrow{(i_r)_*^{-1}} Z^{n+1}(G, N),$$

with i_r as in the proof of Theorem 2.3. So $\mathrm{pro}_r(\rho + p^{r-m}\sigma) \mapsto \rho + p^{r-m}\sigma \mapsto 0 + p^r\eta = i_r(\eta) \mapsto \eta$. Uniqueness follows. □

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2.3. The definition of cochain sequences. Using the ideas of Proposition 2.5, we now define infinite cochain sequences.

DEFINITION 2.6. We use Notation 2.2 and let $n \geq 1$ and $r_0 \geq 0$. We call a sequence $(\alpha_r)_{r\geq r_0}$ of cochains $\alpha_r \in C^n(G, M/p^r N)$ a *cochain sequence* if there are cochains $\rho \in C^n(G, M)$ and $\sigma \in C^n(G, N)$ and an $\omega \in \{0, 1, \dots, r_0\}$ such that

$$\alpha_r = \mathrm{pro}_r(\rho + p^{r-\omega}\sigma) \in C^n(G, M/p^r N) \quad \text{for all } r \geq r_0.$$

Note that the cochain sequences defined by $(\rho, \sigma; r_0, \omega)$ and $(\rho', \sigma'; r_0', \omega')$ are equal if and only if $r_0 = r_0'$ and the cochains $\mathrm{pro}_r(\rho + p^{r-\omega}\sigma)$ and $\mathrm{pro}_r(\rho' + p^{r-\omega'}\sigma')$ are equal as elements of $C^n(G, M/p^r N)$ for all $r \geq r_0$.

Often, r_0 will be clear from the context. We then also write $\alpha_\bullet$ for $(\alpha_r \mid r \geq r_0)$ and $M/p^\bullet N$ for $(M/p^r N \mid r \geq r_0)$. If $\alpha_\bullet$ is induced from $(\rho, \sigma; r_0, \omega)$, then we also write

$$\alpha_\bullet = \mathrm{pro}_\bullet(\rho + p^{\bullet-\omega}\sigma).$$

NOTATION 2.7. Often, r_0 and N will be fixed from the context. We then denote $M_r = M/p^r N$, we write $M_\bullet$ for $(M_r \mid r \geq r_0)$ and we denote by $\mathbb{C}^n_{r_0}(G, M_\bullet)$ the set of all cochain sequences which start at r_0.

2.4. Homological algebra for cochain sequences. Our next aim is to develop some elementary homological algebra for cochain sequences.

NOTATION 2.8. We continue to use the Notation 2.2, imposing minor additional restrictions. We assume from now on that R is a noetherian integral domain and that $p$is a prime number, which, in R, is neither zero nor a unit. Further, M is a finitely generated RG-module which is free as an R-module. Then $\bigcap_r p^r M = \{0\}$, by Krull's theorem [2, 10.17] and $\mathrm{Ann}_N(p) = \{0\}$.

LEMMA 2.9. *The set $\mathbb{C}^n_{r_0}(G, M_\bullet)$ of cochain sequences is an R-module.*

PROOF. Let $\alpha_\bullet$ be defined by $(\rho, \sigma; r_0, \omega)$ and $\beta_\bullet$ by $(\rho', \sigma'; r_0, \omega')$. Then

$$\alpha_r + \beta_r = \mathrm{pro}_r(\rho + \rho' + p^{r-\ell}(p^{\ell-\omega}\sigma + p^{\ell-\omega'}\sigma')) \quad \text{for } \ell = \max(\omega, \omega'),$$

and so $\alpha_\bullet + \beta_\bullet$ is the cochain sequence defined by $(\rho + \rho', p^{\ell-\omega}\sigma + p^{\ell-\omega'}\sigma'; r_0, \ell)$. For $x \in R$, $x\alpha_\bullet$ is the cochain sequence defined by $(x\rho; x\sigma; r_0, \omega)$. □

LEMMA 2.10. *Let $\alpha_\bullet \in \mathbb{C}^n_{r_0}(G, M_\bullet)$ be the cochain sequence defined by $(\rho; \sigma; r_0, \omega)$.*

(1) *Either $\alpha_\bullet = 0$, that is, $\alpha_r = 0$ for all $r \geq r_0$, or $\alpha_r \neq 0$ for all sufficiently large r.*
(2) *$\alpha_\bullet = 0$ if and only if $\rho = 0$ in $C^n(G, M)$ and σ lies in $p^\omega C^n(G, N)$.*

PROOF. If $\rho = 0$ and σ is not divisible by p^ω, then $p^{r-\omega}\sigma$ is not divisible by p^r, and so α_r is nonzero for all r. If $\rho \neq 0$, then there is k such that $\mathrm{pro}_k(\rho) \neq 0$ in $C^n(G, M/p^k N)$, and hence $\alpha_r \neq 0$ for all $r \geq k + \omega$. □

NOTATION 2.11. Let $\alpha_\bullet \in \mathbb{C}^n_{r_0}(G, M_\bullet)$. We define the *level* of $\alpha_\bullet$ to be the smallest value of ω such that $\alpha_\bullet$ is defined by $(\rho, \sigma; r_0, \omega)$ for some ρ, σ.

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Remark 2.12. By definition, the cochain sequence, defined by $(\rho, \sigma; r_0, \omega)$, has level at most ω. Note that $(\rho, \sigma; r_0, \omega)$ and $(\rho, p\sigma; r_0, \omega + 1)$ define the same cochain sequence. Thus the level of the cochain sequence defined by $(\rho, \sigma; r_0, \omega)$ can be strictly smaller than ω.

Definition 2.13. Define $\mathbb{C}^n_{r_0}(G, M_\bullet) \xrightarrow{\Delta} \mathbb{C}^{n+1}_{r_0}(G, M_\bullet)$ by $(\Delta\alpha)_r := \Delta(\alpha_r)$. So if $\alpha_\bullet$ is defined by $(\rho, \sigma; r_0, \omega)$, then $\Delta(\alpha_\bullet)$ is defined by $(\Delta(\rho), \Delta(\sigma); r_0, \omega)$. Further, write $\mathbb{Z}^n_{r_0}(G, M_\bullet) = \ker(\Delta)$ and $\mathbb{B}^{n+1}_{r_0}(G, M_\bullet) = \operatorname{Im}(\Delta)$.

The map Δ is R-linear and satisfies $\Delta^2 = 0$. Thus $\mathbb{B}^n_{r_0}(G, M_\bullet) \subseteq \mathbb{Z}^n_{r_0}(G, M_\bullet)$.

Remark 2.14. By Lemma 2.10, $\Delta(\alpha_\bullet) = 0$ if and only if $\Delta(\rho) = 0$ and $\Delta(\sigma)$ is divisible by p^ω. So we may rephrase Proposition 2.5 as the following corollary.

Corollary 2.15. *Let $n \geq 1$ and $r_0 \geq 2m$. For every $\bar{\rho} \in H^n(G, M)$ and every $\bar{\eta} \in H^{n+1}(G, N)$, there is a cocycle sequence $\alpha_\bullet \in \mathbb{Z}^n_{r_0}(G, M_\bullet)$ of level at most m such that, for every $r \geq r_0$, the cohomology class $\alpha_r + B^n(G, M_r) \in H^n(G, M_r)$ corresponds, under the isomorphism of Theorem 2.3, to $(\bar{\rho}, \bar{\eta}) \in H^n(G, M) \oplus H^{n+1}(G, N)$.* □

Lemma 2.16. *Let $n \geq 1$. Suppose that $\alpha_\bullet \in \mathbb{Z}^n_{r_0}(G, M_\bullet)$ has level $\omega \leq r_0 - m$. The following statements are equivalent:*

(1) *$\alpha_{r_1} \in B^n(G, M_{r_1})$ for some value $r_1 \geq r_0$ of r;*
(2) *$\alpha_r \in B^n(G, M_r)$ for every $r \geq r_0$;*
(3) *$\alpha_\bullet \in \mathbb{B}^n_{r_0}(G, M_\bullet)$; and*
(4) *$\alpha_\bullet = \Delta(\beta_\bullet)$ for some $\beta_\bullet \in \mathbb{C}^{n-1}_{r_0}(G, M_\bullet)$ of level at most $\omega + m$.*

Proof. The implications (4) ⇒ (3) ⇒ (2) ⇒ (1) are clear.

(1) ⇒ (4): Let $\alpha_\bullet = \mathrm{pro}_\bullet(\rho + p^{\bullet-\omega}\sigma)$. Since $\alpha_{r_1} \in B^n(G, M_{r_1})$, there are $\phi \in C^{n-1}(G, M)$ and $\psi \in C^n(G, N)$ such that $\rho + p^{r_1-\omega}\sigma = \Delta(\phi) + p^{r_1}\psi$, and hence

$$p^{r_1-\omega}(\sigma - p^\omega\psi) = \Delta(\phi) - \rho.$$

By Lemma 2.10, we may replace σ by $\sigma - p^\omega\psi$ without altering $\alpha_\bullet$. Hence

$$p^{r_1-\omega}\sigma = \Delta(\phi) - \rho.$$

Now, the right-hand side is a cocycle, since $\alpha_\bullet \in Z^n(G, M_\bullet)$ means that ρ is a cocycle. Hence the left-hand side lies in $Z^n(G, N)$. So $\sigma \in Z^n(G, N)$, by regularity of p, and therefore, since $p^m H^n(G, N) = 0$, there is $\chi \in C^{n-1}(G, N)$ with $\Delta(\chi) = p^m\sigma$. Hence $\rho = \Delta(\lambda)$ for $\lambda = \phi - p^{r_1-\omega-m}\chi \in C^{n-1}(G, M)$. So $\alpha_\bullet = \Delta(\beta_\bullet)$ for $\beta_\bullet = \mathrm{pro}_\bullet(\lambda + p^{\bullet-\omega-m}\chi)$. □

The next result will be needed in the proof of Lemma 4.6.

Lemma 2.17. *Let $n \geq 1$ and $r_1 \geq r_0 \geq 2m$. For each $z \in Z^n(G, M_{r_1})$ there is an $\alpha_\bullet \in \mathbb{Z}^n_{r_0}(G, M_\bullet)$ of level at most m with $\alpha_{r_1} = z$.*

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Proof. Let ξ be the element of $H^n(G, M) \oplus H^{n+1}(G, N)$ corresponding to $z + B^n(G, M_{r_1}) \in H^n(G, M_{r_1})$ under the isomorphism of Theorem 2.3. By Corollary 2.15, there is some $\beta_\bullet = \mathrm{pro}_\bullet(\rho + p^{\bullet - m}\sigma) \in \mathbb{Z}^n_{r_0}(G, M_\bullet)$ such that $\beta_r + B^n(G, M_r)$ corresponds to ξ for every $r \geq r_0$. Hence $z - \beta_{r_1} \in B^n(G, M_{r_1})$. Pick $\lambda \in C^{n-1}(G, M_{r_1})$ with $z = \Delta(\lambda) + \beta_{r_1}$, and choose $\bar{\lambda} \in C^{n-1}(G, M)$ with $\mathrm{pro}_{r_1}(\bar{\lambda}) = \lambda$. For $\rho' = \rho + \Delta(\bar{\lambda}) \in Z^n(G, M)$ we then have $z = \alpha_{r_1}$ for $\alpha_\bullet = \mathrm{pro}_\bullet(\rho' + p^{\bullet - m}\sigma)$. □

3. Coclass families of p-groups

Coclass families are certain infinite families of finite p-groups of fixed coclass. Their construction has been introduced in [10] based on a version of Theorem 2.3. Here we exhibit a construction based on Proposition 2.5. The construction differs from [10] in that it uses cocycles rather than their corresponding cocycle classes and thus is slightly more explicit. This difference will be essential in our later applications.

Every coclass family of p-groups of coclass r is associated with an infinite pro-p-group S of coclass r. The structure of the infinite pro-p-groups of finite coclass is well investigated. For example, it is known that, for each such group S, there exist natural numbers l and d so that the lth term of the lower central series $\gamma_l(S)$ satisfies that $\gamma_l(S) \cong \mathbb{Z}_p^d$, where $\mathbb{Z}_p$ denotes the p-adic numbers and $S/\gamma_l(S)$ is a finite p-group of coclass r. The integer d is an invariant of S called the *dimension* of S. The integer l is not an invariant; in fact, each integer larger than l can be used instead of l. The subgroup $\gamma_l(S)$ is often denoted by T and called a *translation subgroup* of S. Its subgroup series, defined by $T_0 = T$ and $T_{i+1} = [T_i, S]$, satisfies $[T_i : T_{i+1}] = p$ for $i \in \mathbb{N}_0$. Thus the series $T = T_0 > T_1 > \cdots$ is the unique series of S-normal subgroups in T, and T is called a *uniserial* S-module. We refer to [13] for many more details on the structure of the infinite pro-p-groups of coclass r.

Given S and l, we write $S_i = S/\gamma_{l+i}(S)$ for $i \in \mathbb{N}_0$. Then $S_0, S_1, \ldots$ is an infinite sequence of finite p-groups of coclass r. This sequence is called the *main line* associated with S. The main line is not necessarily a coclass family itself, but it always consists of d coclass families and a finite number of other groups. More precisely, there exists an integer $h \geq l$ so that the d infinite sequences $(S_{h+i}, S_{h+i+d}, S_{h+i+2d}, \ldots)$ for $0 \leq i < d$ are coclass families. Note that the group S can be viewed as an extension of S_{h+i+jd} by its natural module T_{h+i+jd}, for each h, i and j, and that the group $S_{h+i+jd+k}$ can be viewed as an extension of S_{h+i+jd} by its natural module $T_{h+i+jd}/T_{h+i+jd+k}$, for each h, i, j and k.

For each coclass family $(G_0, G_1, \ldots)$ associated with the infinite pro-p-group S, there exists an integer k so that each group G_j is a certain extension of S_{h+i+jd} with its natural module $T_{h+i+jd}/T_{h+i+jd+k}$. The extensions can be chosen so that the main line group $S_{h+i+jd+1}$ is not a quotient of G_j. In this case, the integer k is an invariant of the coclass family called its *distance* to the main line.

To describe the groups in a coclass family explicitly, it is more convenient to use a different type of extension construction. Instead of describing a group G_j in a coclass family as extension of an associated main line group S_{h+i+jd} by its natural module

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of fixed size p^k, we describe each G_j as an extension of a fixed main line group S_ℓ for some suitable ℓ by a module of variable size $M_j := T_\ell/T_{\ell+jd}$. It is not difficult to observe that $T_{\ell+jd} = p^j T_\ell$, and thus the group M_j is isomorphic to a direct product of d copies of cyclic groups of order p^j.

We now use Proposition 2.5 to exhibit a complete construction for a coclass family $(G_0, G_1, \ldots)$ associated with the infinite pro-p-group S of coclass r. For this purpose, let $m = \log_p(S_\ell) = r + \ell - 1$ and let $j \geq 3m + 1$. Let $\rho \in Z^2(S_\ell, T_\ell)$ so that S is an extension of S_ℓ with T_ℓ via ρ.

Definition 3.1. There exists $\eta \in Z^3(S_\ell, T_\ell)$ so that G_j is an extension of S_ℓ by M_j via τ_j, where $\tau_\bullet = \mathrm{pro}_\bullet(\rho + p^{\bullet - m}\sigma)$ and $\Delta(\sigma) = p^m \eta$.

The definition of coclass families asserts that, for each coclass family, there exists an η yielding this family. It may happen that different cocycles η_1, η_2 yield coclass families with pairwise isomorphic groups; for example, this is the case if $\eta_1 \equiv \eta_2 \bmod B^3(S_\ell, T_\ell)$. We note that every $\eta \in Z^2(S_\ell, M_j)$ yields a coclass family via the above construction.

The significance of coclass families is underlined by the fact that, for $(p, r) = (2, r)$ or $(p, r) = (3, 1)$, all but a finite number of p-groups of coclass r are contained in a coclass family.

4. Cochain sequences and elementary abelians

We now apply the results of Section 2.4 to the coclass family $G_\bullet$ of Section 3. In the language of Notation 2.8, this means that $M = T$. It would be natural to take $N = T$ as well, but we shall actually take $N = pT$, for the following technical reason: it simplifies Remark 4.4 and, especially, Lemma 5.2 if $M/p^r N$ and M/N have the same elementary abelian subgroups. Hence $M_\bullet = M/p^\bullet N = T/p^{\bullet+1}T$, and $G_{\bullet+1} = M_\bullet.P$ with extension cocycle $\tau_\bullet \in \mathbb{Z}^2_{r_0}(P, M_\bullet)$.

Recall that if $N \trianglelefteq G$ and $U \leq G/N$, then a *lift* of U is a subgroup $\bar{U} \leq G$ such that the projection map $G \to G/N$ maps $\bar{U}$ isomorphically to U.

Suppose that we are given $H \leq G_{r+1}$. Setting $K := H \cap M_r$ and $Q := HM_r/M_r$, we see that H is an extension $H = K \cdot Q$, with $Q \leq P$, and K a Q-submodule of M_r. If K has a complement C in H (which is certainly the case if H is elementary abelian), then $C \leq G_{r+1}$ is a lift of Q.

4.1. Extension theory. We recall some details of extension theory (see, for example, [3, Section 3.7]). Let G be a finite group and M a left $\mathbb{Z}G$-module. Recall that every group extension $\Gamma = M \cdot G$ can be constructed using a 2-cocycle $\tau \in Z^2(G, M)$: the underlying set is $M \times G$, with multiplication

$$(t_1, g_1)(t_2, g_2) = (t_1 + {}^{g_1}t_2 + \tau(g_1, g_2), g_1 g_2).$$

Associativity is equivalent to the cocycle condition. The extension splits as a semidirect product $\Gamma = M \rtimes G$ if and only $\tau \in B^2(G, M)$. If $\tau = \Delta(f)$, then $G(f) = \{(-f(g), g) \mid g \in G\} \leq \Gamma$ is a lift of G, and every lift arises thus.

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Lemma 4.1. *If $f, f' \in C^1(G, M)$ satisfy $\Delta(f) = \tau = \Delta(f')$, then $f' - f \in Z^1(G, M)$ and, moreover,*

$$f' - f \in B^1(G, M) \;\Leftrightarrow\; G(f) \text{ and } G(f') \text{ are conjugate by an element of } M.$$

Proof. This is Proposition 3.7.2 of [3]. Observe, from the proof of that result, that conjugation by elements of M induces every coboundary. □

4.2. Lifting elementary abelians.

Lemma 4.2. *Let $Q \le P$ and suppose $r_0 \ge 2m$. Then the three following statements are equivalent:*

(1) *Q has a lift $\bar{Q}_r \le G_{r+1}$ for all $r \ge r_0$; and*
(2) *Q has a lift $\bar{Q}_r \le G_{r+1}$ for at least one $r \ge r_0$;*
(3) *$\tau_\bullet|_Q = \Delta(f_\bullet)$ for some cochain sequence $f_\bullet \in \mathbb{C}^1_{r_0}(Q, M_\bullet)$ of level at most m.*

Proof. Let $H_r \le G_{r+1}$ be the subgroup with $M_r \le H_r$ and $H_r/M_r = Q$. Then $H_r = M_r \cdot Q$ with extension class $\tau_r|_Q$, and Q has a lift $\bar{Q}_r \le G_{r+1}$ if and only if $\tau_r|_Q$ lies in $B^2(Q, M_r)$. Now apply Lemma 2.16. □

Now suppose that $E \le G_{r+1}$ is elementary abelian. Setting $K = E \cap M_r$ and $U = EM_r/M_r \le P$, as above, we have $E = K \times \bar{U}$ for a lift $\bar{U} \le G_{r+1}$ of U. Recall that $\bar{U} = U(\phi)$ for some $\phi \in C^1(U, M_r)$, with $\Delta(\phi) = \tau_r|_U$.

Notation 4.3. We shall need to refer to several different maps between cohomology modules. Let $L \subseteq M$ be a submodule.

- Inclusion $L \hookrightarrow M$ induces $H^n(G, L) \xrightarrow{\text{inc}} H^n(G, M)$.
- $\mathrm{mul}^r : M/L \to M/p^rL$, $\quad x + L \mapsto p^r x + p^r L$ induces $H^n(G, M/L) \xrightarrow{\mathrm{mul}^r} H^n(G, M/p^rL)$.

Note that $\mathrm{mul}^r \circ \mathrm{mul}^s = \mathrm{mul}^{r+s}$, and that

$$\mathrm{pro}_r(\rho + p^{r-m}\sigma) = \mathrm{pro}_r(\rho) + \mathrm{mul}^{r-m}\,\mathrm{pro}_m(\sigma).$$

Remark 4.4. Since $K \le M_r = T/p^{r+1}T$ is elementary abelian, it follows that

$$K \le \Omega_1(M_r) = p^rT/p^{r+1}T \xrightarrow[\cong]{(\mathrm{mul}^r)^{-1}} T/pT.$$

So $K = \mathrm{mul}^r(W)$ for some $W \le T/pT$. Since E is abelian, $[\bar{U}, K] = 1$, which is equivalent to $W \le (T/pT)^U$.

Notation 4.5. $\mathcal{E}$ is the set of all triples $(U, f_\bullet, W)$ with $U \le P$ an elementary abelian, $f_\bullet \in \mathbb{C}^1_{r_0}(U, M_\bullet)$ is a cochain sequence of level at most $2m$ such that $\Delta(f_\bullet) = \tau_\bullet|_U$ and $W \le (T/pT)^U$.

Lemma 4.6. *Suppose that $r \ge r_0 \ge 2m$. Every elementary abelian $E \le G_{r+1}$ has the form $E = E_r(U, f_\bullet, W) := \mathrm{mul}^r(W) \times U(f_r)$ for some $(U, f_\bullet, W) \in \mathcal{E}$.*

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Proof. We saw above that $E = \mathrm{mul}^r(W) \times U(\phi)$ with $U \leq P$ elementary abelian, $W \leq (T/pT)^U$ and $\phi \in C^1(U, M_r)$ with $\Delta(\phi) = \tau_r|_U$. As $U(\phi)$ is a lift of U in G_{r+1}, there is $f_\bullet \in \mathbb{C}^1_{r_0}(U, M_\bullet)$ of level at most $2m$ with $\Delta(f_\bullet) = \tau_\bullet|_U$, by Lemma 4.2. Hence $\phi - f_r \in Z^1(U, M_r)$, so, by Lemma 2.17, there is $z_\bullet \in \mathbb{Z}^1_{r_0}(U, M_\bullet)$ of level at most m with $z_r = \phi - f_r$. So $(U, f_\bullet + z_\bullet, W) \in \mathcal{E}$, and $E = \mathrm{mul}^r(W) \times U(f_r + z_r)$. □

5. Change of module

The following technical lemma is required in the proofs of Proposition 6.2 and Lemma 7.1. We revert to Notation 2.8, and consider the case of two submodules $L, N \subseteq M$ satisfying the condition $pL \subseteq N \subseteq L$.

Example 5.1. If $(U, f_\bullet, W) \in \mathcal{E}$, then $W \leq T/pT$, and so $W = L/pT$ for some $pT \subseteq L \subseteq T$. Hence $pL \subseteq N \subseteq L$, since $N = pT$.

We shall investigate the cohomology maps induced by the short exact sequence

$$0 \to L/N \xrightarrow{\mathrm{mul}^r} M/p^r N \to M/p^r L \to 0.$$

As we now have to distinguish between two different projection maps, we shall denote them by $M \xrightarrow{\mathrm{pro}^N_r} M/p^r N$ and $M \xrightarrow{\mathrm{pro}^L_r} M/p^r L$.

Lemma 5.2. *Suppose the RG-submodule $L \subseteq M$ satisfies $pL \subseteq N \subseteq L$.*

(1) *Assume $r_0 \geq 1$. Then $j_* \circ i_* = 0$ for the chain maps*

$$C^*(G, L/N) \xrightarrow{i_*} \mathbb{C}^*_{r_0}(G, M/p^\bullet N) \xrightarrow{j_*} \mathbb{C}^*_{r_0}(G, M/p^\bullet L)$$

given by $i_n(c)_r = \mathrm{mul}^r(c)$ and $j_n(\alpha_\bullet)_r = \alpha_r + p^r L$.

(2) *Suppose that $\alpha_\bullet \in \mathbb{Z}^n_{r_0}(G, M/p^\bullet N)$ satisfies $j_n(\alpha_\bullet) \in \mathbb{B}^n_{r_0}(G, M/p^\bullet L)$. If $\alpha_\bullet$ has level $\omega \leq r_0 - m$, then*

$$\alpha_\bullet = i_n(c)_\bullet + \Delta(\mathrm{pro}^N_\bullet(\kappa + p^{\bullet-(\omega+m)}\lambda))$$

for some $c \in Z^n(G, L/N)$, $\kappa \in C^{n-1}(G, M)$ and $\lambda \in C^{n-1}(G, L)$.

Proof. (1) Pick $\bar{c} \in C^n(G, L)$ with $\mathrm{pro}^L_0(\bar{c}) = c$, then $p\bar{c} \in C^n(G, N)$ and

$$i_n(c)_\bullet = \mathrm{pro}^N_\bullet(0 + p^{\bullet-1} \cdot p\bar{c}) \in \mathbb{C}^n_{r_0}(G, M/p^\bullet N) \quad \text{with level } 1 \leq r_0.$$

If $\alpha_\bullet = \mathrm{pro}^N_\bullet(\rho + p^{\bullet-\omega}\sigma) \in \mathbb{C}^n_{r_0}(G, M/p^\bullet N)$, then $j_n(\alpha_\bullet) = \mathrm{pro}^L_\bullet(\rho + p^{\bullet-\omega}\sigma)$.

Clearly i_*, j_* are chain maps. And $j_n i_n = 0$, since c takes values in L.

(2) Let $\alpha_\bullet = \mathrm{pro}^N_\bullet(\rho + p^{\bullet-\omega}\sigma)$, so $j_n(\alpha_\bullet) = \mathrm{pro}^L_\bullet(\rho + p^{\bullet-\omega}\sigma)$. By Lemma 2.16, $j_n(\alpha_\bullet) = \Delta(\gamma_\bullet)$ for some $\gamma_\bullet \in \mathbb{C}^{n-1}_{r_0}(G, M/p^\bullet L)$ of the form

$$\gamma_\bullet = \mathrm{pro}^L_\bullet(\kappa + p^{\bullet-\omega-m}\lambda) \quad \text{with } \kappa \in C^{n-1}(G, M), \lambda \in C^{n-1}(G, L).$$

Applying Lemma 2.10, we have $\rho = \Delta(\kappa)$, and

$$p^m\sigma = \Delta(\lambda) + p^{\omega+m}\bar{c} \quad \text{for some } \bar{c} \in C^n(G, L).$$

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From $\alpha_\bullet \in \mathbb{Z}^n_{r_0}(G, M/p^\bullet N)$, it follows that $\Delta(\sigma)$ takes values in $p^\omega N$, and so $\Delta(\bar{c})$ takes values in N. So $c := \mathrm{pro}^N_0(\bar{c})$ lies in $Z^n(G, L/N)$, and $i_n(c)_r = \mathrm{pro}^N_r(p^r\bar{c})$. Hence

$$\begin{aligned}\alpha_\bullet &= \mathrm{pro}^N_\bullet(\rho + p^{\bullet-\omega}\sigma)\\ &= \mathrm{pro}^N_\bullet(\Delta(\kappa + p^{\bullet-(\omega+m)}\lambda) + p^\bullet\bar{c})\\ &= i_n(c) + \Delta(\mathrm{pro}_\bullet(\kappa + p^{\bullet-(\omega+m)}\lambda)).\end{aligned}$$ □

6. Morphisms in the Quillen category

Notation 6.1. Consider the triple $(U, f_\bullet, W) \in \mathcal{E}$. Recall from Section 2 that G_{r+1} has underlying set $M_r \times P$ and that $U(f_r) = \{(-f_r(u), u) \mid u \in U\}$. So the subgroup $E_r(U, f_\bullet, W) \le G_{r+1}$ of Lemma 4.6 is

$$E_r(U, f_\bullet, W) = \{(p^r w - f_r(u), u) \mid u \in U, w \in W\}.$$

Let $j^f_r : W \times U \to E_r(U, f_\bullet, W)$ be the isomorphism $(w, u) \mapsto (p^r w - f_r(u), u)$.

Proposition 6.2. *Suppose that $r_0 \ge 3m$ and $m \ge 1$. For elements $(U, f_\bullet, W), (U', f'_\bullet, W') \in \mathcal{E}$ the set of isomorphisms $W \times U \to W' \times U'$ of the form*

$$W \times U \xrightarrow{j^f_r} E_r(U, f_\bullet, W) \xrightarrow[\text{in } G_{r+1}]{\text{conjugation}} E_r(U', f'_\bullet, W') \xrightarrow{(j^{f'}_r)^{-1}} W' \times U'$$

is independent of r.

For the proof, we need two lemmas. Observe that mul^{r-r_0} embeds $M_{r_0} = T/p^{r_0+1}T$ in G_{r+1} as $p^{r-r_0}T/p^{r+1}T \le M_r$.

Lemma 6.3. *Suppose that $m \ge 1$ and $r_0 \ge 2m$. Let $(U, f_\bullet, W) \in \mathcal{E}$.*

(1) $\mathrm{Aut}_{M_r}(E_r(U, f_\bullet, W)) = \mathrm{Aut}_{\mathrm{mul}^{r-r_0}(M_{r_0})}(E_r(U, f_\bullet, W))$.

(2) *The subgroup*

$$\mathcal{N} = \{x \in M_{r_0} \mid \mathrm{mul}^{r-r_0}(x) \in N_{G_{r+1}}(E_r(U, f_\bullet, W))\}$$

depends on neither r nor $f_\bullet$; nor does the action of $\mathcal{N}$ on $W \times U$ obtained by using j^f_r to identify $E_r(U, f_\bullet, W)$ with $W \times U$.

Write $\bar{W}$ for the module $pT \subseteq \bar{W} \subseteq T$ with $W = \bar{W}/pT$.

Proof. (1) Conjugation by $t \in T$ fixes $M_r U(f_r)/M_r$ and W pointwise and is described by $\Delta(t) \in B^1(U, M_r)$:

$${}^{(t,0)}(p^r w - f_r(u), u) = (p^r w - \Delta(t)(u) - f_r(u), u).$$

If t normalizes $\mathrm{mul}^r(W) \times U(f_r) \le G_{r+1}$ then $\Delta(t)$ must take values in $p^r\bar{W}$. Hence $\Delta(t) \in p^r Z^1(U, \bar{W}) \subseteq p^{r-m}B^1(U, \bar{W})$. So there is $\bar{v} \in \bar{W}$ such that $\Delta(t) = p^{r-m}\Delta(\bar{v})$, and $\mathrm{pro}_r(p^{r-m}\bar{v}) = \mathrm{mul}^{r-r_0}\,\mathrm{pro}_{r_0}(p^{r_0-m}\bar{v}) \in \mathrm{mul}^{r-r_0}(M_{r_0})$ has the same conjugation action as t.

(2) Conversely, if $v = \bar{v} + p^{r_0+1}T \in M_{r_0}$, then $\mathrm{mul}^{r-r_0}(v)$ normalizes $\mathrm{mul}^r(W) \times U(f_r)$ if and only if $\Delta(p^{r-r_0}\bar{v}) \in Z^1(U, T)$ takes values in $p^r\bar{W}$, that is, if $\Delta(v) = \mathrm{mul}^{r_0}(z)$ for some $z \in Z^1(U, W)$. The action on $W \times U$ is then $(w, u) \mapsto (w - z(u), u)$. □

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Lemma 6.4. *Suppose* $r_0 \geq 2m$ *and* $g \in P$. *Let* $(U, f_\bullet, W), ({}^gU, f'_\bullet, {}^gW) \in \mathcal{E}$. *Define* $\chi_r \in C^1({}^gU, M_r)$ *by*

$$\chi_r(v) = ({}^gf_r)(v) - \tau_r(g, v^g) + \tau_r(v, g) - f'_r(v).$$

Then $\chi_\bullet \in \mathbb{Z}^1_{r_0}({}^gU, M_\bullet)$ *is a cocycle sequence of level at most* $2m$.

Proof. For $c \in C^n(H, M)$ we, of course, define ${}^gc \in C^n({}^gH, M)$ by

$$({}^gc)(k_1, \dots, k_n) = {}^gc(k_1^g, \dots, k_n^g).$$

Everything is of level at most $2m$. Since $\Delta(f'_\bullet) = \tau_\bullet|_{{}^gU}$ and $\Delta({}^gf_\bullet) = {}^g(\tau_\bullet|_U)$

$$\begin{aligned}\Delta(\chi_r)(v_1, v_2) &= {}^g\tau_r(v_1^g, v_2^g) - \tau_r(g, v_2^g) + \tau_r(g, v_1^gv_2^g) - \tau_r(g, v_1^g)\\ &\qquad + \tau_r(v_2, g) - \tau_r(v_1v_2, g) + \tau_r(v_1, g) - \tau_r(v_1, v_2)\\ &= \Delta(\tau_r)(g, v_1^g, v_2^g) - \Delta(\tau_r)(v_1, g, v_2^g) + \Delta(\tau_r)(v_1, v_2, g) = 0.\end{aligned}$$

□

Proof of Proposition 6.2. If ${}^{(t,g)}E_r(U, f_\bullet, W) = E_r(U', f'_\bullet, W')$, then ${}^gU = U'$ and ${}^gW = W'$. So we assume that $g \in P$ is fixed, and consider which $t \in M_r$ satisfy ${}^{(t,g)}E_r(U, f_\bullet, W) = E_r({}^gU, f'_\bullet, {}^gW)$ and which isomorphisms arise in this way.

The map $F : W \times U \to {}^gW \times {}^gU$, given by j_r^f, then conjugation by (t, g) and then $(j_r^{f'})^{-1}$ must have the form $F(w, u) = ({}^gw - \pi({}^gu), {}^gu)$ for some $\pi \in Z^1({}^gU, {}^gW)$. So we consider g, π to be fixed and ask for which values of r there is $t + p^{r+1}T \in M_r$ realizing this F. The condition on t, i can be phrased as

$$(t, g)(p^rw - f_r(u), u) = (p^r \cdot {}^gw - p^r\pi({}^gu) - f'_r({}^gu), {}^gu)(t, g).$$

Equality in ${}^gU$ is immediate. We are left with the condition in M_r given by

$$t + p^r \cdot {}^gw - {}^g(f_r(u)) + \tau_r(g, u) = p^r({}^gw - \pi({}^gu)) - f'_r({}^gu) + {}^{({}^gu)}t + \tau_r({}^gu, g).$$

So, with χ_r as in Lemma 6.4,

$$\begin{aligned}p^r\pi({}^gu) &= ({}^gf)({}^gu) - \tau_r(g, u) + \tau_r({}^gu, g) - f'_r({}^gu) + \Delta(t)({}^gu)\\ &= (\chi_r + \Delta(t))({}^gu).\end{aligned}$$

That is, F is realizable for this r if and only if

$$p^r\pi - \chi_r \in B^1({}^gU, M_r). \tag{6.1}$$

Since π takes values in ${}^gW = {}^g\bar{W}/pT$, a necessary condition for any such F to be realizable is that

$$\chi_r + p^r \cdot {}^g\bar{W} \in B^1({}^gU, T/p^r \cdot {}^g\bar{W}). \tag{6.2}$$

Since $\chi_\bullet \in \mathbb{Z}^1_{r_0}({}^gU, M_\bullet)$ has level at most $2m$ and $r_0 \geq 2m + m$, we deduce, from Lemma 2.16, that (6.2) is either satisfied for all r_0 or for none.

If (6.2) is satisfied, then we apply Lemma 5.2 with $G = {}^gU$, $\alpha_\bullet = \chi_\bullet$ and $L = {}^g\bar{W}$, and hence $L/N = {}^gW$. Note that $\chi_\bullet$ has level at most $2m \leq r_0 - m$. We conclude that there are $c \in Z^1({}^gU, {}^gW)$, $\kappa \in C^0({}^gU, T)$ and $\lambda \in C^0({}^gU, {}^g\bar{W})$ with $\chi_\bullet = \mathrm{mul}^\bullet(c) + \Delta(\mathrm{pro}_\bullet(\kappa + p^{\bullet - 3m}\lambda))$. We also conclude that if we take $\pi = c$, then (6.1) is solvable for all r: that is, this one map $F : W \times U \to {}^gW \times {}^gU$ is independent of r. But all other maps for this value of g correspond to a M_r-automorphism of $U \times W$ followed by F, and we saw in Lemma 6.3 that these isomorphisms are independent of r too. □

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Corollary 6.5. *Suppose that $e \geq 3m$ and $m \geq 1$. For elements $(U, f_\bullet, W), (U', f'_\bullet, W') \in \mathcal{E}$ the following statements are equivalent:*

(1) *$E_r(U, f_\bullet, W)$ and $E_r(U', f'_\bullet, W')$ are G_{r+1}-conjugate for some r; and*
(2) *$E_r(U, f_\bullet, W)$ and $E_r(U', f'_\bullet, W')$ are G_{r+1}-conjugate for every r.*

Proof. They are G_{r+1}-conjugate if and only if the set of isomorphisms in Proposition 6.2 is nonempty. But this set does not depend on r. □

7. Wrapping up the main theorem

Lemma 7.1. *Suppose $r_0 \geq 2m$. Let $(U, f_\bullet, W) \in \mathcal{E}$. For each $V \leq W \times U$ there is $(U', f'_\bullet, W') \in \mathcal{E}$ such that, for all r, $j_r^f(V) = E_r(U', f'_\bullet, W')$. Moreover, the map*

$$\kappa^V : W' \times U' \xrightarrow{j_r^{f'}} E_r(U', f'_\bullet, W') \hookrightarrow E_r(U, f_\bullet, W) \xrightarrow{(j_r^f)^{-1}} W \times U$$

has image V and is independent of r.

Proof. Take $W' = V \cap W$ and $U' = \{u \in U \mid uW \in VW/W\} \leq U$. Then $V/W' \cong U'$ and W' is a direct factor of V, so there is $c \in Z^1(U', W)$ with $V = \{(w - c(u), u) \mid w \in W', u \in U'\}$. Then

$$j_r^f(V) = \{(p^{i+e-1}w - p^{i+e-1}c(u) - f_r(u), u) \mid w \in W', u \in U'\}.$$

In the terminology of Lemma 5.2, $f'_\bullet = f_\bullet|_{U'} + i_1(c)_\bullet$. In particular, $\kappa^V(w, u) = (w - c(u), u)$. □

Corollary 7.2. *Suppose that $r_0 \geq 3m$ and $m \geq 1$. For elements $(U, f_\bullet, W), (U', f'_\bullet, W') \in \mathcal{E}$, the set of homomorphisms $W \times U \to W' \times U'$ of the form*

$$W \times U \xrightarrow{j_r^f} E_r(U, f_\bullet, W) \xrightarrow[\text{in } \mathscr{A}_p(G_{r+1})]{\text{morphism}} E_r(U', f'_\bullet, W') \xrightarrow{(j_r^{f'})^{-1}} W' \times U'$$

is independent of r.

Proof. Every such map is an isomorphism to some $V \leq W' \times U'$. From Lemma 7.1, for some $(U'', f''_\bullet, W'')$, $j_r^{f'}(V) = E_r(U'', f''_\bullet, W'')$ for all r. From Proposition 6.2, the set $\mathcal{I}_V$ of isomorphisms of the form

$$W \times U \xrightarrow{j_r^f} E_r(U, f_\bullet, W) \xrightarrow[\text{in } G_{r+1}]{\text{conjugation}} E_r(U'', f''_\bullet, W'') \xrightarrow{(j_r^{f''})^{-1}} W'' \times U''$$

is independent of r. But $\phi \mapsto \kappa^V \circ \phi$ is a bijection from $\mathcal{I}_V$ to the set of homomorphisms of the form

$$W \times U \xrightarrow{j_r^f} E_r(U, f_\bullet, W) \xrightarrow[\text{in } \mathscr{A}_p(G_{r+1})]{\text{morphism}} E_r(U', f'_\bullet, W') \xrightarrow{(j_r^{f'})^{-1}} W' \times U'$$

whose image is V. □

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Proposition 7.3. *Suppose $e \geq 3m$ and $m \geq 1$. Choose a subset $\mathcal{E}_0 \subseteq \mathcal{E}$ such that, for every conjugacy class C of elementary abelian subgroups in G_{r_0+1}, there is exactly one $(U, f_\bullet, W) \in \mathcal{E}_0$ such that $E_{r_0}(U, f_\bullet, W)$ lies in C. Define C_r to be the full subcategory of the Quillen category $\mathscr{A}_p(G_{r+1})$ on the $E_r(U, f_\bullet, W)$ with $(U, f_\bullet, W)$ in $\mathcal{E}_0$. Then:*

(1) *C_r is a skeleton of $\mathscr{A}_p(G_{r+1})$ for every $r \geq r_0$; and*
(2) *the categories C_r are all isomorphic to each other.*

Hence the Quillen categories $\mathscr{A}_p(G_{r+1})$ are all equivalent to each other.

Proof. $\mathcal{E}_0$ exists, by Lemma 4.6.

(1) We need to show that each conjugacy class C in G_{r+1} contains $E_r(U, f_\bullet, W)$ for precisely one $(U, f_\bullet, W) \in \mathcal{E}_0$. From Corollary 6.5, there is, at most, one such triple. From Lemma 4.6, there is some $(U', f'_\bullet, W') \in \mathcal{E}$ such that $E_r(U', f'_\bullet, W')$ lies in C. By construction of $\mathcal{E}_0$, there is $(U, f_\bullet, W) \in \mathcal{E}_0$ such that $E_0(U, f_\bullet, W)$, $E_0(U', f'_\bullet, W')$ are G_0-conjugate. So $E_r(U, f_\bullet, W)$ lies in C, by Corollary 6.5.

(2) For $r, r' \geq r_0$ and $(U, f_\bullet, W) \in \mathcal{E}_0$ have isomorphism

$$\lambda^f_{rr'} : E_r(U, f_\bullet, W) \xrightarrow{(j^f_r)^{-1}} W \times U \xrightarrow{j^f_{r'}} E_{r'}(U, f_\bullet, W),$$

with $\lambda^f_{r'r} = (\lambda^f_{rr'})^{-1}$. For a morphism $E_r(U, f_\bullet, W) \xrightarrow{\phi} E_r(U', f'_\bullet, W')$ in C_r, define $F(\phi)$ in $C_{r'}$ by

$$F(\phi) : E_{r'}(U, f_\bullet, W) \xrightarrow{\lambda^f_{r'r}} E_r(U, f_\bullet, W) \xrightarrow{\phi} E_r(U', f'_\bullet, W') \xrightarrow{\lambda^{f'}_{rr'}} E_{r'}(U', f'_\bullet, W').$$

This is a bijection

$$C_r(E_r(U, f_\bullet, W), E_r(U', f'_\bullet, W')) \to C_{r'}(E_{r'}(U, f_\bullet, W), E_{r'}(U', f'_\bullet, W')),$$

by Corollary 7.2, and it is functorial since $F(\mathrm{Id}_{W \times U_r(f)}) = \lambda^f_{rr'} \,\mathrm{Id}\, \lambda^f_{r'r} = \mathrm{Id}_{W \times U_{r'}(f)}$. Also, for $E_r(U', f'_\bullet, W') \xrightarrow{\psi} E_r(U'', f''_\bullet, W'')$ in C_r,

$$F(\psi)F(\phi) = \lambda^{f''}_{rr'} \psi \lambda^{f'}_{r'r} \circ \lambda^{f'}_{ir'} \phi \lambda^f_{r'r} = \lambda^{f''}_{rr'} \psi\phi \lambda^f_{r'r} = F(\psi\phi).$$

This establishes (2). The last part follows from (1) and (2). □

8. Examples

8.1. Main line maximal class groups. If p is an odd prime, then $\mathcal{G}(p, 1)$ consists of one infinite tree, together with the isolated point C_{p^2}: so there is only one uniserial p-adic space group of coclass one. We recall the construction of the main line groups from [13, Example 3.1.5(ii)].

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The pth local cyclotomic number field is $K = \mathbb{Q}_p(\theta)$, where θ has minimal polynomial $\Phi_p(X) = (X^p - 1)/(X - 1)$. The ring of integers in K is $\mathcal{O} = \mathbb{Z}_p[\theta]$, which is a free $\mathbb{Z}_p$-module of rank $p - 1$ with basis $1, \theta, \dots, \theta^{p-2}$. The coclass one uniserial p-adic space group is then $G := \mathcal{O} \rtimes C_p$, where the generator τ of C_p acts as multiplication by θ: that is, ${}^{\tau}v = \theta v$ for $v \in \mathcal{O}$.

The valuation ring $\mathcal{O}$ has unique maximal ideal $\alpha\mathcal{O}$, where $\alpha = \theta - 1$. So $\gamma_i(G) = \alpha^{i-1}\mathcal{O}$ for $i \geq 2$; and, by considering $\Phi_p(X + 1)$, one observes that $p\mathcal{O} = \alpha^{p-1}\mathcal{O}$. Since $1, \alpha, \dots, \alpha^{p-2}$ is a $\mathbb{Z}_p$-basis of $\mathcal{O}$, this means that $\mathcal{O}/\alpha\mathcal{O} \cong \mathbb{F}_p$, and hence $\gamma_i(G)/\gamma_{i+1}(G) \cong C_p$ for $i \geq 2$.

The main line groups are the quotients $G_i = G/\gamma_i(G)$. These main line groups fall into $p - 1$ coclass families, where, for $0 \leq r \leq p - 2$, the ith group in the rth family is $G_{r+(p-1)i}$. From [8] (see, also, Proposition 7.3), we know that all groups in one coclass family have equivalent Quillen categories. But here a stronger result holds: all $p - 1$ coclass families have the same equivalence class of Quillen categories.

Lemma 8.1. *For this group $G = \mathcal{O} \rtimes C_p$, the Quillen category of $G/\gamma_i(G)$ is independent (up to equivalence of categories) of i for $i \geq p + 1$.*

Remark 8.2. For $p = 3$, the first author and King [9] have shown that $G/\gamma_5(G)$, $G/\gamma_6(G)$ and $G/\gamma_7(G)$ have isomorphic cohomology rings, and that these differ from the cohomology ring of $G/\gamma_4(G) \cong 3^{1+2}_+$.

Proof. If $v \in \mathcal{O}$, then $(v\tau)^p = (\Phi_{p-1}(\theta) \cdot v)\tau^p = 1$, and so $v\tau$ has order p. Since $p\mathcal{O} = \alpha^{p-1}\mathcal{O}$, there are two kinds of order p elements of $G/\gamma_i(G)$:

- elements of the form $v\tau^r\gamma_i(G)$, with $v \in \mathcal{O}$ and $1 \leq r \leq p - 1$; and
- elements of $\gamma_{i-p+1}(G)/\gamma_i(G) \cong (C_p)^{p-1}$.

Moreover, the conjugacy class of $v\tau$ in G is $\{w\tau \mid w \in v + \alpha\mathcal{O}\}$, and the centralizer of $v\tau\gamma_i(G)$ in $G/\gamma_i(G)$ is elementary abelian of rank two, generated by $v\tau\gamma_i(G)$ and $\gamma_{i-1}(G)/\gamma_i(G)$. So, as τ acts on $\gamma_{i-p+1}(G)/\gamma_i(G) = \alpha^{i-p}\mathcal{O}/\alpha^{i-1}\mathcal{O}$ as multiplication by $1 + \alpha$, the objects of the Quillen category form the following equivalence classes:

- the class of $\langle v_j\tau\gamma_i(G), \gamma_{i-1}(G)/\gamma_i(G)\rangle \cong C_p^2$ for some fixed transversal $v_1, \dots, v_p$ of $\mathcal{O}/\alpha\mathcal{O}$;
- the class of $\langle v_j\tau\gamma_i(G)\rangle \cong C_p$ for the same transversal $v_1, \dots, v_p$; and
- the conjugacy classes of subgroups of $\gamma_{i-p+1}(G)/\gamma_i(G) \cong \mathcal{O}/\alpha^{p-1}\mathcal{O} \cong C_p^{p-1}$ under the action of C_p, given by multiplication by $1 + \alpha$.

So the equivalence classes of objects admit a description which is independent of i. From this description and the fact that $\langle v_j\tau\gamma_i(G), \gamma_{i-1}(G)/\gamma_i(G)\rangle$ has normalizer $\langle v_j\tau\gamma_i(G), \gamma_{i-2}(G)/\gamma_i(G)\rangle$, it follows that the morphisms between these representatives also admit a description which is independent of i. □

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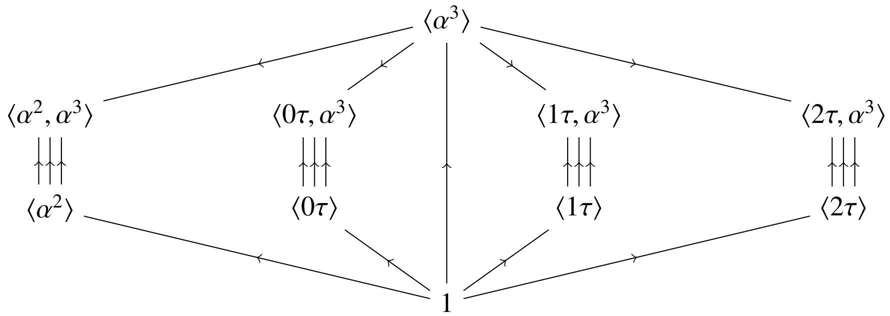
As $p \in \alpha\mathcal{O}$, we may always take the transversal $v_1, \dots, v_p$ to be $0, 1, \dots, p-1$. For $p = 3$ and $i = 4$, the Quillen category has skeleton



where the three automorphisms of each rank two elementary abelian are omitted, for clarity. Specifically, the three maps $\langle 2\tau \rangle \to \langle 2\tau, \alpha^3 \rangle$ are $2\tau \mapsto (2 + \lambda\alpha^3)\tau$ for $\lambda = 0, 1, 2$, and three automorphisms of $\langle 2\tau, \alpha^3 \rangle$ fix α^3 and act on 2τ as one of these three maps.

8.2. A more substantial example. Together with Leedham-Green, Newman and O'Brien, the first author studied the 3-groups of coclass two in [11]. In particular, they construct the skeleton groups in the four coclass trees (out of sixteen) whose branches have unbounded depth. Here we consider the skeleton groups in one of these unbounded depth trees: that is, the tree associated to the pro-3-group, which they denote as R (see their Theorem 4.2(a)).

We briefly recall the construction of the skeleton groups $R_{j-3,\gamma,m}$ from [11, Section 5]. Let $j \geq 7$. Let $K = \mathbb{Q}_3(\theta)$ be the ninth local cyclotomic number field, so θ is a root of $\Phi_9(X) = X^6 + X^3 + 1$. Let $\mathcal{O}$ be the ring of integers in K; then $\mathcal{O} = \mathbb{Z}_3[\theta]$ is free as a $\mathbb{Z}_3$-module, with basis $1, \theta, \dots, \theta^5$. Moreover, $\mathcal{O}$ is a local ring, with maximal ideal $\mathfrak{p} = (\theta - 1)\mathcal{O}$. Observing that $(\theta - 1)^6$ and 3 are associates, one sees that $3\mathcal{O} = \mathfrak{p}^6$.

We now recall the twisting map $\mathfrak{p} \wedge \mathfrak{p} \to \mathcal{O}$, which we shall denote by γ_0. Note, however, that, in [11], it is called ϑ. It is the map

$$\gamma_0(x \wedge y) = \sigma_2(x)\sigma_{-1}(y) - \sigma_{-1}(x)\sigma_2(y),$$

where the automorphism $\sigma_r \in \mathrm{Gal}(K/\mathbb{Q}_3)$ is given by $\sigma_r(\theta) = \theta^r$. Lemma 5.1 of [11] shows that

$$\gamma_0(\mathfrak{p}^i \wedge \mathfrak{p}^j) = \mathfrak{p}^{i+j+\varepsilon} \quad \text{for } \varepsilon = \begin{cases} 3 & i \equiv j \bmod 3, \\ 2 & \text{otherwise.} \end{cases}$$

Pick $j \geq 7$ and set $T = \mathfrak{p}^{j-3}$, $T_\ell = \mathfrak{p}^{j-3+\ell}$. Then $\gamma_0(T \wedge T) = T_j$, and $\gamma_0(T_j \wedge T) = T_k$ for

$$k = \begin{cases} 2j & 3 \mid j, \\ 2j - 1 & 3 \nmid j. \end{cases}$$

Now pick a unit $c \in \mathcal{O}^\times$ and set $\gamma = c\gamma_0$. For any $m \in \{j, j+1, \dots, k\}$, one defines $T_{j-3,\gamma,m}$ to be the group with underlying set T/T_m and product

$$(x + T_m) * (y + T_m) = (x + y + \tfrac{1}{2}\gamma(x \wedge y)) + T_m.$$

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Finally, one sets $R_{j-3,\gamma,m} = T_{j-3,\gamma,m} \rtimes C$, where $C = \langle\tau\rangle$ has order nine and acts on T via ${}^{\tau}v = \theta v$ for $v \in T$. Note that $|R_{j-3,\gamma,m}| = 3^{m+2}$.

Lemma 8.3. *Let $v, w \in T$.*

(1) *$(v + T_m)^r = rv + T_m$ in $T_{j-3,\gamma,m}$ for all $r \in \mathbb{Z}$.*
(2) *The order three elements of $R_{j-3,\gamma,m}$ are:*

- *elements of the form $(v + T_m)\tau^{3r}$, with $v \in T$ and $r \in \{1, 2\}$; and*
- *elements of the form $v + T_m$ with $v \in T_{m-6}$.*

(3) *If $v + T_m$ has order three, then $\gamma(v \wedge w) \in T_m$ for all $w \in T$. Hence $\Omega_1(T_{j-3,\gamma,m}) \leq Z(T_{j-3,\gamma,m})$.*

Proof. (1) This follows by induction, since $\gamma(v \wedge rv) = r\gamma(v \wedge v) = 0$.

(2) Firstly, $[(v + T_m)\tau^3]^3 = (1 + \theta^3 + \theta^6)v + T_m = 0$. Secondly, $(v + T_m)^3 = 3v + T_m$. This is zero for $v \in 3^{-1}T_m = T_{m-6}$.

(3) Suppose that $v \in T_{m-6}$ and $w \in T$. Then $\gamma(v \wedge w)$ lies in $T_{j+m-9+\varepsilon}$, with $\varepsilon \in \{2, 3\}$. Since $\varepsilon \geq 2$ and $j \geq 7$, this means that $\gamma(v \wedge w)$ lies in T_m. □

Lemma 8.4.

(1) *The orbit of $(v + T_m)\tau^3$ under conjugation by $T_{j-3,\gamma,m}$ is*

$$\{(v' + T_m)\tau^3 \mid v \in v + T_3\}.$$

(2) *$(v + T_m)\tau^3$ and $(w + T_m)\tau^3$ are conjugate in $R_{j-3,\gamma,m}$ if and only if $v + T_3$ and $w + T_3$ lie in the same orbit under the action of C on T/T_3.*
(3) *The action of $R_{j-3,\gamma,m}$ on T_{m-6}/T_m factors through C and coincides with the action of C on T/T_6 via the isomorphism $v + T_6 \mapsto (\theta - 1)^{m-6}v + T_m$.*
(4) *$C_{T_{j-3,\gamma,m}}((v + T_m)\tau^3) = T_{m-3}/T_m$.*

Proof. (1) Since T_m and the image of γ lie in $T_j \subseteq T_7$,

$$\begin{aligned} {}^{(w+T_m)}[(v + T_m)\tau^3] &= [(w + T_m) * (v + T_m) * (-\theta^3 w + T_m)]\tau^3 \\ &\in (v + (1 - \theta^3)w + T_7)\tau^3. \end{aligned}$$

Since $\mathfrak{p} = (\theta - 1)\mathcal{O}$ and $3\mathcal{O} = \mathfrak{p}^6$, it follows that $(1 - \theta^3)T = (1 - \theta)^3 T = \mathfrak{p}^3 T = T_3$.

So, for each $v' \in v + T_3$, we find $w \in T$ with ${}^{(w+T_m)}[(v + T_m)\tau^3] = (v'' + T_m)\tau^3$ and $v'' \in v' + T_7$. If we now adjust w by adding $u \in T_r$, then, since $\gamma(T \wedge T_r) = T_{j+r-3+\varepsilon} \subseteq T_{r+6}$, we alter v' by an element of $(1 - \theta^3)u + T_{r+6}$. So if the error $v'' - v'$ lies in T_{s+3}, then, with one correction, we can reduce to an error in T_{s+6}. Iterating reduces the error to an element of T_m.

(2) This follows from (1).

(3) The action factors, by Lemma 8.3(3). The second statement follows, since C acts as multiplication by θ.

(4) This follows from (3), since T_3/T_6 is the subspace of T/T_6 consisting of elements fixed by τ^3. □

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Let d be the number of orbits for the action of C on T/T_3. One easily verifies that $d = 11$. Pick $v_1, \dots, v_d \in T$ such that the $v_i + T_3$ form a set of orbit representatives for this action.

Lemma 8.5. *Every maximal elementary abelian subgroup of $R_{j-3,\gamma,m}$ is conjugate to precisely one of the following:*

(1) *d rank four groups of the form $V_i = \langle (v_i + T_m)\tau^3 \rangle \times T_{m-3}/T_m$; or*
(2) *$V_0 = T_{m-6}/T_m$ of rank six.*

If $U \le V_i$ is not contained in V_0, then it is not conjugate to a subgroup of any other V_j.

Proof. Any elementary abelian outside T_{m-6}/T_m must contain some element of the form $(v + T_m)\tau^3$ and is therefore contained in $\langle (v + T_m)\tau^3 \rangle \times C_{T_{j-3,\gamma,m}}((v + T_m)\tau^3)$: that is, $\langle (v + T_m)\tau^3 \rangle \times T_{m-3}/T_m$. Since $m \ge j \ge 7$ and, therefore, $m - 3 \ge 3$, no two of the rank four elementary abelians in (1) are conjugate. This argument also demonstrates the last part. □

Theorem 8.6. *Up to equivalence of categories, the Quillen category of the skeleton group $R_{j-3,\gamma,m}$ is independent of j, γ, m.*

Proof. V_0 is a normal subgroup, and Lemma 8.4(3) describes the conjugation action. So, by the last part of Lemma 8.5, it suffices to show that if $U \le V_i$ is not contained in V_0, then the set of homomorphisms $U \to V_i$ lying in the Quillen category is independent of j, γ, m.

So $U = \langle (v + T_m)\tau^3 \rangle \times A$ for some $A \le T_{m-3}/T_m$ and some $v \in v_i + T_{m-3}$. Consider conjugation by $(u + T_m)\tau^r$. By Lemma 8.4, this can only send $(v + T_m)\tau^3$ to an element of V if $\theta^r v_i$ lies in $v_i + T_3$ and, if $\theta^r v_i$ does lie there, then, by adjusting u, we may send $(v + T_m)\tau^3$ to any element of the form $(v' + T_m)\tau^3$ with $v' \in v_i + T_{m-3}$. Moreover, the restriction to A of conjugation by $(u + T_m)\tau^r$ only depends on r. □

8.3. The generalized quaternion groups. Let G be a finite group, and k a field of characteristic p. Write

$$\bar{H}^*(G, k) = \lim_{E \in \mathscr{A}_p(G)} H^*(E, k).$$

Quillen [16, Theorem 6.2] proved that the induced homomorphism

$$\phi_G : H^*(G, k) \to \bar{H}^*(G, k)$$

induces a homeomorphism between prime ideal spectra.

Our result shows that if G_r is a coclass family, then $\bar{H}^*(G_r, k)$ is independent of r. However, this does *not* mean that the map ϕ_{G_r} is an isomorphism for large r. The (generalized) quaternion groups Q_{2^n} $(n \ge 3)$ provide a good example.

The quaternion groups form a coclass sequence. The mod-2 cohomology ring $H^*(Q_{2^n}, \mathbb{F}_2)$ is well known

$$H^*(Q_8, \mathbb{F}_2) \cong \mathbb{F}_2[x, y, z]/(x^2 + xy + y^2, x^2y + xy^2)$$
$$H^*(Q_{2^n}, \mathbb{F}_2) \cong \mathbb{F}_2[x, y, z]/(x^2 + xy, y^3) \quad (n \ge 4),$$

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with $x, y \in H^1$ and $z \in H^4$. To our knowledge, the earliest references are [4, pages 253–254] for the additive structure and [15, page 244] for the ring structure. By 1987, Rusin [17, page 316] could quote the result without needing to give a reference.

Since $H^1(G, \mathbb{F}_2) = \mathrm{Hom}(G, \mathbb{F}_2)$ and all order two elements lie in the Frattini subgroup, it follows that $x, y \in \ker(\phi_{Q_{2^n}})$. In fact, $\bar{H}^*(Q_{2^n}, \mathbb{F}_2) \cong \mathbb{F}_2[z]$, since z restricts to the central C_2 as $t^4 \in H^*(C_2, \mathbb{F}_2) \cong \mathbb{F}_2[t]$ (see Rusin's construction of z as a top Stiefel–Whitney class [17, page 316]). So both $H^*(Q_{2^n}, \mathbb{F}_2)$ and $\bar{H}^*(Q_{2^n}, \mathbb{F}_2)$ are constant for $n \geq 4$, but $\phi_{Q_{2^n}}$ is never injective.

In fact, one can demonstrate that $\phi_{Q_{2^n}}$ is never injective without even knowing the cohomology of Q_{2^n}. Recall that a class $x \in H^n(G, k)$ is called *essential* if its restriction to every proper subgroup $H < G$ vanishes: so if G is not elementary abelian, then every essential class lies in the kernel of ϕ_G. Now, Adem and Karagueuzian showed [1] that $H^*(G, \mathbb{F}_p)$ is Cohen–Macaulay and has nonzero essential elements if and only if G is a p-group and all order p elements are central. As Q_{2^n} satisfies this group-theoretic condition, it follows that $\ker(\phi_{Q_{2^n}}) \neq 0$.

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BETTINA EICK, Institut Computational Mathematics, TU Braunschweig,
38106 Braunschweig, Germany
e-mail: beick@tu-bs.de

DAVID J. GREEN, Institut für Mathematik, Friedrich-Schiller-Universität Jena,
07737 Jena, Germany
e-mail: david.green@uni-jena.de

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