

EXTENSION OF CONTINUOUS MAPPINGS AND H_1 -RETRACTS

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Abstract

We prove that any continuous mapping $f : E \rightarrow Y$ on a completely metrizable subspace E of a perfect paracompact space X can be extended to a Lebesgue class one mapping $g : X \rightarrow Y$ (that is, for every open set V in Y the preimage $g^{-1}(V)$ is an F_σ -set in X) with values in an arbitrary topological space Y .

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1. Introduction

A mapping $f : X \rightarrow Y$ from a topological space X to a topological space Y is called a *Lebesgue class α mapping* (or a *mapping of the α th Lebesgue class*) if for every closed set F in Y the set $f^{-1}(F)$ is of the multiplicative class α in X . The family of all such mappings $f : X \rightarrow Y$ we denote by $H_\alpha(X, Y)$. Moreover, we write $f \in H_1(X, Y)$ if for every open set V in Y the preimage $g^{-1}(V)$ is an F_σ -set in X .

Obviously, if X or Y is a perfect space then any continuous mapping $f : X \rightarrow Y$ is of the first Lebesgue class.

Classification of mappings naturally leads to the problem of the extension of mappings from a subset of a topological space to the whole space with preservation of the mapping class or its estimation. So, such classical results as the Tietze theorem [4, p. 116] or the Dugundji theorem [3] give the possibility of the extension of a continuous mapping to a continuous mapping.

Many mathematicians (F. Hausdorff [8], W. Sierpiński [13], G. Alexits [1], H. Hahn [5], K. Kuratowski [11]) have considered the extension of real-valued functions of some Lebesgue class.

Kuratowski [11] proved that every mapping $f \in H_\alpha(E, Y)$ on a subset E of a metric space X with values in a complete metric separable space Y can be extended to a mapping $g : B \rightarrow Y$ of class α such that the set $B \supseteq E$ is of the multiplicative class $\alpha + 1$. Moreover, if E is of the multiplicative class $\alpha > 0$ then f can be extended to

a Lebesgue mapping of class α on the whole space X . Consequently, every mapping of the Lebesgue class $\alpha \geq 0$ on a set $E \subseteq X$ can be extended to a mapping of the Lebesgue class $\alpha + 1$ on X . In particular, the following result holds.

THEOREM 1.1. *Let X be a metric space, Y be a complete metric separable space and $E \subseteq X$. Then every continuous mapping $f : E \rightarrow Y$ can be extended to a mapping $g \in H_1(X, Y)$.*

It follows from the above that the problem of the extension of a continuous function (Lebesgue class one function) to a continuous function (Lebesgue class one function) essentially differs from the problems of the extension of functions preserving their class. For example, if $X = \mathbb{R}$ and $E = \mathbb{Q}$ then not every continuous function $f : E \rightarrow \mathbb{R}$ can be extended to a continuous function defined on X ; moreover, it is easy to construct an everywhere discontinuous function $f : E \rightarrow \mathbb{R}$ which is of the first Lebesgue class and cannot be extended to a function of the first Lebesgue class on X . On the other hand, Theorem 1.1 implies that every continuous function $f : E \rightarrow \mathbb{R}$ can be extended to a function $g : X \rightarrow \mathbb{R}$ of the first Lebesgue class.

In connection with Theorem 1.1 the following question arises.

QUESTION 1.2. Is it possible to omit the assumption of separability on space Y in Theorem 1.1?

Hansell studied the problem of the extension of Lebesgue mappings with nonseparable metrizable ranges using the notion of σ -discrete mappings as introduced by Stone [14].

Recall that a family \mathcal{A} of subsets of a topological space X is said to be *discrete* if for every point $x \in X$ there exists a neighborhood U which intersects with at most one set from \mathcal{A} .

A family \mathcal{A} is said to be σ -*discrete* if it can be written as a countable union of discrete families.

The family \mathcal{B} of subsets of a topological space X is said to be *a base for a mapping* $f : X \rightarrow Y$ if for every open set V in Y there exists a subfamily $\mathcal{B}_V \subseteq \mathcal{B}$ such that $f^{-1}(V) = \cup \mathcal{B}_V$. If, moreover, the system \mathcal{B} is σ -discrete then it is said to be *a σ -discrete base for f* and f is said to be *a σ -discrete mapping*. The family of all σ -discrete mappings we denote by $\Sigma(X, Y)$.

Obviously, every mapping with a second countable range space is σ -discrete. Also it is easy to see that every continuous mapping with metrizable domain or range is σ -discrete since a metrizable space has a σ -discrete base [4].

The paper of Hansell [6] mentioned above contains the following result.

THEOREM 1.3 [6, Theorem 9]. *Let X be a paracompact space, Y a complete metric space, $E \subseteq X$ and $f : E \rightarrow Y$ a σ -discrete Lebesgue mapping of class α . Then f can be extended to a Lebesgue mapping $g : B \rightarrow Y$ of class α so that the set $B \supseteq E$ is of multiplicative class $\alpha + 1$.*

The following question naturally arises.

QUESTION 1.4. Is it possible to replace the set B in Theorem 1.3 with the whole space X ?

The most recent (to the best of our knowledge) result on the extension of Lebesgue functions is due to Kalenda and Spurný.

THEOREM 1.5 [9, Theorem 29]. *Let E be a Lindelöf subspace of a completely regular space X , Y a complete metric separable space and*

- (i) E be hereditarily Baire or
- (ii) E be G_δ in X .

Then every mapping $f \in H_1(E, Y)$ can be extended to a mapping $g \in H_1(X, Y)$.

At the same time it is interesting to study when we can extend mappings with values in an arbitrary topological space.

In Section 2 we introduce and study the notion of H_1 -retract which is tightly connected with the problem of the extension of continuous mappings to Lebesgue class one mappings with values in an arbitrary topological space (analogously, the notion of a retract is connected with the extension of continuous mappings with preservation of continuity).

Furthermore, in Section 3 we prove that every continuous mapping $f : E \rightarrow Y$ on a completely metrizable subspace E of a perfect paracompact space X with values in an arbitrary topological space Y can be extended to a Lebesgue class one mapping $g : X \rightarrow Y$. This result implies a positive answer to Question 1.2. In addition, we give a negative answer to Question 1.4.

2. H_1 -retracts and their properties

Let X be a topological space and $E \subseteq X$. Recall [2] that a set E is said to be a *retract* of X if there exists a continuous mapping $r : X \rightarrow E$ such that $r(x) = x$ for all $x \in E$. The mapping r is called a *retraction* of X onto E . It is easy to see that a set $E \subseteq X$ is a retract of X if and only if for any topological space Y every continuous mapping $f : E \rightarrow Y$ can be extended to a continuous mapping $g : X \rightarrow Y$.

We call a subset E of a topological space X an H_1 -retract of X if there exists a mapping $r \in H_1(X, E)$ such that $r(x) = x$ for all $x \in E$. We call the mapping r an H_1 -retraction of X onto E .

The following properties of H_1 -retracts immediately follow from the definition.

PROPOSITION 2.1. *Let X be a topological space. A set $E \subseteq X$ is an H_1 -retract of X if and only if for an arbitrary space Y every continuous mapping $f : E \rightarrow Y$ can be extended to a Lebesgue class one mapping $g : X \rightarrow Y$.*

PROPOSITION 2.2. *Let E be an H_1 -retract of a topological space X . Then E is a perfect space.*

A subset A of a topological space X is said to be an *ambiguous set* if A is simultaneously F_σ and G_δ in X .

PROPOSITION 2.3. *Let X be a metrizable space and E be an H_1 -retract of X . Then E is G_δ in X .*

PROOF. Let $r : X \rightarrow E$ be an H_1 -retraction of X onto E . It is easy to see that $E = \{x \in X \mid r(x) = x\}$.

Consider the diagonal $\Delta = \{(x', x'') \in X \times X \mid x' = x''\}$ of the space X^2 and the mapping $h : X \rightarrow X \times X$, $h(x) = (r(x), x)$. Since $r \in H_1(X, E) \subseteq H_1(X, X)$ and the mapping $g : X \rightarrow X$, $g(x) = x$ is continuous, according to [7, Theorem 1] the mapping $h : X \rightarrow X \times X$ is of the first Lebesgue class. Since Δ is closed in $X \times X$, the set $E = h^{-1}(\Delta)$ is G_δ in X . \square

Note that an H_1 -retract may, in general, be even a nonmeasurable set. Moreover, the following example shows that the assumption of metrizability of X in the previous proposition is essential.

EXAMPLE 2.4. There exists a nonmeasurable H_1 -retract E of a perfect separable linear ordered compact space X .

PROOF. Let $X = [0, 1] \times \{0, 1\}$ be endowed with the lexicographic order, that is $(x, i) < (y, j)$ if $x < y$ or $x = y$ and $i < j$, $i, j \in \{0, 1\}$. Note that X satisfies necessary conditions (see [4, p. 318]).

Consider a set $E = \{(x, 0) \mid x \in [0, 1]\}$. A mapping $r : X \rightarrow E$, $r(x, i) = (x, 0)$, is of the first Lebesgue class.

It remains to prove that E is nonmeasurable.

For a set $A \subseteq X$ denote $A^+ = \{x \in [0, 1] \mid (x, 1) \in A\}$ and $A^- = \{x \in [0, 1] \mid (x, 0) \in A\}$. It is not hard to prove that for any open or closed set A in X we have $|A^+ \Delta A^-| \leq \aleph_0$. This implies that $|B^+ \Delta B^-| \leq \aleph_0$ for any measurable set B . But $E^+ = \emptyset$ and $E^- = [0, 1]$. Hence, E is a nonmeasurable set. \square

PROPOSITION 2.5. *Let X and Y be topological spaces, E be an ambiguous subset of X and $f : E \rightarrow Y$ be a Lebesgue class one mapping. Then there exists a Lebesgue class one mapping $g : X \rightarrow Y$ such that $g|_E = f$.*

COROLLARY 2.6. *Let X be a topological space and E be a perfect ambiguous subset of X . Then E is an H_1 -retract of X .*

We call a subset E of a topological space X a Coz_δ -set if there exists a sequence of continuous functions $f_n : X \rightarrow [0, 1]$ such that $E = \bigcap_{n=1}^{\infty} f_n^{-1}((0, 1))$. We call the complement to a Coz_δ -set a Zer_σ -set. We call a set which is simultaneously Coz_δ and Zer_σ a *functionally ambiguous set*.

PROPOSITION 2.7. *Let E_1, \dots, E_n be disjoint H_1 -retracts of topological space X and let E_i be Coz_δ in X for every $i \in \{1, \dots, n\}$. Then the union $E = \bigcup_{i=1}^n E_i$ is an H_1 -retract of X .*

PROOF. First we obtain that for every finite family of disjoint Coz_δ -sets E_1, \dots, E_n there exist disjoint functionally ambiguous sets B_1, \dots, B_n such that $E_i \subseteq B_i$ for every $i \in \{1, \dots, n\}$ and $X = \bigcup_{i=1}^n B_i$.

Let $n = 2$ and E_1, E_2 be disjoint Coz_δ -sets. Then the complements $E_i^c = X \setminus E_i$, $i = 1, 2$, are Zer_σ and $E_1^c \cup E_2^c = X$. From [10, Lemma 3.2] it follows that there exist functionally ambiguous sets B_1 and B_2 such that $B_1^c \subseteq E_1^c$, $B_2^c \subseteq E_2^c$, $B_1^c \cup B_2^c = X$ and $B_1^c \cap B_2^c = \emptyset$. Then $E_1 \subseteq B_1$, $E_2 \subseteq B_2$, $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = X$.

Let $n > 2$ and the assumption holds when we have $n - 1$ sets. There exist disjoint functionally ambiguous sets $\tilde{B}_1, \dots, \tilde{B}_{n-1}$ such that $E_i \subseteq \tilde{B}_i$ if $1 \leq i \leq n - 2$, $E_{n-1} \cup E_n \subseteq \tilde{B}_{n-1}$ and $X = \bigcup_{i=1}^{n-1} \tilde{B}_i$. Moreover, there exist disjoint functionally ambiguous sets C and D such that $E_{n-1} \subseteq C$, $E_n \subseteq D$ and $C \cup D = X$. Set $B_i = \tilde{B}_i$ for $i = 1, \dots, n - 2$, $B_{n-1} = \tilde{B}_{n-1} \cap C$ and $B_n = \tilde{B}_{n-1} \cap D$.

Let $r_i : X \rightarrow E_i$ be H_1 -retractions, $1 \leq i \leq n$. For every $x \in X$ define $r(x) = r_i(x)$ if $x \in B_i$ for some $i \in \{1, \dots, n\}$. Clearly, $r \in H_1(X, E)$ and $r(x) = x$ if $x \in E$. \square

3. Extension of continuous mappings to the first class mappings from completely metrizable subspaces

In this section we prove the main results of this paper. All topological spaces will be considered to be Hausdorff.

We say that a family $\mathcal{A} = (A_i : i \in I)$ of sets A_i refines a family $\mathcal{B} = (B_j : j \in J)$ of sets B_j if for every $i \in I$ there exists $j \in J$ such that $A_i \subseteq B_j$. We write this as $\mathcal{A} \preceq \mathcal{B}$.

LEMMA 3.1. *Let X be a perfect paracompact space and \mathcal{G} be a locally finite cover of X by ambiguous sets. Then there exists a disjoint locally finite cover of X by ambiguous sets which refines \mathcal{G} .*

PROOF. Without loss of generality we may assume that $\mathcal{G} = \{G_\alpha \mid 0 \leq \alpha < \beta\}$, where β is some ordinal.

Denote $A_0 = G_0$. For every $0 < \alpha < \beta$ let $A_\alpha = G_\alpha \setminus \bigcup_{\xi < \alpha} G_\xi$. According to Michael's theorem [4, p. 430], the set $\bigcup_{\xi < \alpha} G_\xi$ is ambiguous as a locally finite union of ambiguous sets. Then the set A_α is also ambiguous. Clearly, the family $\mathcal{A} = (A_\alpha : 0 \leq \alpha < \beta)$ is to be found. \square

The next theorem is the main result of our paper.

THEOREM 3.2. *Let X be a perfect paracompact space and $E \subseteq X$ be a completely metrizable subspace of X . Then E is an H_1 -retract of X .*

PROOF. Let d be a metric on E such that (E, d) is a complete metric space and d induces the topology in E .

For every $n \in \mathbb{N}$ consider a cover \mathcal{V}_n of the set E by open balls with radius $1/(2^{n+2})$. For every ball $V \in \bigcup_{n=1}^\infty \mathcal{V}_n$ choose an open set U_V in X so that $V = E \cap U_V$.

For every $n \geq 1$ let $\mathcal{G}_n = (U_{V_1} \cap \dots \cap U_{V_n} : V_1 \in \mathcal{V}_1, \dots, V_n \in \mathcal{V}_n)$ and $G_n = \bigcup_{G \in \mathcal{G}_n} G$.

Since X is perfect, the set G_n is an F_σ -set in X . It follows from [4, p. 457] that G_n is paracompact space. Then there exists a locally finite in G_n cover \mathcal{U}_n of G_n by open sets in G_n , which refines \mathcal{G}_n . According to Lemma 3.1, there exists a disjoint locally finite in G_n cover of G_n by ambiguous sets in G_n , which refines \mathcal{U}_n . Remove from this cover those sets that do not intersect with E and denote this new system by \mathcal{W}_n . Let $P_n = \bigcup \mathcal{W}_n$. Note that $P_n \subseteq G_n$. Fix an arbitrary set W_n from \mathcal{W}_n and denote $W_n^0 = W_n \cup (X \setminus P_n)$.

Index the elements of the system $\{\mathcal{W}_n^0\} \cup (\mathcal{W}_n \setminus \{W_n\})$ and obtain the family $\mathcal{X}_n = (X_{n,i} : i \in I_n)$.

Constructed in such a way, the sequence $(\mathcal{X}_n)_{n=1}^\infty$ of families \mathcal{X}_n satisfies the following properties:

- (i) $X = \bigcup_{i \in I} X_{n,i}$;
- (ii) $X_{n,i} \cap X_{n,j} = \emptyset, i \neq j$;
- (iii) $X_{n,i} \cap E \neq \emptyset$ for all $i \in I_n$.
- (iv) family $(X_{n,i} \cap G_n : i \in I_n)$ is locally finite in G_n ;
- (v) $|\{i \in I_n \mid X_{n,i} \setminus G_n \neq \emptyset\}| = 1$;
- (vi) $\text{diam}(X_{n,i} \cap E) \leq 1/(2^{n+1})$ for every $i \in I_n$.

Since all the elements of the system \mathcal{W}_n are ambiguous sets in the open subset G_n of a perfect space X , all the elements of system \mathcal{W}_n are ambiguous sets in X . In addition, since \mathcal{W}_n is locally finite in G_n , Michael’s theorem [4, p. 430] implies that P_n is also an ambiguous set in X . This implies that

- (vii) $X_{n,i}$ is ambiguous in X for all $i \in I_n$.

For every $n \in \mathbb{N}$ let

$$E_{n,i} = X_{n,i} \cap E$$

and for all $i_1 \in I_1, \dots, i_n \in I_n$ let

$$B_{i_1 \dots i_n} = E_{1,i_1} \cap E_{2,i_2} \cap \dots \cap E_{n,i_n},$$

$$C_{i_1 \dots i_n} = X_{1,i_1} \cap X_{2,i_2} \cap \dots \cap X_{n,i_n}.$$

Then:

- (1) $E = \bigcup_{i_1 \in I_1, \dots, i_n \in I_n} B_{i_1 \dots i_n}$ and $X = \bigcup_{i_1 \in I_1, \dots, i_n \in I_n} C_{i_1 \dots i_n}$ for every $n \in \mathbb{N}$;
- (2) $B_{i_1 \dots i_n} \cap B_{j_1 \dots j_n} = \emptyset$ and $C_{i_1 \dots i_n} \cap C_{j_1 \dots j_n} = \emptyset$ if $(i_1, \dots, i_n) \neq (j_1, \dots, j_n)$;
- (3) if $m \geq n$ and $C_{i_1 \dots i_n} \cap C_{j_1 \dots j_m} \neq \emptyset$ then $i_1 = j_1, \dots, i_n = j_n$;
- (4) $C_{i_1 \dots i_n} \cap E = B_{i_1 \dots i_n}$ for every $n \in \mathbb{N}$ and $i_1 \in I_1, i_2 \in I_2, \dots, i_n \in I_n$;
- (5) $B_{i_1 \dots i_n}$ is an ambiguous set in E and $C_{i_1 \dots i_n}$ is an ambiguous set in X for all n and $i_1 \in I_1, i_2 \in I_2, \dots, i_n \in I_n$.

Moreover,

- (6) for every $n \in \mathbb{N}$ and any set $I' \subseteq I_1 \times \dots \times I_n$ the set $A = \bigcup_{(i_1, \dots, i_n) \in I'} C_{i_1 \dots i_n}$ is ambiguous in X .

According to (iv), for every $k \in \mathbb{N}$ the family $(X_{k,i} \cap G_k : i \in I_k)$ is locally finite in G_k , and (v) implies that the family $(X_{k,i} \cap (X \setminus G_k) : i \in I_k)$ is locally finite in $X \setminus G_k$. Therefore, taking into consideration the fact that the sequence $(G_n)_{n=1}^\infty$ decreases, we obtain that for an arbitrary set

$$D \in \{X \setminus G_1, G_1 \setminus G_2, G_2 \setminus G_3, \dots, G_{n-1} \setminus G_n, G_n\} = \{D_0, \dots, D_n\}$$

and for every $k \in \{1, 2, \dots, n\}$ the family $(X_{k,i} \cap D : i \in I_k)$ is locally finite in D . Then we have that the family $(C_{i_1, \dots, i_n} : i_1 \in I_1, \dots, i_n \in I_n)$ is also locally finite in D . Hence, the family $(C_{i_1, \dots, i_n} \cap D : (i_1, \dots, i_n) \in I')$ is locally finite in D . Furthermore, since all the sets D_0, \dots, D_n are ambiguous in X and (5) holds, all the sets $A_k = \bigcup_{(i_1, \dots, i_n) \in I'} C_{i_1, \dots, i_n} \cap D_k$ are ambiguous in X and $A = \bigcup_{k=0}^n A_k$ is an ambiguous set in X .

For every n and $i \in I_n$ choose an arbitrary point $y_{n,i} \in E_{n,i}$. For every $x \in E$ let $\psi_n(x) = y_{n,i_n}$ if $x \in B_{i_1 \dots i_n}$. Note that, according to (1) and (2), mappings $\psi_n : E \rightarrow E$ are correctly defined. We now show that the sequence $(\psi_n)_{n=1}^\infty$ uniformly converges to the identical mapping $\psi : E \rightarrow E, \psi(x) = x$.

Fix $x \in E$ and $n \in \mathbb{N}$. Then there exist $i_1 \in I_1, \dots, i_n \in I_n$ such that $x \in B_{i_1 \dots i_n}$. Then $\psi_n(x) = y_{n,i_n}$. Since $B_{i_1 \dots i_n} \subseteq E_{n,i_n}, x \in E_{n,i_n}$ and $y_{n,i_n} \in E_{n,i_n}$. According to (vi), we have that $\text{diam} E_{n,i_n} \leq 1/(2^{n+1})$. Then

$$d(\psi(x), \psi_n(x)) = d(x, y_{n,i_n}) \leq \frac{1}{2^{n+1}}.$$

Note that

$$d(\psi_m(x), \psi_n(x)) \leq \frac{1}{2^{m+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n} \quad \text{for all } m \geq n \text{ and } x \in E. \quad (*)$$

For every n and multi-index $(i_1 \dots i_n) \subseteq I_1 \times \dots \times I_n$ denote

$$\ell(i_1 \dots i_n) = \max\{1 \leq k \leq n \mid B_{i_1 \dots i_k} \neq \emptyset\}.$$

For all $n \in \mathbb{N}$ and $x \in X$ let $r_n(x) = y_{\ell(i_1 \dots i_n), i_{\ell(i_1 \dots i_n)}}$ if $x \in C_{i_1 \dots i_n}$. Properties (1) and (2) imply that all the mappings $r_n : X \rightarrow E$ are correctly defined.

We now prove that the sequence $(r_n)_{n=1}^\infty$ satisfies inequality (*) for all $x \in X$.

Let $x_0 \in X$ and $m \geq n$. Then there exist $i_1 \in I_1, \dots, i_n \in I_n$ and $j_1 \in I_1, \dots, j_m \in I_m$ such that $x_0 \in C_{i_1 \dots i_n} \cap C_{j_1 \dots j_m}$. Property (3) implies that $i_1 = j_1, \dots, i_n = j_n$.

If $B_{i_1 \dots i_n} \neq \emptyset$ then $r_n(x_0) = y_{n,i_n}$. Let $k = \ell(j_1 \dots j_m)$. Then $r_m(x_0) = y_{k,j_k}$. Clearly, $k \geq n$. Choose any point $x \in B_{j_1 \dots j_k}$. Since $B_{j_1 \dots j_k} = B_{i_1 \dots i_n j_{n+1} \dots j_k} \subseteq B_{i_1 \dots i_n}$, $\psi_n(x) = y_{n,i_n}$ and $\psi_k(x) = y_{k,j_k}$. Inequality (*) implies that

$$d(r_n(x_0), r_m(x_0)) = d(y_{n,i_n}, y_{k,j_k}) = d(\psi_n(x), \psi_k(x)) < \frac{1}{2^n}.$$

If $B_{i_1 \dots i_n} = \emptyset$ then $\ell(i_1 \dots i_n) = \ell(j_1 \dots j_m)$. Now we have that $r_n(x_0) = r_m(x_0)$ and $d(r_n(x_0), r_m(x_0)) = 0$. Hence, sequence $(r_n)_{n=1}^\infty$ satisfies (*) for all $x \in X$.

Since X is a completely metrizable space, there exists a mapping $r : X \rightarrow E$ such that the sequence $(r_n)_{n=1}^\infty$ uniformly converges to r on X . Moreover, since $r_n|_E = \psi_n$ and $\psi(x) = \lim_{n \rightarrow \infty} \psi_n(x)$ for all $x \in E$, we have that $r|_E = \psi$, that is $r(x) = x$ for every $x \in E$.

Since a uniform limit of the Lebesgue class one mappings is a Lebesgue class one mapping [12, p. 395], it remains to prove that $r_n \in H_1(X, E)$ for all $n \in \mathbb{N}$.

Since for any $n \in \mathbb{N}$ and $i_1, \dots, i_n \in I_1 \times \dots \times I_n$ such that $C_{i_1, \dots, i_n} \neq \emptyset$ the mapping $r_n|_{C_{i_1, \dots, i_n}}$ is constant, we have that for an arbitrary set $B \subseteq E$

$$r_n^{-1}(B) = \bigcup_{(i_1, \dots, i_n) \in I'} C_{i_1, \dots, i_n},$$

where $I' = \{(i_1, \dots, i_n) \in I_1 \times \dots \times I_n \mid r_n(C_{i_1, \dots, i_n}) \subseteq B\}$. Therefore, according to (6), $r_n^{-1}(B)$ is an ambiguous set in X . In particular, all the mappings r_n are of the first Lebesgue class. □

COROLLARY 3.3. *Let X be a completely metrizable space and $E \subseteq X$. The set E is an H_1 -retract of X if and only if E is G_δ in X .*

PROOF. *Sufficiency.* This immediately follows from 2.3.

Necessity. According to the Aleksandrov–Hausdorff theorem [4, p. 407], the space E is completely metrizable. Hence, Theorem 3.2 implies that E is an H_1 -retract of X . □

The following corollary gives a positive answer to Question 1.2.

COROLLARY 3.4. *Let X be a metrizable space, Y be a completely metrizable space, $A \subseteq X$ and $f : A \rightarrow Y$ be a continuous mapping. Then there exists a Lebesgue class one mapping $g : X \rightarrow Y$ such that $g|_A = f$.*

PROOF. Denote by \widehat{X} the completion of X . According to [4, p. 405], there exists a G_δ -subset \widehat{A} of \widehat{X} and a continuous mapping $h : \widehat{A} \rightarrow Y$ such that $A \subseteq \widehat{A}$ and $h|_A = f$. According to Corollary 3.3, \widehat{A} is an H_1 -retract of \widehat{X} . Then there exists a mapping $\widehat{h} \in H_1(\widehat{X}, Y)$ such that $\widehat{h}|_{\widehat{A}} = h$.

Let $g = \widehat{h}|_X$. Then $g : X \rightarrow Y$ is the desired extension of f . □

Since every completely metrizable separable space is hereditarily Baire and Lindelöf, the result of Kalenda and Spurný implies the following fact.

THEOREM 3.5. *Let E be a completely metrizable separable subspace of a completely regular space X . Then E is an H_1 -retract of X .*

At first sight this theorem gives a solution to the problem of the extension of a continuous mapping to a mapping of the first Lebesgue class with values in an arbitrary (not necessary separable) topological space, analogously to as in Theorem 3.2. However, since a continuous image of a separable space E is also separable, in fact

separability of Y is present here imperceptibly and we cannot obtain Corollary 3.4 from Theorem 3.5.

The following example shows that the assumption that X is perfect in Theorem 3.2 and the assumption that E is separable in Theorem 3.5 cannot be omitted. Moreover, this example gives a negative answer to Question 1.4.

EXAMPLE 3.6. There exist a completely metrizable subspace E of a compact space X and a continuous function $f : E \rightarrow [0, 1]$ which cannot be extended to a Lebesgue class one function on X .

PROOF. Let E be an uncountable discrete space and $X = \alpha E = E \cup \{\infty\}$ be the Aleksandrov compactification of E .

Choose two uncountable disjoint subsets E_1 and E_2 of E so that $E = E_1 \sqcup E_2$ and consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in E_1, \\ 0 & \text{if } x \in E_2. \end{cases}$$

The function $f : E \rightarrow [0, 1]$ is continuous and hence a σ -discrete function of the first Lebesgue class.

Note that for every continuous function (and for every Baire one function that is a pointwise limit of continuous functions) $g : X \rightarrow [0, 1]$ there exists at most countable set $X_0 \subseteq X$ such that $g(x) = g(\infty)$ for all $x \in X \setminus X_0$. It follows that a function f cannot be extended to a Baire one function $g : X \rightarrow [0, 1]$, provided E_1 and E_2 are uncountable sets.

According to [15, Theorem 3.7], the class $H_1(X, [0, 1])$ coincides with the class of all Baire one functions $g : X \rightarrow [0, 1]$. Therefore, the function f cannot be extended to a Lebesgue class one function on X . \square

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