J. Austral. Math. Soc. (Series A) 32 (1982), 114–128

FINITE INVERSE PERFECT SEMIGROUPS AND THEIR CONGRUENCES

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(Received 18 June 1980; revised 2 February 1981)

Communicated by T. E. Hall

Abstract

In this paper we characterize the structure of finite inverse perfect semigroups and study congruences on those semigroups, in particular we study those semigroups that have modular lattice of congruences.

1980 *Mathematics subject classification* (*Amer. Math. Soc*): 20 M 10.

0. Introduction

Let S be a semigroup and ρ a congruence on S. If $a \in S$, $a\rho$ denotes the ρ-class containing a. Following Vagner (1968), a congruence ρ is called *perfect* if $(a\rho)(b\rho) = (ab)\rho$ for all $a, b \in S$, as sets. A semigroup S is called perfect if all congruences on S are perfect. Groups are familiar examples of perfect semigroups. Perfect semigroups were studied by Fortunatov (1970, 1972, 1974, 1977). In this paper we completely characterize the structure of finite inverse perfect semigroups and study congruences on those semigroups, especially we study those that have modular lattice of congruences. In Section 1 we determine the structure of finite inverse perfect semigroups. There are two types: one type is a chain of groups, the other type is a chain of groups with a Brandt semigroup as an ideal component. The results for type 1 are due to Fortunatov (1972). In Section 2 we consider their construction and the isomorphism conditions. We discuss how to obtain their congruences in Section 3. Unlike groups, finite inverse semigroups do not all have modular lattices of congruences even if they are perfect. In Section 4 we completely determine finite inverse perfect semigroups with modular lattice of congruences. For preliminary knowledge about inverse semigroups, completely simple semigroups, ideal extensions, semilattice of groups, the reader is referred to Clifford and Preston (1961) and Petrich (1973).

1. Structure

LEMMA 1.1. *Let* $S = \bigcup_{i \in \Gamma} S_i$ *be the greatest semilattice decomposition of* S. *If* S *is an inverse semigroup, each* S_i *is a semilattice-indecomposable inverse semigroup.*

PROOF. By Tamura (1956) or Petrich (1973), each S_i is semilattice-indecomposable. We can easily see S_i is an inverse semigroup.

LEMMA 1.2. (Fortunatov (1972)) *Perfectness is preserved under homomorphic images.*

LEMMA 1.3. (Fortunatov (1972)) *A (lower) semilattice* S *is perfect if and only if it is a chain.*

LEMMA 1.4. *Let* S *be a perfect semigroup. If* I *is an ideal of* S *and* $I \neq S$, $I \neq \{0\}$, *then* I *is a completely prime ideal of* S.

PROOF. Suppose $x, y \in S \setminus I$ and $xy \in I$. Since $\{x\}\{y\} \neq I$, S is not perfect. Hence $S \setminus I$ is a subsemigroup of I.

Let S be a finite inverse semigroup and $S = \bigcup_{i \in \Gamma} S_i$ be the greatest semilattice decomposition of S. Assume S is perfect. By Lemmas 1.2 and 1.3, or by Fortunatov (1972), Γ is a chain, say $\Gamma = \{0, 1, \ldots, n\}$ with the usual ordering.

LEMMA 1.5. S_i *is a group for all* $i \in \Gamma$, $i \neq 0$, *and* S_0 *is either a group or a Brandt semigroup.*

PROOF. Suppose S_i, $i > 0$, is not simple, let $J \neq S_i$ be an ideal of S_i (namely, $1 \leqslant |J| < |S_i|$). Then $\bar{I} = \bigcup_{j<i} S_j \cup S_i J S_i$ is an ideal of S, and $\bar{I} \neq S$. By Lemma 1.4 $\bar{I}$ is a completely prime ideal of S, hence J is a completely prime ideal of S_i. This contradicts semilattice-indecomposability of S_i. Thus S_i is completely simple because of finiteness. Since S_i is an inverse semigroup, S_i is a

group for all $i \in \Gamma$, $i \neq 0$. In case S has no zero, let K be the minimal ideal of S. In case S has zero, let K be a 0-minimal ideal of S with $K \cap S_0 \neq \emptyset$. In both cases $K \subseteq S_0$. Suppose $K \neq S_0$. By the same argument as $\bar{I}$ above, K is a completely prime ideal of S_0, a contradiction. Hence $K = S_0$. Since S_0 is a completely [0 −] simple inverse semigroup, we conclude that S_0 is either a group or a Brandt semigroup.

In this paper a Brandt semigroup whose ground group is G and whose sandwich matrix is the $m \times m$ identity matrix is denoted by $\mathfrak{M}^0(m, G)$. Since S_0 is semilattice-indecomposable, we adopt the convention that a Brandt semigroup $\mathfrak{M}^0(m, G)$ satisfies $m > 1$. The universal relation and the equality relation are respectively denoted by

$$\omega(X) = X \times X, \qquad \iota(X) = \{(x, x): x \in X\}.$$

Let $\Omega_r(S_j)$ denote the semigroup of right translations of S_j which are linked with some left translations of S_j. The system of homomorphisms used in the definition of the binary operation on $\bigcup_{i=0}^{n} S_i$ are given by

$$\begin{aligned} i > j \quad \varphi_{ij}&: S_i \to \Omega_r(S_j), \quad p \to p\varphi_{ij}, \quad p \in S_i \\ p\varphi_{ij}&: S_j \to S_j, \qquad x \to x(p\varphi_{ij}), \quad x \in S_j \end{aligned}$$

with a dual action for $\psi_{ij} \in \Omega_l(S_j)$, the semigroup of left translations of S_j. If $x \in S_i$ and $y \in S_j$ for $i \geqslant j$ then $yx = y(x\varphi_{ij})$ and $xy = (\psi_{ij}x)y$. In case $i, j > 0$, S_i and S_j are groups and $p\varphi_{ij}$ is interpreted as an element of S_j since $p\varphi_{ij}$ is an inner right translation of S_j, and hence $\varphi_{ij}: S_i \to S_j$ is a homomorphism of S_i into S_j. For $\bigcup_{i=1}^{n} S_i$ the set $\{\varphi_{ij}: i > j\}$ is the so-called transitive system, namely, it satisfies

(i) φ_{ii} is the identical map of S_i,

(ii) $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$, $i \geqslant j \geqslant k$.

However condition (ii) will be extended to S_0 even if S_0 is not a group (see Lemma 2.2). Since S_i ($i = 0, 1, \ldots$) is left reductive, for $p \in S_i$ and $x \in S_j$, there is a unique $z \in S_i$ such that

$$(p\varphi_{ij})(x\varphi_{jj}) = z\varphi_{jj} \quad \text{and} \quad z = (\psi_{ij}p)x.$$

Thus ψ_{ij} is determined by φ_{ij}.

LEMMA 1.6. *$S = \bigcup_{i=0}^{n} S_i$ is composed by permutations.*

PROOF. S_0 is an ideal of S. By perfectness, $S_0 p = S_0$ hence $p\varphi_{i0}$ is a permutation on S_0 for all $p \in S_i$ for all $i \neq 0$, since S_0 is finite. Obviously $p\varphi_{ij}$ is a permutation for all $p \in S_i$, all $i > j > 0$.

LEMMA 1.7. (Fortunatov (1972)) *For all $i > j > 0$ (for all $i > j \geqslant 0$ if S_0 is a group), φ_{ij} is surjective.*

S is called type 1 if S_0 is a group; type 2 if S_0 is a Brandt semigroup.

We want to prove that if $S = \bigcup_{i \in \Gamma} S_i$ satisfies Lemmas 1.5, 1.6 and 1.7 then S is perfect. This has been proved for type 1 semigroups by Fortunatov (1972), so we need only prove it for type 2 semigroups.

Let $S_0 = \mathfrak{M}^0(m, G) = \{0\} \cup \{(k, x, l): x \in G, k, l = 1, \ldots, m\}$ where G is a finite group and $(k_1, x, l_1)(k_2, y, l_2) = (k_1, xy, l_2)$ if $l_1 = k_2$; 0 if $l_1 \neq k_2$. Let ρ be a congruence on S.

LEMMA 1.8. *Assume $i > 0$ and $a\rho b$ for some $a \in S_0$, $b \in S_i$. Then*

$$\omega\Big(\bigcup_{l<i} S_l\Big) \subseteq \rho.$$

PROOF. For all $x \in S_0$, $x(a\varphi_{00})\rho x(b\varphi_{i0})$. By Lemma 1.6 $b\varphi_{i0}$ is a permutation on S_0. There is $z \in S_0$, $z \neq 0$, such that $za = 0$ but $zb \neq 0$. It follows that $x\rho b$ for all $x \in S_0$. On the other hand $I = \{c \in S: x\rho c$ for all $x \in S_0\}$ is an ideal of S and $I \cap S_i \neq \varnothing$. Since S_i is a group, $\bigcup_{l<i} S_l \subseteq I$.

LEMMA 1.9. (Clifford and Preston (1967), Tamura (1960)). *Let $\rho_0 = \rho | S_0$. If $\rho_0 \neq \omega(S_0)$ then ρ_0 is determined by a group-congruence τ on G such that $0\rho_0 0$ and*

$$(k_1, x, l_1)\rho_0(k_2, y, l_2) \text{ if and only if } k_1 = k_2,\ x\tau y,\ l_1 = l_2.$$

We denote ρ_0 by $\rho_0(\tau)$.

LEMMA 1.10. *If $b \in S_i$, $a \in S_j$, $x \in S_k$ and if $x\rho ab$ then $x \in (a\rho)(b\rho)$.*

PROOF. If $i, j, k \neq 0$, the argument for Type 1 is available. By Lemmas 1.8 and 1.9 we need consider only the folowing cases:

(I) $a\rho = b\rho = \bigcup_{l<k} S_l$ where $k = \max(i, j)$;

(II) $a\rho = \bigcup_{l<j} S_l$ and $b\rho \cap S_0 = \varnothing$; (II′) $a\rho \cap S_0 = \varnothing$ and $b\rho = \bigcup_{l<i} S_l$;

(III) $a\rho \subseteq S_0$ and $b\rho \cap S_0 = \varnothing$; (III′) $a\rho \cap S_0 = \varnothing$ and $b\rho \subseteq S_0$;

(IV) $a\rho \subseteq S_0$ and $b\rho \subseteq S_0$.

For (I), perfectness follows from the fact that $ab \in \bigcup_{l<k} S_l$ and $S_l^2 = S_l$ for all l. For (II), necessarily $i > j$; $a\rho$ is an ideal of S. Let $y \in b\rho$. By Lemma 1.6, y acts on S_l by permutation for all $l \leqslant j$, and hence acts on $a\rho$ by permutation.

Then we have

$$(ab)\rho \subseteq a\rho = (a\rho)y \subsetneq (a\rho)(b\rho).$$

For (III), let $y \in b\rho$. Certainly $(a\rho)y \subseteq (ab)\rho$, but since y acts on S_0 by permutation, we see that $(a\rho)y = (ab)\rho$ for all $y \in b\rho$ by Lemma 1.9. Hence $(a\rho)(b\rho) = (ab)\rho$. Case (IV) follows since completely 0-simple semigroups are perfect by Fortunatov (1972).

Summarizing Lemmas 1.5 to 1.10 we have

THEOREM 1.11. *Let S be a finite inverse semigroup and $S = \bigcup_{i\in\Gamma} S_i$ be the greatest semilattice decomposition of S where Γ denotes a lower semilattice. Then S is perfect if and only if*

(1.11.1) *Γ is a chain: $\Gamma = \{0, 1, \ldots, n\}$ with operation $i \cdot j = \min\{i, j\}$,*

(1.11.2) *S_0 is either a group or a Brandt semigroup and S_i is a group for all $i > 0$,*

(1.11.3) *S is composed by permutations,*

(1.11.4) *for all $i > j > 0$ (for all $i > j \geqslant 0$ if S_0 is a group), $\varphi_{ij}\colon S_i \to S_j$ is surjective.*

2. Construction and isomorphism criterion

A type 1 semigroup S is determined by a transitive system of surjective homomorphisms $\{\varphi_{ij}\}$ of finite groups S_i into S_j; S is denoted by

$$S = \left(\bigcup_{i=1}^{n} S_i, \varphi_{ij}, i \geqslant j \right).$$

Let another type 1 semigroup be

$$S' = \left(\bigcup_{i=1}^{n'} S'_i, \varphi'_{ij}, i \geqslant j \right).$$

THEOREM 2.1. *$S \cong S'$ if and only if $n = n'$ and for each i there is an isomorphism h_i of S_i onto S'_i such that*

$$\varphi_{ij} h_j = h_i \varphi'_{ij} \quad \text{for } i \geqslant j.$$

LEMMA 2.2. *Let $S = \bigcup_{i=0}^{n} S_i$ be of type 2.*

(2.2.1) *Let $i > j > 0$ and $k > j > 0$, and $p \in S_i$, $q \in S_k$. If $p\varphi_{ij} = q\varphi_{kj}$ then $p\varphi_{i0} = q\varphi_{k0}$.*

(2.2.2) *$\varphi_{ij}\varphi_{j0} = \varphi_{i0}$ for $i > j > 0$.*

https://doi.org/10.1017/S1446788700024459 Published online by Cambridge University Press

PROOF OF (2.2.1). The map $p \mapsto p\varphi_{i0}$ induces a homomorphism of $\bigcup_{i \geqslant 1} S_i$ into the group of permutations of S_0. Now $p\varphi_{ij} = q\varphi_{kj}$ implies $xp = xq$ for all $x \in S_j$; then $(x\varphi_{j0})(p\varphi_{i0}) = (x\varphi_{j0})(q\varphi_{k0})$. Since $x\varphi_{j0}$ is a permutation of S_0, we have $p\varphi_{i0} = q\varphi_{k0}$.

PROOF OF (2.2.2). Let $p\varphi_{ij} = r\varphi_{jj}$, $r \in S_j$. By (2.2.1) $p\varphi_{i0} = r\varphi_{j0}$. For $z \in S_0$, $z(p\varphi_{i0}) = z(r\varphi_{j0}) = z(p\varphi_{ij})\varphi_{j0}$, hence $p\varphi_{i0} = p\varphi_{ij}\varphi_{j0}$ for all $p \in S_i$.

The aim now is to construct type 2 semigroups from type 1 semigroups and Brandt semigroups. Let F_M be a permutation group over $M = \{1, 2, \ldots, m\}$, G a finite group and $\mathcal{G}_M(G)$ the group of mappings of M into G with operation $*$ defined by

$$j(g_1 * g_2) = (jg_1)(jg_2), \qquad g_1, g_2 \in \mathcal{G}_M(G), j \in M.$$

Let $(\bigcup_{i=1}^{n} S_i, \varphi_{ij}, i \geqslant j)$ be of type 1. For S_1, choose $M = \{1, \ldots, m\}$, F_M and a homomorphism f of S_1 onto F_M. This is well known as a representation of S_1 by permutations (for example see Wielandt (1964)). Of course the representation $S_1 \to F_M$ is not required to be transitive. Given F_M and f, we can choose a finite group G and a mapping g of S_1 onto $\mathcal{G}_M(G)$ such that

$$(2.3) \qquad (ag)*((af)(bg)) = (ab)g \quad \text{for all } a, b \in S_1.$$

This is always possible since the trivial group can be chosen as G. We do not consider the determination of all G, $\mathcal{G}_M(G)$ and g satisfying (2.3), but note that if e is the identity element of S_1 then eg is necessarily the map which carries all elements of M to the identity element of G.

Given F_M, G, f and g, define a Brandt semigroup $S_0 = \mathfrak{M}^0(m, G)$, where $m = |M|$, and define φ_{10} by

$$\varphi_{10}\colon S_1 \to \Omega_r(S_0), \quad a\varphi_{10} = (ag, af),$$

where $(ag, af)\colon S_0 \to S_0$ is defined by

$$(i, x, j) \mapsto (i, x \cdot (j(ag)), j(af)).$$

By (2.3) we see that φ_{10} is a homomorphism of S_1 into $\Omega_r(S_0)$. We denote φ_{10} by $\varphi_{10} = (g, f)$.

By means of φ_{10} we can construct a composition $S = \bigcup_{i=0}^{n} S_i$ of type 2 of S_0 and $\bigcup_{i=1}^{n} S_i$. All semigroups of type 2 can be obtained in this manner. S is denoted by

$$S = \left(\bigcup_{i=0}^{n} S_i, \varphi_{10}, \varphi_{ij}, i > j \right),$$

or

$$S = \left(\bar{S}_1, G, m, g, f\right), \quad S_0 = \mathfrak{M}^0(m, G),$$

where

$$\bar{S}_1 = \left(\bigcup_{i=1}^{n} S_i, \varphi_{ij}, i > j \right), \quad \text{and} \quad \varphi_{10} = (g, f).$$

Let $S' = (\bar{S}_1', G', m', g', f')$, $\bar{S}_1' = (\bigcup_{i=1}^n S_i', \varphi_{ij}', i > j)$, $\varphi_{10}' = (g', f')$.

THEOREM 2.4. *$S \cong S'$ if and only if the following conditions are satisfied*:

(2.4.1) $\bar{S}_1 \cong \bar{S}_1'$;

(2.4.2) $m = m'$;

(2.4.3) *there is an isomorphism γ of G onto G' and a permutation α of $M = \{1, 2, \ldots, m\}$ such that for all $a \in S_1$*

$$(af) \cdot \alpha = \alpha \cdot (ah_1)f'$$

and

$$(ag) \cdot \gamma = \alpha \cdot (ah_1)g'$$

where h_1: $S_1 \to S_1'$ is defined in Theorem 2.1.

PROOF. First we have $S \cong S'$ if and only if $\bar{S}_1 \cong \bar{S}_1'$ and there is an isomorphism h_0: $S_0 \to S_0'$ such that

$$h_0^{-1}(a\varphi_{10})h_0 = (ah_1)\varphi_{10}' \quad \text{for all } a \in S_1.$$

On the other hand h_0 is determined by a permutation α of M and an isomorphism γ of G to G' such that

$$(i, x, j)h_0 = (i\alpha, x\gamma, j\alpha).$$

One can prove the theorem by using these.

3. Congruences

In this section we study congruences on finite inverse perfect semigroups.

LEMMA 3.1. (Birkhoff (1967)) *A congruence on a finite chain can be obtained as a disjoint union of intervals.* (*Singleton intervals are admitted.*)

If Γ is a finite chain, namely, $\Gamma = \{1, 2, \ldots, n\}$ with operation $i \cdot j = \min\{i, j\}$, then each congruence σ on Γ is associated with a partition $\{I_j$: $1 \leqslant j \leqslant m\}$ where

$$I_j = \{x \in \Gamma: i_j \leqslant x \leqslant i_{j+1} - 1\}, \qquad j = 1, \ldots, m-1,$$
$$I_m = \{x \in \Gamma: i_m \leqslant x \leqslant n\}$$

and

$$1 = i_1 < i_2 < \cdots < i_m < n.$$

The i_j is called the *initial* of I_j, and the initial of the class $k\sigma$ containing k is denoted by $\bar{k}$, that is, $\bar{k} \in k\sigma$ and $\bar{k} \leqslant x$ for all $x \in k\sigma$. The set of initials with respect to σ is denoted by $\mathfrak{I}_\sigma$.

The next result for a type 1 semigroup $S = (\bigcup_{i=1}^{n} S_i, \varphi_{ij}, i \geqslant j)$ specializes results of Reilly and Scheiblich (1967).

THEOREM 3.2. *ρ is an idempotent separating congruence on S if and only if $\rho = \bigcup_{i=1}^{n} \rho_i$ where ρ_i is a congruence on S_i so that $x\rho_i y$ implies $x\varphi_{ij}\rho_j y\varphi_{ij}$ for all i and j, $0 \leqslant j \leqslant i \leqslant n$ and $x, y \in S_i$.*

Assume σ is a congruence on Γ and $\rho = \bigcup_{i\in\Gamma} \rho_i$ is an idempotent-separating congruence on S. For each $l \in \mathfrak{I}_\sigma$, define η_l on $\bigcup_{i\in l\sigma} S_i$ as follows: $x\eta_l y$ if and only if $x \in S_i$, $y \in S_j$, $i, j \in l\sigma$, $x\varphi_{il}\rho_l y\varphi_{jl}$. Now define η by $\eta = \bigcup_{l\in\mathfrak{I}_\sigma} \eta_l$, denoted by $\eta = (\sigma; \rho_i, i \in \Gamma)$.

THEOREM 3.3. *The η is a congruence on S. Every congruence on S can be obtained in this manner.*

PROOF. The proof of compatibility is routine. To prove the second part, let ρ be a congruence on S. Let $\rho_i = \rho|S_i$. Then $\bigcup \rho_i$ is a congruence on S. Define σ on Γ by $i\sigma j$ if and only if $x\rho y$ for some $x \in S_i$ and $y \in S_j$. Let $\eta = (\sigma; \rho_i, i \in \Gamma)$. Assume $a\rho b$ where $a \in S_i$, $b \in S_j$, $i \geqslant j$. Let $l \in \mathfrak{I}_\sigma$ such that $i, j \in l\sigma$. Let e_l denote the identity element of S_l. Then

$$a\varphi_{il} = e_l(a\varphi_{il}) = e_l a\rho_l e_l b = e_l(b\varphi_{jl}) = b\varphi_{jl}$$

hence $a\eta b$. Conversely assume $a\eta b$, $a \in S_i$, $b \in S_j$, $i, j \in l\sigma$. Since $i\sigma l$, $x\rho y$ for some $x \in S_i$, $y \in S_l$. Let $a = xu$, $u \in S_i$. Then $a\varphi_{il} = e_l(a\varphi_{il}) = e_l a\rho e_l yu = yu\rho a$. Similarly $b\varphi_{jl}\rho b$. Hence $a\rho b$ follows.

COROLLARY 3.4. *If $\rho_i = \iota(S_i)$ for each $i \in l\sigma$, then η_l is the smallest group congruence on $\bigcup_{i\in l\sigma} S_i$. In particular $\eta = (\omega(\Gamma); \iota(S_i), i \in \Gamma)$ is the smallest group congruence on S, and $\eta|S_1 = \iota(S_1)$* (see Howie (1976)).

COROLLARY 3.5. *I is an ideal of $S = (\bigcup_{i\in\Gamma} S_i, \varphi_{ij}, i \geqslant j)$ if and only if there is $\xi = (\sigma; \rho_i, i \in \Gamma)$ on S such that $I = \bigcup_{i\in l\sigma} S_i$ and $\rho_i = \omega(S_i)$ for all $i \in 1\sigma$ (or equivalently by Theorem* 3.3, $\rho_1 = \omega(S_1)$).

https://doi.org/10.1017/S1446788700024459 Published online by Cambridge University Press

The ideal I is called the ideal due to ξ, denoted by $I(\xi)$.

Let $S = (\bar{S}, G, m, g, f)$, $S_0 = \mathfrak{M}^0(m, G)$ where $\bar{S} = (\bigcup_{i\in\Gamma} S_i, \varphi_{ij}, i \geqq j)$, $\Gamma = \{1, 2, \ldots, n\}$.

Let ξ be a congruence on $\bar{S}$, say $\xi = (\sigma; \rho_i, i \in \Gamma)$. Let $\bar{S}^*_\xi = \bar{S} \setminus I(\xi)$ if $\rho_1 = \omega(S_1)$. For any ξ, choose a congruence $\rho_0 = \rho_0(\tau)$ on S_0 such that $a\rho_1 b$, $a, b \in S_1$, implies $af = bf$ and $j(ag)\tau j(bg)$ for all $j \in M = \{1, \ldots, m\}$; equivalently,

(3.6) $$a\rho_1 b \text{ implies } z(a\varphi_{10})\rho_0 z(b\varphi_{10}) \quad \text{for all } z \in S_0.$$

Then we say a pair (ρ_0, ρ_1) satisfies condition (3.6), or a pair (ρ_0, ξ) satisfies condition (3.6).

THEOREM 3.7. *For any congruence ξ on $\bar{S}$ and a pair (ρ_0, ξ) satisfying* (3.6), *define $\eta_1 = \rho_0 \cup \xi$. If $\xi = (\sigma; \rho_i, i \in \Gamma)$ and $\rho_1 = \omega(S_1)$, define η_2 by $\eta_2 = \omega(I(\xi) \cup S_0) \cup (\xi|\bar{S}^*_\xi)$. Then η_1 and η_2 are congruences on S. Every congruence on S is of the form of η_1 or η_2. The expressions for η_1 and η_2 are unique.*

η_1 is called a congruence of *the first kind*; η_2 is of *the second kind.*

PROOF. Let η be a congruence on S and $\xi = \eta|\bar{S}$. By Lemma 1.8, if $x\eta y$ for some $x \in S_0$ and $y \notin S_0$ then η has the form η_2. Condition (3.6) follows from compatibility. Conversely it is obvious that η_2 is a congruence on S. If no element of S_0 is η-related to an element not in S_0 and if $\rho_0 = \eta|S_0 \neq \omega(S_0)$ then $\rho_0 = \rho_0(\tau)$ by Lemma 1.9. Compatibility yields the condition (3.6) on ρ_0. Conversely let $\eta_1 = \xi \cup \rho_0$. We need verify only compatibility. It is easy to see that ρ_0 is compatible with every right and left translation of S_0, hence ρ_0 is compatible with all elements $z \in S$. Assume $a\xi b$, $a \in S_i$, $b \in S_j$, $i, j \geqq 1$. Let $c \in S_1$. Then $a\xi b$ implies $ca\xi cb$, that is, $c(a\varphi_{i1})\rho_1 c(b\varphi_{j1})$ whence $a\varphi_{i1}\rho_1 b\varphi_{j1}$ since ρ_1 is a group congruence. Let $z \in S_0$. By using (3.6) and (2.2.2),

$$za = z(a\varphi_{i0})\rho_0 z(b\varphi_{j0}) = zb \quad \text{for all } z \in S_0$$

whence $za\rho_0 zb$. Next we want to show $az\rho_0 bz$ for all $z \in S_0$. Recall $\psi_{i0}a$ is a left translation linked with $a\varphi_{i0}$. For $x \in S_0$, $x(az) = x((\psi_{i0}a)z) = (x(a\varphi_{i0}))z\rho_0(x(b\varphi_{j0}))z = x((\psi_{j0}b)z) = x(bz)$. Since by Lemma 1.6, $az = 0$ if and only if $bz = 0$ we assume $az \neq 0$ and $bz \neq 0$ for some z, so $x(az) \neq 0$ and $x(bz) \neq 0$ for some x. Since $\rho_0 \neq \omega(S_0)$, the elements x, az and bz have the form $x = (k, u, l)$, $az = (l, v, s)$, $bz = (l, w, s)$ by Lemma 1.9. Let $x' = (l, u^{-1}, k)$. Then $x'x(az)\rho_0 x'x(bz)$ implies $az\rho_0 bz$ as desired.

REMARK. As seen in the proof of Theorem 3.7, it follows that $a\rho_1 b$ implies $(\psi_{10}a)z\rho_0(\psi_{10}b)z$ for all $z \in S_0$. Hence only (3.6) is required.

COROLLARY 3.8. *If $\bar{S}$ is a group, all nonuniversal congruences on S are of the first kind.*

PROOF. Suppose $a\eta b$ for some $a \in S_0$ and some $b \in \bar{S}$. By Lemma 1.8, $\omega(S_0) \subseteq \eta$. Since the set $\{b \in \bar{S}: x\eta b \text{ for } x \in S_0\}$ is an ideal of $\bar{S}$, it must coincide with $\bar{S}$, and hence $\eta = \omega(S)$.

COROLLARY 3.9. *If $|S_1| = 1$, any pair (ρ_0, ξ) satisfies condition (3.6) for a congruence ξ on $\bar{S}$ and a congruence ρ_0 on S_0.*

4. Modularity

Let L be a lattice with join $\vee$ and meet $\wedge$, and L^1 the lattice obtained by adjoining a new element $\mathbf{1}$ to L such that $\mathbf{1}$ is the greatest element of L^1, and join and meet in L are preserved.

LEMMA 4.1. (Birkhoff (1967)) *L^1 is modular if and only if L is modular. Modularity in lattices is preserved under sublattices, homomorphic images nad direct products.*

A semigroup D is called modular if the lattice $L(D)$ of congruences on D is a modular lattice. In this section we determine the structure of finite inverse perfect modular semigroups.

LEMMA 4.2. *A homomorphic image of a modular semigroup is modular.*

Let $D^0 = D \cup \{0\}$ be D with zero 0 adjoined.

LEMMA 4.3. *For any semigroup D, D^0 is modular if and only if D is modular.*

PROOF. For each $\xi \in L(D)$, define a congruence $(\xi, 0)$ on D^0 by

$$(\xi, 0) = \{(0, 0)\} \cup \xi.$$

$L(D)$ is isomorphic to a sublattice of $L(D^0)$ under $\xi \mapsto (\xi, 0)$. Hence if $L(D^0)$ is modular, $L(D)$ is modular. If $\xi \in L(D)$ such that D/ξ has zero 0 we let $I(\xi)$ denote the pre-image of 0 under $D \to D/\xi$, and define a congruence $(\xi, 1)$ on D^0 by

$$(\xi, 1) = \omega(I(\xi) \cup \{0\}) \cup (\xi | D_\xi^*)$$

where $D_\xi^* = D \setminus I(\xi)$. Every congruence on D^0 can be uniquely expressed as either $(\xi, 0)$ or $(\xi, 1)$ for $\xi \in L(D)$. Thus

$$L(D^0) = \{(\xi, 0): \xi \in L(D)\} \cup \{(\xi, 1): \xi \in L(D), D/\xi \text{ has zero}\}.$$

Let $L(2)$ denote the lattice of two elements: $L(2) = \{0, 1\}$. It can be shown that if $\xi, \eta \in L(D)$ and $i, j \in L(2)$, then

$$(\xi, i) \cap (\eta, j) = (\xi \cap \eta, \min\{i, j\}),$$
$$(\xi, i) \vee (\eta, j) = (\xi \vee \eta, \max\{i, j\}).$$

Thus $L(D^0)$ is isomorphic to a sublattice of the direct product $L(D) \times L(2)$. Hence if $L(D)$ is modular, $L(D^0)$ is modular.

LEMMA 4.4. *Let I be a completely prime ideal of a semigroup D and $D \setminus I \neq \varnothing$. If D is modular, then $D \setminus I$ is modular.*

PROOF. Let ρ be a congruence on $D \setminus I$. For ρ, define a congruence ρ^0 on D by $\rho^0 = p \cup \omega(I)$. Then $L(D \setminus I)$ is isomorphic to a sublattice of $L(D)$ under $\rho \to \rho^0$. The conclusion follows.

Consider the following finite semigroups:
(M1) *a chain*;
(M2) *a group*;
(M3) *a chain-group composition*.
We explain (M3). Let G be a group and let $G_0 = G$, $G_1 = G_0 \cup \{0_1\}$ be G_0 with zero 0_1 adjoined. By induction, let G_i be G_{i-1} with zero 0_i adjoined. Then G_n is called a group with a chain $\Gamma = \{1, \dots, n\}$ adjoined below, that is, the ideal extension of Γ by G^0 by means of identity translations of Γ. We call it the *identity-composition of a chain by* a group or simply, a *chain-group composition*.

THEOREM 4.5. *Assume S is a finite inverse perfect semigroup of type* 1. *Then S is modular if and only if S is one of* M1, M2 *and* M3.

PROOF. *Sufficiency*. Applying Lemma 4.3, we can show, by induction on the length of the chain, that M1 and M3 are modular. The result for M2 is well known.

Necessity. Let $S = (\bigcup_{i=1}^n S_i, \varphi_{ij}, i > j)$. By Theorem 1.11, $|S_1| < |S_2| \leqslant \cdots \leqslant |S_n|$. Assume $n > 2$ and $|S_1| = \cdots = |S_{l-1}| = 1$ but $|S_l| \neq 1$ for some $l \leqslant n-2$, whence $|S_{n-1}| \neq 1$, $|S_n| \neq 1$. Let I be the ideal of S defined by $I = \bigcup_{i=1}^{n-2} S_i$. Then $S/I \cong (S_{n-1} \cup S_n)^0$. If S is modular, then by Lemmas 1.4

and 4.4, $S_{n-1} \cup S_n$ is modular. To prove necessity it is sufficient to prove the following:

(4.6) *If S is a perfect semigroup which is a semilattice of two non-trivial groups then S is not modular.*

Let $S = (S_1 \cup S_2,\ \varphi_{ij},\ i \geqslant j,\ i, j = 1, 2)$, $|S_1| > 1$, $|S_2| > 1$. Let ξ be the smallest semilattice congruence on S, namely, $\xi = \omega(S_1) \cup \omega(S_2)$. Let η be the smallest group congruence on S (see Corollary 3.4). In addition to ξ and η, we define ζ by

$$\zeta = \omega(S_1) \cup (\eta | S_2).$$

It is easy to see that ζ is a congruence. Since $\eta | S_1 = \iota(S_1)$, $\eta | S_2 \subsetneqq \omega(S_2)$, and hence $\zeta \subsetneqq \xi$. We see

$$\xi \vee \eta = \zeta \vee \eta = \omega(S),$$
$$\xi \cap \eta = \zeta \cap \eta = \iota(S_1) \cup (\eta | S_2).$$

Therefore S is not modular. Thus (4.6) has been proved.

Let S be a finite inverse perfect semigroup of type 2. If S is not a Brandt semigroup, $S = S_0 \cup \bar{S}$, where $S_0 = \mathfrak{M}^0(m, G)$ is a Brandt semigroup and $\bar{S} = (\bigcup_{i=1}^n S_i,\ \varphi_{ij},\ i \geqslant j)$ is of type 1. Assume S is modular. Then $\bar{S}$ is modular by Lemma 4.4, hence $\bar{S}$ is either M1, M2 or M3. Then we have

THEOREM 4.7. *A finite inverse perfect semigroup S of type 2 is modular if and only if S is one of the following*:

(M4) *a Brandt semigroup*;

(M5) *a Brandt-group composition*;

(M6) *a Brandt-chain composition*;

(M7) *a Brandt-chain-group composition.*

We call S a Brandt-chain composition if $\bar{S}$ is M1; a Brandt-group composition if $\bar{S}$ is M2; a Brandt-chain-group composition if $\bar{S}$ is M3. Each S is an ideal extension of a Brandt semigroup S_0 by $\bar{S}^0$ by means of permutations. In particular, S in case of M6 or M7 is an ideal extension of S_0 by $\bar{S}^0$ by means of identity translations.

PROOF. We want to show that all these are modular.

M4. Let $S = S_0 = \mathfrak{M}^0(m, G)$. By Lemma 1.9, non-universal congruences on S_0 are determined by congruences on G. So $L(S) \cong (L(G))^1$. By Lemma 4.1 $(L(G))^1$ is modular, and hence S is modular.

M5. Let $S = S_0 \cup S_1$ where $S_0 = \mathfrak{M}^0(m, G)$ and S_1 is a group. We will prove that $L(S) \setminus \{\omega(S)\}$ is isomorphic to a sublattice of $L(S_0) \times L(S_1)$, whence $L(S)$ is modular. By Corollary 3.8 all congruences on S are of the first kind except $\omega(S)$. By Theorem 3.7 if $\eta \in L(S)$ and $\eta \neq \omega(S)$, η has the form $\eta = \rho_0 \cup \rho_1$, where $\rho_0 \in L(S_0)$, $\rho_1 \in L(S_1)$ and the pair satisfies condition (3.6). With a slight change of notation, let $\rho_1^{(0)}, \rho_2^{(0)} \in L(S_0)$, $\rho_1^{(1)}, \rho_2^{(1)} \in L(S_1)$. Congruences on a group and a Brandt semigroup are permutable, that is, $\rho_1^{(0)} \cdot \rho_2^{(0)} = \rho_2^{(0)} \cdot \rho_1^{(0)}$ and $\rho_1^{(1)} \cdot \rho_2^{(1)} = \rho_2^{(1)} \cdot \rho_1^{(1)}$ as the product of relations. Hence $\rho_1^{(0)} \vee \rho_2^{(0)} = \rho_1^{(0)} \cdot \rho_2^{(0)}$, and $\rho_1^{(1)} \vee \rho_2^{(1)} = \rho_1^{(1)} \cdot \rho_2^{(1)}$. Then we can show that if pairs $(\rho_1^{(0)}, \rho_1^{(1)})$ and $(\rho_2^{(0)}, \rho_2^{(1)})$ satisfy condition (3.6), then the pairs $(\rho_1^{(0)} \cdot \rho_2^{(0)}, \rho_1^{(1)} \cdot \rho_2^{(1)})$ and $(\rho_1^{(0)} \cap \rho_2^{(0)}, \rho_1^{(1)} \cap \rho_2^{(1)})$ satisfy (3.6). Note $\rho_1^{(0)} \cap \rho_2^{(0)} \neq \varnothing$ on S_0 and $\rho_1^{(1)} \cap \rho_2^{(1)} \neq \varnothing$ on S_1. Hence if $\rho_1^{(0)} \cup \rho_1^{(1)}, \rho_2^{(0)} \cup \rho_2^{(1)} \in L(S)$, then $\rho_1^{(0)} \cdot \rho_2^{(0)} \cup \rho_1^{(1)} \cdot \rho_2^{(1)}$, $(\rho_1^{(0)} \cap \rho_2^{(0)}) \cup (\rho_1^{(1)} \cap \rho_2^{(1)}) \in L(S)$. Immediately we have

$$\left(\rho_1^{(0)} \cup \rho_1^{(1)}\right) \vee \left(\rho_2^{(0)} \cup \rho_2^{(1)}\right) = \left(\rho_1^{(0)} \vee \rho_2^{(0)}\right) \cup \left(\rho_1^{(1)} \vee \rho_2^{(1)}\right),$$
$$\left(\rho_1^{(0)} \cup \rho_1^{(1)}\right) \cap \left(\rho_2^{(0)} \cup \rho_2^{(1)}\right) = \left(\rho_1^{(0)} \cap \rho_2^{(0)}\right) \cup \left(\rho_1^{(1)} \cap \rho_2^{(1)}\right).$$

Thus $L(S) \setminus \{\omega(S)\}$ is isomorphic to a sublattice of $L(S_0) \times L(S_1)$, and hence $L(S)$ is modular by Lemma 4.1.

M6 and M7. Let $S = S_0 \cup \bar{S}$, $S_0 = \mathfrak{M}^0(m, G)$ and $\bar{S} = \bigcup_{i=1}^n S_i$, where $n > 1$, $S_i = \{s_i\}$ for $1 \leqslant i < n$ and S_n is a group (possibly trivial).

Let $L_1(S)$ and $L_2(S)$ denote the sets of congruences on S of the first kind and second kind, respectively.

Because of identity composition, any pair (ρ_0, η) of $\rho_0 \in L(S_0)$ and $\eta \in L(\bar{S})$ satisfies condition (3.6). Therefore we have $L_1(S) \cong L(S_0) \times L(\bar{S})$.

As $S_1 = \{s_1\}$, every congruence η on $\bar{S}$ has the property that $\bar{S}/\eta$ has zero, and hence a congruence η_2 on S of the second kind can be obtained from every congruence η on $\bar{S}$. It is easily shown that $L_2(S) \cong L(\bar{S})$. Then we have

$$L_1(S) \cong \left\{(\xi, \eta) \colon \xi \in L(S_0), \eta \in L(\bar{S})\right\} \quad \text{under} \quad \xi \cup \eta \mapsto (\xi, \eta),$$
$$L_2(S) \cong \left\{(\mathbf{1}, \eta) \colon \eta \in L(\bar{S})\right\} \quad \text{under} \quad \zeta \mapsto \left(\mathbf{1}, \zeta | \bar{S}\right),$$

and also

$$(\xi_1, \eta_1) \vee (\mathbf{1}, \eta_2) = (\mathbf{1}, \eta_1 \vee \eta_2),$$
$$(\xi_1, \eta_1) \wedge (\mathbf{1}, \eta_2) = (\xi_1, \eta_1 \wedge \eta_2).$$

Therefore we have

$$L(S) = L_1(S) \cup L_2(S) \cong (L(S_0))^1 \times L(\bar{S}).$$

Since $L(S_0)$ and $L(\bar{S})$ are modular, $L(S)$ is modular. This completes the proof of Theorem 4.7.

In summary

THEOREM 4.8. *Let S be a finite inverse perfect semigroup. Then S is modular if and only if S does not contain more than one non-trivial group component in the greatest semilattice decomposition of S; equivalently, S is one of the following*:

(M1) *a chain*;

(M2) *a group*;

(M3) *a chain-group composition*;

(M4) *a Brandt semigroup*;

(M5) *a Brandt-group composition*;

(M6) *a Brandt-chain composition*;

(M7) *a Brandt-chain-group composition.*

REMARK ON MODULARITY OF M1. Without assuming finiteness, the lattice of congruences on any chain is modular [distributive]. A congruence on a chain as a semilattice is its partition into non-overlapping segments by Lemma 3.1, and its partition gives also a congruence on the chain as a lattice (Exercise 1, page 138, Birkhoff (1967)). By Funayama and Nakayama's theorem (Theorem 9, page 138, Birkhoff (1967)), the lattice of congruences on any lattice is complete Browerian, but a Browerian lattice is distributive by Theorem 18, page 45, Birkhoff (1967). Incidentally we have that the lattice of congruences on a lattice $S(\vee, \wedge)$ equals the lattice of congruences on the join [meet] semilattice $S(\vee)$ [$S(\wedge)$] if and only if $S(\vee, \wedge)$ is a chain.

ACKNOWLEDGEMENT. The authors express their heartfelt thanks to the referee for his kind suggestion.

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