Canad. J. Math. Vol. **52** (1), 2000 pp. 119–122

Corrigendum to "Spectral Theory for the Neumann Laplacian on Planar Domains with Horn-Like Ends"

Julian Edward

Abstract. Errors to a previous paper (Canad. J. Math. (2) **49**(1997), 232–262) are corrected. A non-standard regularisation of the auxiliary operator A appearing in Mourre theory is used.

The purpose of this note is to correct errors in the paper "Spectral Theory for the Neumann Laplacian on planar domains with horn-like ends" [2].

The argument used in [2] was an application of the Mourre Method as presented in [1]. For the notation that follows, the reader is referred to [2] and [1]. Recall that the planar domain in question is transformed to a connected domain Ω consisting of a union of an open set with compact closure and the end $\{(r,s), r \in (0,\infty), s \in (-1,1)\}$. The Mourre Method requires the existence of operators H, H_0, and A acting on $L^2(\Omega)$. In [2], the operators H, H_0 are second order differential operators on Ω, with associated mixed boundary conditions on the end of the form

$$(\partial u/\partial s + a_{\pm} u)|_{s=\pm 1} = 0, \tag{1}$$

with $a_{\pm} = a(r, \pm 1)$ vanishing as $r \to \infty$. The auxiliary operator used for the Mourre theory in [2] was the following:

Definition 1 $A = Pr\chi_R D_r + D_r \chi_R r P$.

The argument used in [2] fails for the technical reason that the mixed boundary condition in Eq. 1 is not invariant under differentiation in r. One consequence of this is that the following part of Mourre Hypothesis 3 fails:

$$D(A) \cap D(H_0 A) \text{ is a core for } H_0. \tag{2}$$

Because of this, the regularisation A_λ of the operator A used in [2] no longer satisfies Lemma 4.5 of that book. (This lemma is used in the proofs of both the non-accumulation of eigenvalues and the absence of singular continuous spectrum).

Furthermore, it was recently noted in [5] that the hypotheses presented in [1] are insufficient to support the conclusion of Lemma 4.5 of [1].

Another consequence of the non-invariance of boundary conditions is that there is some difficulty in defining the composition HA. This difficulty was overlooked in the arguments used in [2] to obtain estimates for $[H, A]$.

Received by the editors May 20, 1999.
AMS subject classification: 35P25, 58G25, 47F05.

https://doi.org/10.4153/CJM-2000-005-x Published online by Cambridge University Press

To resolve these difficulties, first we redefine A as follows. The new definition of A is:

Definition 2 $A = Pr\chi_R D_r P + PD_r \chi_R r P$.

Since P will map smooth functions of bounded support into $D(H) = D(H_0)$, the same now holds for A, so Eq. 2 is now satisfied. Furthermore HA is well defined when applied to smooth functions of bounded support, so (noting Lemma 8 in [2]) the proofs of Mourre Hypotheses 1–4 and the Mourre Estimate appearing in [2, pp. 253–258] are easily adapted to the new choice of A.

To address the concerns raised in [5], we define the following "regularisation" of A, which was also used by this author in [3]. This regularisation also applies in a similar geometric setting in [4], where the main results were also obtained using the Mourre theory of [1].

Let τ be a compactly supported, smooth function on $[0, \infty)$ such that $\tau = 1$ in a neighbourhood of 0. For fixed $T > 0$, we define a cutoff function on the end by

$$\tau_T(r, s) = \tau\left(\frac{r}{T + R}\right);$$

the function is then extended to all Ω by setting $\tau_T = 1$ on the compact part of Ω. (Here R is the positive constant defined in [2] which also appears in the definitions above). Define A_T by

$$A_T = PD_r r\chi_R \tau_T P + P\tau_T r\chi_R D_r P.$$

Lemma 1

A) For $u \in D(A)$, $A_T u \to Au$ in W^0 as $T \to \infty$.
B) A_T is a bounded mapping from W^s to W^{s-1} for $s \in [-1, 2]$, $\forall T > 0$.

Proof Part A is clear. To prove part B, note first that by Lemmas 3 and 8 in [2], D_r is a bounded map from W^s to W^{s-1}, $s \in [-1, 2]$, while P is bounded on W^j. Thus it suffices to show that $\chi_R r\tau_T$ is bounded on W^s, $s \in [-2, 2]$. Note first that since $\chi_R r\tau_T$ depends only on r, multiplication by this function preserves the boundary conditions associated with W^2. It is now an exercise in differentiation to show that $\chi_R r\tau_T$ is a bounded map from W^2 to W^2. By duality, we then have boundedness on W^{-2}, and by interpolation on W^s, $s \in [-2, 2]$.

The following proposition is now a straightforward calculation:

Proposition 2

A: $[H, A_T]$ is a bounded map from W^2 to W^{-1}, with bounds uniform in T. Furthermore, as $T \to \infty$, $[H, A_T] \to [H, A]$ strongly as a mapping from W^2 to W^{-1}.
B: $\big[[H, A_T], A_T\big]$ is bounded as a map from W^2 to W^{-2}, with bounds uniform in T. Furthermore, as $T \to \infty$, $\big[[H, A_T], A_T\big] \to \big[[H, A], A\big]$ strongly as a mapping from W^2 to W^{-1}.

Because of Lemma 1 and Proposition 1, the arguments in [1, pp. 66–74] will now carry through in our setting, with A_T playing the role of A_λ. Some details are given in the Appendix. Theorem 1 in [2] now follows arguing exactly as in [2].

1 Appendix

In this section, we indicate the modifications necessary for the arguments in [1, pp. 66–74]. The remarks below should be read with a copy of [1] in hand.

Theorem 4.6 The argument carries through easily with A_λ replaced by A_T. We then let $T \to \infty$ and use Proposition 1A of this paper.

Lemma 4.12 The integral expression for $[e^{itH}, A_T]$ along with Proposition 1A prove that

$$\|[A_T, e^{itH}]\| \le Ct$$

with C independant of T. The subsequent bound on $[A_T, g(H)]$ follows by the book's arguments. Note that all equations in the argument can be viewed as operator equations (as opposed to quadratic form equations) because $A_T \colon W^j \to W^{j-1}$.

The estimates for $[A_T, (H+i)^{-1}]$ follow easily from Prop. 1A.

Arguing as in the book, we arrive at the estimate

$$\|[A_T, f(H)]\|_{-1,1} \le C \tag{3}$$

with C independant of T. Using a quadratic form argument, the lemma now follows by letting $T \to \infty$. However, all that will really be needed later is Equation 3 above.

Lemma 4.13 One first estimates $[A_T, M^2]$ and then let $T \to \infty$. Again, all that is really necessary will be the uniform bound on $[A_T, M^2]$.

Lemma 4.14 This lemma carries through with no modification.

Lemma 4.15 In the analysis of the term Q_3, it is again necessary to get the desired estimates with A_T, and then let $T \to \infty$. Again we use Prop. 1A, B, of this paper.

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Department of Mathematics
Florida International University
Miami, FL 33199
USA
email: edwardj@fiu.edu