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CHEN'S PROBLEM ON MIXED FOLIATE CR-SUBMANIFOLDS

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We prove that the simply connected compact mixed foliate CR-submanifold in a hyperbolic complex space form is either a complex submanifold or a totally real submanifold. This is the problem posed by Chen.

1. PRELIMINARIES

Let $\overline{M}$ be an m-dimensional hyperbolic complex space form, that is a Kaehler manifold of constant holomorphic sectional curvature-4. The curvature tensor $\overline{R}$ of $\overline{M}$ is given by

$$\overline{R}(X, Y)Z = -\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\} \tag{1.1}$$

where J is the almost complex structure on $\overline{M}$ and g is the Hermitian metric.

A $(2p+q)$-dimensional submanifold M of $\overline{M}$ is called a CR-*submanifold* if there exists a pair of orthogonal complementary distributions D and $D^\perp$ such that $JD = D$ and $JD^\perp \subset \nu$ where ν is the normal bundle of M and $\dim D = 2p$, $\dim D^\perp = q$[1]. A CR-submanifold is said to be proper if neither $D = \{0\}$ nor $D^\perp = \{0\}$. We shall denote by $\overline{\nabla}$ ∇, $\nabla^\perp$ the Riemannian connections on $\overline{M}$, M and the normal bundle respectively. They are related by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)\,, \overline{\nabla}_X N = -A_N X + \nabla^\perp_X N,\ N \in \nu \tag{1.2}$$

where $h(X, Y)$ and $A_N X$ are the second fundamental forms satisfying

$$g(h(X, Y), N) = g(A_N X, Y).$$

Denote by R and $R^\perp$ the curvature tensors associated with ∇ and $\nabla^\perp$ respectively. Then the equations of Codazzi and Ricci are given by

$$\overline{R}(X, Y, Z, N) = g\big((\overline{\nabla}_X h(Y, Z)) - (\overline{\nabla}_Y h)(X, Z), N\big) \tag{1.3}$$

$$\overline{R}(X, Y; N, N') = R^\perp(X, Y; N, N') - g([A_N, A_{N'}](X), Y) \tag{1.4}$$

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where $(\overline{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y)$ $X, Y, Z \in *(M)$ and $N, N' \in \nu$.

A CR-submanifold is said to be *mixed foliate* if

(i) D is integrable and

(ii) $h(X, Y) = 0$, $X \in D$, $Y \in D^{\perp}$.

For a mixed foliate CR-submanifold the following hold (see [**2**])

(1.5) $\quad h(X, JY) = h(JX, Y)$ for $X, Y \in D$;

(1.6) $\quad A_N X \in D$ for $X \in D$; and $A_N X \in D^{\perp}$ for $X \in D^{\perp}$;

(1.7) $\quad A_N JX = -JA_N X$ for $X \in D$;

(1.8) $\quad A_{JX} Y = A_{JY} X$ for $X, Y \in D^{\perp}$;

(1.9) $\quad \nabla_X Y \in D^{\perp}$ for $X, Y \in D^{\perp}$; $\nabla_X Y \in D$ for $X \in D^{\perp}$, $Y \in D$;
and $\nabla^{\perp}_X JY \in JD^{\perp}$ for $X \in D$, $Y \in D^{\perp}$.

The normal bundle splits as $\nu = JD \oplus \mu$ where μ is a J-invariant sub-bundle of ν.

2. Main Theorem

We shall prove the following theorem which was proved by Chen [**2**] for the case $\dim M \leqslant 5$ and was conjectured for the general case.

THEOREM. *Let M be a simply connected compact mixed foliate CR-submanifold of a hyperbolic complex space form $\overline{M}(-4)$. Then M is either a complex submanifold or a totally real submanifold of $\overline{M}$.*

First we prove the following

LEMMA. *Let M be a proper mixed foliate CR-submanifold of a hyperbolic complex space form $\overline{M}(-4)$. Then for $X, Y \in D$, $h(X, Y) \in JD^{\perp}$.*

PROOF: For $X, Y \in D$ and $Z \in D^{\perp}$ it follows from (1.1) that $[\overline{R}(X, Y)Z]^{\perp} = -2g(X, JY)JZ$. Using this and $X, Y \in D$ in the Codazzi equation (1.3) we get

$$-2g(X, JY)JZ = h(X, \nabla_Y Z) - h(Y, \nabla_X Z). \tag{2.1}$$

Taking the inner product with $JW \in JD^{\perp}$ and replacing X by JX we have, using (1.2), (1.6) and (1.7), that

$$\begin{aligned} -2g(JX, JY)g(JZ, JW) &= g(h(JX, \nabla_Y Z), JW) - g(h(Y, \nabla_{JX} Z), JW) \\ &= g(A_{JW} JX, \overline{\nabla}_Y Z) - g(A_{JW} Y, \overline{\nabla}_{JX} Z) \\ &= g(A_{JW} X, \overline{\nabla}_Y JZ) - g(A_{JW} Y, \overline{\nabla}_{JX} Z) \\ &= g(A_{JW} X, -A_{JZ} Y) - g(JA_{JW} Y, -(A_{JZ} JX)) \\ &= -g(A_{JW} X, A_{JZ} Y) - g(A_{JW} Y, A_{JZ} X). \end{aligned}$$

Thus we get:

$$2g(X,\, Y)g(Z,\, W) = g(A_{JW}X,\, A_{JZ}Y) + g(A_{JW}Y,\, A_{JZ}X). \tag{2.2}$$

From (1.9) we get $R^{\perp}(X,\, Y)JZ \in JD^{\perp}$ for $X, Y \in D$. Thus for $N \in \mu$, the Ricci equation (1.4) and equation (1.1) gives

$$g([A_{JZ},\, A_N](X),\, Y) = 0. \tag{2.3}$$

Taking the inner product in (2.1) with $N \in \mu$ and using similar techniques to those used in (2.2) we get

$$g(A_N X,\, A_{JZ}Y) + g(A_{JZ}X,\, A_N Y) = 0. \tag{2.4}$$

Combining (2.3) and (2.4) we get

$$A_{JZ}(A_N X) = 0 \quad \text{for } X \in D.$$

Replacing X by $A_N X$ in (2.2) and using the above equation we get

$$2g(A_N X,\, Y)g(Z,\, W) = 0.$$

Now suppose that M is a proper CR-submanifold. Then the above equation gives $g(h(X,\, Y),\, N) = 0$, that is $h(X,\, Y) \in JD^{\perp}$ for $X, Y \in D$. ∎

PROOF OF THE THEOREM: Let M be a proper mixed foliate CR-submanifold of $\overline{M}(-4)$. For any unit vector $Z \in D^{\perp}$ and $X \in D$ we easily get from (1.6) and (2.2) that $A_{JZ}^2 X = X$. Now take a non-zero vector $Y_0 \in D$ and consider the set $\{Jh(X,\, Y_0) \colon X \in D\}$. By the above lemma, this is a subset of $D^{\perp}$. We claim that $D^{\perp} = \{Jh(X,\, Y_0) \colon X \in D\}$, for if not then there exists a unit vector $Z \in D^{\perp}$ such that $g(Jh(X,\, Y_0),\, Z) = 0$, $X \in D$. This gives $g(A_{JZ}X,\, Y_0) = 0$. In particular for $X = A_{JZ}Y_0$, we get $g\big(A_{JZ}^2 Y_0,\, Y_0\big) = g(Y_0,\, Y_0) = 0$, a contradiction, and hence our claim is established. We can then define the linear map;
$T_{Y_0} \colon D_p \xrightarrow{\text{onto}} D_p^{\perp}$ for $p \in M$ given by $T_{Y_0}X = Jh(X,\, Y_0)$. Now we prove that $\dim\ker T_{Y_o} = 2p - 1$. For $X \perp Y_0$ we get $0 = g(Y_0,\, X) = g\big(Y_0,\, A_{JZ}^2 X\big) = g(-Jh(A_{JZ}X,\, Y_0),\, X)$, which implies $Jh(A_{JZ}X,\, Y_0) = 0$, that is, $A_{JZ}X \in \ker T_{Y_0}$. Also with the same argument we would get $A_{JZ}Y_0 \notin \ker T_{Y_0}$. Since $A_{JZ}^2 \mid_D = I$, $A_{JZ} \mid_D$ is non-singular, and thus $\dim\ker T_{Y_0} = 2p - 1$. Then using the fundamental theorem of homomorphism we get $\dim D^{\perp} = 1$.

Now let ξ be the unit vector in $D^{\perp}$ which is globally defined on M. We shall prove that its dual 1-form η is closed. Note that for every vector field V on M, we can write

$$JV = PV + \eta(V)J\xi \quad \text{where } PV \in D.$$

Then using (1.2) in $\overline{\nabla}_X JY = J\overline{\nabla}_X Y$ and equating the $JD^\perp$ components we get

$$(\nabla_X \eta)(Y)J\xi = -h(X, PY)$$

where we have used the definition of mixed foliate CR-submanifold and $h(X, Y) \in JD^\perp$ for $X, Y \in D$.

Now using (1.5) we get $d\eta = 0$. Since $\eta(\xi) = 1$ and M is compact, η cannot be exact. Thus $[\eta]$ is a non-trivial cohomology class in the cohomology group $H^1(M, \mathbf{R})$. This is contradiction to the simply connectedness of M. Hence M cannot be proper and this completes the proof of the theorem $\square$

References

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