7

Tilting in Exact Categories

Contents

We discuss tilting objects in exact categories. An object T of an exact category $\mathcal{A}$ is a tilting object if it has no self-extensions and generates $\mathcal{A}$ as a thick subcategory. There is an analogous notion of a tilting object in a triangulated category and we see that T is tilting in $\mathcal{A}$ if and only if it is tilting when viewed as an object of the derived category $\mathbf{D}^b(\mathcal{A})$.

Any tilting object in $\mathcal{A}$ gives rise to a cotorsion pair for $\mathcal{A}$, and we characterise such cotorsion pairs. In fact, a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is determined either by $\mathcal{X}$ or by $\mathcal{Y}$. The subcategory $\mathcal{X}$ is resolving and contravariantly finite, while $\mathcal{Y}$ is coresolving and covariantly finite. This yields a correspondence between equivalence classes of tilting objects and appropriate subcategories.

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7.1 Cotorsion Pairs

We introduce cotorsion pairs for exact categories and study their basic properties. A cotorsion pair is given by a pair of subcategories, and of particular interest are subcategories whose objects are defined via resolutions or coresolutions.

Thick Subcategories and Resolutions

Let $\mathcal{A}$ be an exact category. A full additive subcategory $\mathcal{C} \subseteq \mathcal{A}$ is *thick* if it is closed under direct summands and satisfies the following *two out of three property*: an admissible exact sequence $0 \to X \to Y \to Z \to 0$ lies in $\mathcal{C}$ if two of X, Y, Z are in $\mathcal{C}$. Given a class of objects $\mathcal{C} \subseteq \mathcal{A}$, we write $\operatorname{Thick}(\mathcal{C})$ for the smallest thick subcategory of $\mathcal{A}$ that contains $\mathcal{C}$.

Let $\mathcal{C} \subseteq \mathcal{A}$ be a full additive subcategory. A *finite $\mathcal{C}$-resolution* of an object A in $\mathcal{A}$ is an admissible exact sequence (that is, an acyclic complex)

$$0 \longrightarrow X_r \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

such that $X_i \in \mathcal{C}$ for all i. We write $\operatorname{Res}(\mathcal{C})$ for the full subcategory of objects in $\mathcal{A}$ that admit a finite $\mathcal{C}$-resolution. A *finite $\mathcal{C}$-coresolution* is defined dually, and we write $\operatorname{Cores}(\mathcal{C})$ for the full subcategory of objects in $\mathcal{A}$ that admit a finite $\mathcal{C}$-coresolution.

Self-Orthogonal Subcategories

Let $\mathcal{A}$ be an exact category. A full additive subcategory $\mathcal{C} \subseteq \mathcal{A}$ is *self-orthogonal* if it is closed under direct summands and $\operatorname{Ext}^n(X, Y) = 0$ for all X, Y in $\mathcal{C}$ and $n \neq 0$.

We wish to resolve objects in $\mathcal{A}$ via objects from a self-orthogonal subcategory $\mathcal{C}$. A basic tool is the derived category $\mathbf{D}(\mathcal{A})$. In fact, the inclusion $\mathcal{C} \to \mathcal{A}$ is exact and induces an exact functor $\mathbf{D}(\mathcal{C}) \to \mathbf{D}(\mathcal{A})$.

Lemma 7.1.1. *Let $\mathcal{A}$ be an exact category and $\mathcal{C} \subseteq \mathcal{A}$ a self-orthogonal subcategory. Then the canonical functor*

$$\mathbf{K}^b(\mathcal{C}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{C}) \longrightarrow \mathbf{D}(\mathcal{A})$$

is fully faithful and identifies $\mathbf{K}^b(\mathcal{C})$ with a thick subcategory of $\mathbf{D}(\mathcal{A})$.

Proof The functor $\mathbf{K}^b(\mathcal{C}) \to \mathbf{D}^b(\mathcal{C})$ is an equivalence since $\mathcal{C}$ is split exact, and $\mathbf{D}^b(\mathcal{C}) \to \mathbf{D}(\mathcal{A})$ is fully faithful by a dévissage argument (Lemma 3.1.8). Thus the composite identifies $\mathbf{K}^b(\mathcal{C})$ with a triangulated subcategory of $\mathbf{D}(\mathcal{A})$; it is thick because $\mathcal{C}$ is closed under direct summands. □

Thick subcategories have been defined for exact and for triangulated categories. The next lemma shows that these two notions are compatible.

Let us write

$$\Phi\colon \mathcal{A} \longrightarrow \mathbf{D}(\mathcal{A})$$

for the inclusion that identifies $\mathcal{A}$ with the complexes concentrated in degree zero. We call a complex X in $\mathcal{A}$ *bounded* if $X^n = 0$ for $|n| \gg 0$.

Lemma 7.1.2. *Let $\mathcal{A}$ be an exact category and $\mathcal{C} \subseteq \mathcal{A}$ a self-orthogonal subcategory. For an object $A \in \mathcal{A}$ the following are equivalent.*

(1) $A \in \operatorname{Thick}(\mathcal{C})$.

(2) $\Phi(A) \in \operatorname{Thick}(\Phi(\mathcal{C}))$.

(3) *There is a bounded complex X in $\mathcal{C}$ that admits acyclic truncations*

$$\cdots \longrightarrow X^{-3} \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow \operatorname{Coker} d^{-2} \longrightarrow 0 \longrightarrow \cdots$$

and

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{Ker} d^{0} \longrightarrow X^{0} \longrightarrow X^{1} \longrightarrow X^{2} \longrightarrow \cdots$$

which induce an admissible exact sequence

$$0 \longrightarrow \operatorname{Coker} d^{-2} \longrightarrow \operatorname{Ker} d^{0} \longrightarrow A \longrightarrow 0. \tag{7.1.3}$$

(4) *There is a bounded complex X in $\mathcal{C}$ that admits acyclic truncations*

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{Ker} d^{1} \longrightarrow X^{1} \longrightarrow X^{2} \longrightarrow X^{3} \longrightarrow \cdots$$

and

$$\cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow X^{0} \longrightarrow \operatorname{Coker} d^{-1} \longrightarrow 0 \longrightarrow \cdots$$

which induce an admissible exact sequence

$$0 \longrightarrow A \longrightarrow \operatorname{Coker} d^{-1} \longrightarrow \operatorname{Ker} d^{1} \longrightarrow 0. \tag{7.1.4}$$

Proof (1) $\Rightarrow$ (2): This is clear, since any admissible exact sequence in $\mathcal{A}$ induces an exact triangle in $\mathbf{D}(\mathcal{A})$.

(2) $\Rightarrow$ (3): An object X in $\operatorname{Thick}(\Phi(\mathcal{C}))$ is a bounded complex with $X^n \in \mathcal{C}$ for all $n \in \mathbb{Z}$, by Lemma 7.1.1. The truncations exist when X is quasi-isomorphic to $\Phi(A)$.

(3) $\Leftrightarrow$ (4): Reverse arrows and signs of the degrees.

(3) $\Rightarrow$ (1): Clear. □

We have the following immediate consequence.

Proposition 7.1.5. *For a self-orthogonal subcategory $\mathcal{C} \subseteq \mathcal{A}$ we have*

$$\Phi^{-1}(\mathrm{Thick}(\Phi(\mathcal{C}))) = \mathrm{Thick}(\mathcal{C}). \qquad \square$$

For a class of objects $\mathcal{C} \subseteq \mathcal{A}$ we set

$${}^{\perp}\mathcal{C} = \{X \in \mathcal{A} \mid \mathrm{Ext}^n(X,Y) = 0 \text{ for all } Y \in \mathcal{C},\ n > 0\}$$

and

$$\mathcal{C}^{\perp} = \{Y \in \mathcal{A} \mid \mathrm{Ext}^n(X,Y) = 0 \text{ for all } X \in \mathcal{C},\ n > 0\}.$$

Lemma 7.1.6. *Let $\mathcal{A}$ be an exact category and $\mathcal{C} \subseteq \mathcal{A}$ a self-orthogonal subcategory. Then we have the following equalities:*

$${}^{\perp}\mathcal{C} \cap \mathrm{Res}(\mathcal{C}) = \mathcal{C} = \mathrm{Cores}(\mathcal{C}) \cap \mathcal{C}^{\perp}$$

$${}^{\perp}\mathcal{C} \cap \mathrm{Thick}(\mathcal{C}) = \mathrm{Cores}(\mathcal{C}) \qquad \textit{and} \qquad \mathcal{C}^{\perp} \cap \mathrm{Thick}(\mathcal{C}) = \mathrm{Res}(\mathcal{C}).$$

Proof We show the first equality. Then the second follows by duality. The inclusion ${}^{\perp}\mathcal{C} \cap \mathrm{Res}(\mathcal{C}) \supseteq \mathcal{C}$ is clear. Thus we fix $A \in {}^{\perp}\mathcal{C} \cap \mathrm{Res}(\mathcal{C})$. An induction on the length n of a $\mathcal{C}$-resolution shows that A is in $\mathcal{C}$. The case $n = 0$ is clear. If $n > 0$, consider an exact sequence $\eta\colon 0 \to A' \to C \to A \to 0$ with $C \in \mathcal{C}$. Then $A' \in {}^{\perp}\mathcal{C} \cap \mathrm{Res}(\mathcal{C})$, and $A' \in \mathcal{C}$ by the inductive hypothesis. Thus the sequence η splits, and A is in $\mathcal{C}$ since $\mathcal{C}$ is closed under direct summands.

Next we verify the third equality. Then the last follows by duality. We have ${}^{\perp}\mathcal{C} \supseteq \mathrm{Cores}(\mathcal{C})$ since ${}^{\perp}\mathcal{C}$ contains $\mathcal{C}$ and is closed under kernels of admissible epimorphisms. The inclusion $\mathrm{Thick}(\mathcal{C}) \supseteq \mathrm{Cores}(\mathcal{C})$ is clear. Now fix A in ${}^{\perp}\mathcal{C} \cap \mathrm{Thick}(\mathcal{C})$. We apply Lemma 7.1.2 and choose a bounded complex X in $\mathcal{C}$ that is quasi-isomorphic to $\Phi(A)$. We have $\mathrm{Ker}\, d^1 \in {}^{\perp}\mathcal{C}$, and then the sequence (7.1.4) implies $\mathrm{Coker}\, d^{-1} \in {}^{\perp}\mathcal{C}$ since ${}^{\perp}\mathcal{C}$ is extension closed. From the first equality it follows that $\mathrm{Coker}\, d^{-1} \in \mathcal{C}$. Then

$$0 \longrightarrow A \longrightarrow \mathrm{Coker}\, d^{-1} \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots$$

yields a finite $\mathcal{C}$-coresolution of A. $\square$

The category $\mathrm{Proj}\,\mathcal{A}$ of projective objects in $\mathcal{A}$ is a particular example of a self-orthogonal subcategory.

Proposition 7.1.7. *Let $\mathcal{A}$ be an exact category and $\mathcal{P} \subseteq \mathrm{Proj}\,\mathcal{A}$ a full additive subcategory closed under direct summands. Then* $\mathrm{Thick}(\mathcal{P}) = \mathrm{Res}(\mathcal{P})$.

Proof The inclusion $\mathrm{Res}(\mathcal{P}) \subseteq \mathrm{Thick}(\mathcal{P})$ is clear. Thus we may assume that $\mathcal{A} = \mathrm{Thick}(\mathcal{P})$. Clearly, $\mathcal{P}^{\perp} = \mathcal{A}$. Then $\mathcal{A} = \mathrm{Res}(\mathcal{P})$ by Lemma 7.1.6. $\square$

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Corollary 7.1.8. *Let Λ be a ring. Then a Λ-module X viewed as a complex concentrated in degree zero belongs to $\mathbf{D}^{\mathrm{perf}}(\Lambda)$ if and only if X admits a finite length projective resolution*

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

such that each P_i is finitely generated.

Proof Combine Proposition 7.1.5 and Proposition 7.1.7. □

Cotorsion Pairs

Let $\mathcal{A}$ be an exact category and let $\mathcal{X}, \mathcal{Y}$ be full subcategories of $\mathcal{A}$. Then $(\mathcal{X}, \mathcal{Y})$ is a (hereditary and complete) *cotorsion pair* for $\mathcal{A}$ if

$$\mathcal{X}^{\perp} = \mathcal{Y} \qquad \text{and} \qquad \mathcal{X} = {}^{\perp}\mathcal{Y}$$

and every object $A \in \mathcal{A}$ fits into admissible exact sequences

$$0 \longrightarrow Y_A \longrightarrow X_A \longrightarrow A \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow A \longrightarrow Y^A \longrightarrow X^A \longrightarrow 0$$

with $X_A, X^A \in \mathcal{X}$ and $Y_A, Y^A \in \mathcal{Y}$.

Remark 7.1.9. Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair for $\mathcal{A}$ and set $\mathcal{C} = \mathcal{X} \cap \mathcal{Y}$.

(1) We have $X_A \in \mathcal{C}$ if $A \in \mathcal{Y}$, and $Y^A \in \mathcal{C}$ if $A \in \mathcal{X}$.

(2) The above exact sequences are uniquely determined up to isomorphism in the quotient category $\mathcal{A}/\mathcal{C}$ (that is obtained from $\mathcal{A}$ by annihilating all morphisms that factor through an object in $\mathcal{C}$). In fact, the assignment $A \mapsto X_A$ gives a right adjoint of the inclusion $\mathcal{X}/\mathcal{C} \to \mathcal{A}/\mathcal{C}$, while the assignment $A \mapsto Y^A$ gives a left adjoint of the inclusion $\mathcal{Y}/\mathcal{C} \to \mathcal{A}/\mathcal{C}$.

Proposition 7.1.10. *Let $\mathcal{A}$ be an exact category and let $\mathcal{C} \subseteq \mathcal{A}$ be a self-orthogonal subcategory such that* $\operatorname{Thick}(\mathcal{C}) = \mathcal{A}$. *Then*

$$({}^{\perp}\mathcal{C}, \mathcal{C}^{\perp}) = (\operatorname{Cores}(\mathcal{C}), \operatorname{Res}(\mathcal{C}))$$

is a cotorsion pair for $\mathcal{A}$ and ${}^{\perp}\mathcal{C} \cap \mathcal{C}^{\perp} = \mathcal{C}$.

Proof Combine Lemma 7.1.2 and Lemma 7.1.6. □

Resolving and Coresolving Subcategories

Let $\mathcal{A}$ be an exact category and let $\mathcal{X}, \mathcal{Y}$ be full subcategories of $\mathcal{A}$. The subcategory $\mathcal{X}$ is *resolving* if $\mathcal{X}$ is closed under extensions, direct summands, kernels of admissible epimorphisms, and for each object $A \in \mathcal{A}$ there is an

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admissible epimorphism $X \to A$ with $X \in \mathcal{X}$. Dually, the subcategory $\mathcal{Y}$ is *coresolving* if $\mathcal{Y}$ is resolving when viewed as a full subcategory of $\mathcal{A}^{\mathrm{op}}$.

Given an object $A \in \mathcal{A}$, a morphism $X \to A$ with $X \in \mathcal{X}$ is called a *right $\mathcal{X}$-approximation* of A if the induced map $\operatorname{Hom}(X', X) \to \operatorname{Hom}(X', A)$ is surjective for every object $X' \in \mathcal{X}$. The subcategory $\mathcal{X}$ is *contravariantly finite* if every object $A \in \mathcal{A}$ admits a right $\mathcal{X}$-approximation. Dually, a morphism $A \to Y$ with $Y \in \mathcal{Y}$ is called a *left $\mathcal{Y}$-approximation* of A if the induced map $\operatorname{Hom}(Y, Y') \to \operatorname{Hom}(A, Y')$ is surjective for every object $Y' \in \mathcal{Y}$. The subcategory $\mathcal{Y}$ is *covariantly finite* if every object $A \in \mathcal{A}$ admits a left $\mathcal{Y}$-approximation.

Example 7.1.11. A full subcategory $\mathcal{X} \subseteq \mathcal{A}$ is contravariantly finite if the inclusion admits a right adjoint $p \colon \mathcal{A} \to \mathcal{X}$. In that case the counit $p(A) \to A$ yields a right $\mathcal{X}$-approximation for each object $A \in \mathcal{A}$. Dually, $\mathcal{Y} \subseteq \mathcal{A}$ is covariantly finite if the inclusion admits a left adjoint $q \colon \mathcal{A} \to \mathcal{X}$, and then the unit $A \to q(A)$ yields a left $\mathcal{Y}$-approximation for $A \in \mathcal{A}$.

Lemma 7.1.12. *Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair for $\mathcal{A}$.*

(1) *The subcategory $\mathcal{X} \subseteq \mathcal{A}$ is resolving and contravariantly finite.*
(2) *The subcategory $\mathcal{Y} \subseteq \mathcal{A}$ is coresolving and covariantly finite.*

Proof Clear. □

A morphism $\alpha \colon X \to Y$ is called *right minimal* if every endomorphism $\phi \colon X \to X$ with $\alpha\phi = \alpha$ is invertible. Dually, α is *left minimal* if every endomorphism $\psi \colon Y \to Y$ with $\psi\alpha = \alpha$ is invertible. Note that any morphism $\phi \colon X \to Y$ in a Krull–Schmidt category admits a decomposition $X = X' \oplus X''$ such that $\phi|_{X'}$ is right minimal and $\phi|_{X''} = 0$. There is an analogue for left minimal morphisms.

The following is known as *Wakamatsu's lemma*.

Lemma 7.1.13 (Wakamatsu)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be extension closed subcategories of $\mathcal{A}$ and $A \in \mathcal{A}$.*

(1) *Let $0 \to Y \to X \xrightarrow{\phi} A \to 0$ be an exact sequence in $\mathcal{A}$ such that ϕ is a right minimal $\mathcal{X}$-approximation. Then $\operatorname{Ext}^1(X', Y) = 0$ for all $X' \in \mathcal{X}$.*
(2) *Let $0 \to A \xrightarrow{\phi} Y \to X \to 0$ be an exact sequence in $\mathcal{A}$ such that ϕ is a left minimal $\mathcal{Y}$-approximation. Then $\operatorname{Ext}^1(X, Y') = 0$ for all $Y' \in \mathcal{Y}$.*

Proof We prove (1), and (2) is dual. An exact sequence $0 \to Y \to E \to X' \to 0$ gives rise to the following commutative diagram with exact rows and

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columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & \tilde{X} & \longrightarrow & X' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \tilde{\phi}} & & & & \\
 & & A & = & A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We have $\tilde{X} \in \mathcal{X}$ since $\mathcal{X}$ is extension closed, and $\tilde{\phi}$ factors through ϕ since ϕ is a right $\mathcal{X}$-approximation. Then the minimality of ϕ implies that $X \to \tilde{X}$ is a split monomorphism. The approximation ϕ induces the following exact sequence

$$0 \longrightarrow \operatorname{Hom}(X', Y) \longrightarrow \operatorname{Hom}(X', X) \longrightarrow \operatorname{Hom}(X', A) \xrightarrow{\alpha} \operatorname{Ext}^1(X', Y) \xrightarrow{\beta} \operatorname{Ext}^1(X', X) \longrightarrow \operatorname{Ext}^1(X', A) \longrightarrow \cdots$$

and we have shown that $\beta = 0$. On the other hand, $\alpha = 0$ since $\operatorname{Hom}(X', \phi)$ is surjective. Thus $\operatorname{Ext}^1(X', Y) = 0$. □

Lemma 7.1.14. *Let $\mathcal{A}$ be an exact category and $\mathcal{Y} \subseteq \mathcal{A}$ a full additive subcategory. Then $({}^{\perp}\mathcal{Y}, \mathcal{Y})$ is a cotorsion pair for $\mathcal{A}$ if and only if the following holds.*

(1) *Each object $A \in \mathcal{A}$ fits into an admissible exact sequence*

$$0 \longrightarrow A \longrightarrow Y^A \longrightarrow X^A \longrightarrow 0$$

with $X^A \in {}^{\perp}\mathcal{Y}$ and $Y^A \in \mathcal{Y}$.

(2) *For each object $A \in \mathcal{A}$ there is an admissible epimorphism $X \to A$ with $X \in {}^{\perp}\mathcal{Y}$.*

(3) *The subcategory $\mathcal{Y} \subseteq \mathcal{A}$ is closed under direct summands.*

Proof For an object $A \in \mathcal{A}$ we need to construct an admissible exact sequence $0 \to Y_A \to X_A \to A \to 0$. To this end choose an admissible epimorphism

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$X \to A$ with $X \in {}^{\perp}\mathcal{Y}$ and form the following pushout diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y^U & \longrightarrow & X' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Then the bottom row is the desired exact sequence. It is easily checked that $({}^{\perp}\mathcal{Y})^{\perp} = \mathcal{Y}$, since $\mathcal{Y}$ is closed under direct summands. □

In order to apply the above lemma, we make the following observation. Let $\mathcal{A}$ be an exact category with enough projective objects and $\mathcal{X} \subseteq \mathcal{A}$ a resolving subcategory. Then a dimension shift argument shows for $Y \in \mathcal{A}$ that

$$Y \in \mathcal{X}^{\perp} \quad \iff \quad \operatorname{Ext}^1(X, Y) = 0 \text{ for all } X \in \mathcal{X}$$

since $\operatorname{Ext}^{p+q}(X, Y) \cong \operatorname{Ext}^p(\Omega^q X, Y)$ for $p, q \geq 1$.

Corollary 7.1.15. *Let $\mathcal{A}$ be an exact category with enough projective objects and suppose that $\mathcal{A}$ is a Krull–Schmidt category. Then the assignment*

$$\mathcal{Y} \longmapsto ({}^{\perp}\mathcal{Y}, \mathcal{Y})$$

induces a bijection between the covariantly finite coresolving subcategories of $\mathcal{A}$ and the cotorsion pairs for $\mathcal{A}$.

Proof Any cotorsion pair yields a covariantly finite coresolving subcategory by Lemma 7.1.12. Conversely, if $\mathcal{Y} \subseteq \mathcal{A}$ is covariantly finite and coresolving, then the assumptions in Lemma 7.1.14 are satisfied, thanks to Lemma 7.1.13 and the fact that $\mathcal{A}$ is a Krull–Schmidt category. Thus $({}^{\perp}\mathcal{Y}, \mathcal{Y})$ is a cotorsion pair for $\mathcal{A}$. □

Corollary 7.1.16. *Let $\mathcal{A}$ be an exact category with enough projective and enough injective objects. Suppose also that $\mathcal{A}$ is a Krull–Schmidt category. Then the assignments*

$$\mathcal{X} \longmapsto \mathcal{X}^{\perp} \quad \textit{and} \quad {}^{\perp}\mathcal{Y} \longleftarrow\!\shortmid \mathcal{Y}$$

induce mutually inverse bijections between the contravariantly finite resolving subcategories of $\mathcal{A}$ and the covariantly finite coresolving subcategories of $\mathcal{A}$.

Proof Apply Corollary 7.1.15 and the dual assertion. □

Example 7.1.17. (1) Let Λ be an Artin algebra. Then $\operatorname{mod}\Lambda$ is an abelian Krull–Schmidt category with enough projective and enough injective objects.

(2) Let Λ be an Artin algebra and suppose that Λ is Gorenstein. Then the category of finitely generated Λ-modules of finite projective dimension is an

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exact Krull–Schmidt category with enough projective and enough injective objects.

7.2 Tilting in Exact Categories

We introduce tilting objects in exact categories and discuss the connection with cotorsion pairs. Also, we show that each tilting object gives rise to a derived equivalence. The correspondence between tilting objects and cotorsion pairs is very explicit: a tilting object T corresponds to the pair $({}^{\perp}T, T^{\perp})$. The correspondence is of particular interest for modules over Artin algebras. Also, we characterise the subcategories which are of the form ${}^{\perp}T$ or $T^{\perp}$.

Tilting Objects

Before giving the definition of a tilting object, let us point out that there is a plethora of different definitions in the literature. Each definition depends on its context. There are definitions for module categories, abelian categories, triangulated categories etc. Also, a definition may require the existence of set-indexed coproducts.

Let $\mathcal{A}$ be an exact category. An object T is a *tilting object* if $\operatorname{Ext}^n(T,T) = 0$ for all $n \neq 0$ and $\operatorname{Thick}(T) = \mathcal{A}$.

For an object X in $\mathcal{A}$ we denote by $\operatorname{add} X$ the full subcategory consisting of the direct summands of finite direct sums of copies of X.

Proposition 7.2.1. *Let $\mathcal{A}$ be an exact category and $\mathcal{C} = \operatorname{add} T$ for an object $T \in \mathcal{A}$. Then T is tilting if and only if $(\operatorname{Cores}(\mathcal{C}), \operatorname{Res}(\mathcal{C}))$ is a cotorsion pair for $\mathcal{A}$. In that case we have*

$$ {}^{\perp}T = \operatorname{Cores}(\mathcal{C}), \qquad T^{\perp} = \operatorname{Res}(\mathcal{C}), \qquad {}^{\perp}T \cap T^{\perp} = \mathcal{C}. $$

Proof Apply Proposition 7.1.10. □

The definition of a tilting object in an exact category is compatible with the definition of a tilting object in a triangulated category. Let $\mathcal{T}$ be a triangulated category with suspension $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$. An object T is a *tilting object* if $\operatorname{Hom}(T, \Sigma^n T) = 0$ for all $n \neq 0$ and $\operatorname{Thick}(T) = \mathcal{T}$.

Proposition 7.2.2. *Let $\mathcal{A}$ be an exact category. An object T in $\mathcal{A}$ is a tilting object if and only if it is a tilting object of $\mathbf{D}^b(\mathcal{A})$ when viewed as a complex concentrated in degree zero.*

Proof Set $\mathcal{C} = \operatorname{add} T$ and apply Proposition 7.1.5. □

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A tilting object gives rise to a derived equivalence.

Theorem 7.2.3. *Let $\mathcal{A}$ be an exact and idempotent complete category. For an object T with $\Lambda = \operatorname{End}(T)$, the following are equivalent.*

(1) *The object T is a tilting object in $\mathcal{A}$.*
(2) *The functor $\operatorname{Hom}(T,-)$ induces a triangle equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^{\text{perf}}(\Lambda)$ that makes the following square commutative.*

$$\begin{array}{ccc} \mathbf{K}^b(\operatorname{add} T) & \xrightarrow{\operatorname{Hom}(T,-)} & \mathbf{K}^b(\operatorname{proj} \Lambda) \\ \downarrow \wr & & \downarrow \wr \\ \mathbf{D}^b(\mathcal{A}) & \xrightarrow{\sim} & \mathbf{D}^{\text{perf}}(\Lambda) \end{array}$$

(3) *There is a triangle equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^{\text{perf}}(\Lambda)$ that maps T to Λ.*

Proof (1) $\Rightarrow$ (2): Suppose T is tilting. The functor $\operatorname{Hom}(T,-)$ induces an equivalence

$$\operatorname{add} T \xrightarrow{\sim} \operatorname{proj} \Lambda$$

while the vertical functors are fully faithful by Lemma 7.1.1. The functor on the right is surjective on objects by definition, and the functor on the left by Proposition 7.2.2.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): The object Λ is a tilting object in $\mathbf{D}^{\text{perf}}(\Lambda)$. This property is preserved under a triangle equivalence, but also under the embedding $\mathcal{A} \to \mathbf{D}^b(\mathcal{A})$ by Proposition 7.2.2. □

We will see another proof of the equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^{\text{perf}}(\Lambda)$ when we discuss tilting objects in $\mathbf{D}^b(\mathcal{A})$; see Proposition 9.1.20.

Next we consider Grothendieck groups and derive a consequence from the fact that a triangle equivalence preserves Grothendieck groups. Recall that $K_0(\mathcal{A})$ denotes the Grothendieck group of an exact category, and that $K_0(\Lambda) = K_0(\operatorname{proj} \Lambda)$ for any ring Λ.

Corollary 7.2.4. *Let $\mathcal{A}$ be an exact and idempotent complete category. Given a tilting object T with $\Lambda = \operatorname{End}(T)$, then $\operatorname{Hom}(T,-)$ induces an isomorphism*

$$K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\Lambda).$$

Proof We have isomorphisms

$$K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\mathbf{D}^b(\mathcal{A})) \xrightarrow{\sim} K_0(\mathbf{D}^{\text{perf}}(\Lambda)) \xrightarrow{\sim} K_0(\Lambda),$$

where the first and the third follow from Lemma 4.1.17 and the middle one from Theorem 7.2.3. □

Of particular interest are module categories. Let Λ be a right coherent ring and consider the abelian category $\operatorname{mod}\Lambda$ of finitely presented Λ-modules. If $T \in \operatorname{mod}\Lambda$ is a tilting object and $\Gamma = \operatorname{End}(T)$, then $\operatorname{Hom}(T,-)$ induces a triangle equivalence

$$\mathbf{D}^b(\operatorname{mod}\Lambda) \xrightarrow{\sim} \mathbf{D}^{\mathrm{perf}}(\Gamma)$$

by the above theorem. For example, Λ_Λ is a tilting object in $\operatorname{mod}\Lambda$ if and only if every finitely presented Λ-module has finite projective dimension. This reflects the fact that the inclusion $\operatorname{proj}\Lambda \to \operatorname{mod}\Lambda$ induces a triangle equivalence $\mathbf{D}^{\mathrm{perf}}(\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$ if and only if every finitely presented Λ-module has finite projective dimension (Corollary 4.2.9).

The existence of a tilting object imposes some immediate constraints on objects and morphisms in $\mathcal{A}$ and $\mathbf{D}^b(\mathcal{A})$.

Lemma 7.2.5. *Let $\mathcal{A}$ be an exact category and suppose there is a tilting object in $\mathcal{A}$ or in $\mathbf{D}^b(\mathcal{A})$. Then for each pair of objects $X, Y \in \mathbf{D}^b(\mathcal{A})$ we have*

$$\operatorname{Hom}(X, \Sigma^n Y) = 0 \quad \text{for} \quad |n| \gg 0.$$

In particular, for all $X, Y \in \mathcal{A}$ we have $\operatorname{Ext}^n(X,Y) = 0$ for $n \gg 0$.

Proof Suppose there is a tilting object T in $\mathbf{D}^b(\mathcal{A})$. This includes the case that there is a tilting object in $\mathcal{A}$, by Proposition 7.2.2. It follows from the definition of a tilting object that T is homologically finite, so for all $Y \in \mathbf{D}^b(\mathcal{A})$ we have $\operatorname{Hom}(T, \Sigma^n Y) = 0$ for $|n| \gg 0$. The homologically finite objects form a thick subcategory (Example 3.1.7) and therefore all objects in $\mathbf{D}^b(\mathcal{A})$ are homologically finite. □

We have further consequences when $\mathcal{A}$ is a length category.

Lemma 7.2.6. *Let $\mathcal{A}$ be a length category and $T \in \mathbf{D}^b(\mathcal{A})$ a tilting object. Then $\mathcal{A}$ has only finitely many isomorphism classes of simple objects and*

$$\operatorname{gl.dim}\mathcal{A} = \inf_{\substack{S,S' \\ \text{simple}}} \{i \in \mathbb{N} \mid \operatorname{Ext}^{i+1}(S,S') = 0\} < \infty.$$

Proof The length of $H = \bigoplus_n H^n T$ gives a bound for the number of isomorphism classes of simple objects in $\mathcal{A}$. More precisely, let $\mathcal{B} \subseteq \mathcal{A}$ denote the Serre subcategory generated by the composition factors of H. Then T belongs to the thick subcategory of objects $X \in \mathbf{D}^b(\mathcal{A})$ with $H^n X \in \mathcal{B}$ for all n. Thus $\mathcal{B} = \mathcal{A}$.

Having only finitely many simple objects in $\mathcal{A}$, the bound for $\operatorname{gl.dim}\mathcal{A}$ follows from the previous lemma. □

Let us consider another class of exact categories.

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Lemma 7.2.7. *Let $\mathcal{A}$ be a Frobenius category and $T \in \mathcal{A}$ a tilting object. Then* $\operatorname{Proj}\mathcal{A} \subseteq \operatorname{add} T$.

Proof Every projective (and injective) object belongs to ${}^{\perp}T \cap T^{\perp} = \operatorname{add} T$. □

Tilting Modules

We consider an exact category $\mathcal{A}$ and study its tilting objects. A useful assumption is that $\mathcal{A}$ contains a projective tilting object. For example, this holds for a ring Λ when $\mathcal{A}$ equals the category of Λ-modules X having a resolution

$$0 \longrightarrow P_r \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

such that each P_i is finitely generated projective. We set

$$\mathcal{P}(\Lambda) := \operatorname{Res}(\operatorname{proj}\Lambda)$$

and note that $\mathcal{P}(\Lambda) = \operatorname{Thick}(\Lambda)$ by Proposition 7.1.7. Let us give a criterion for when $\mathcal{P}(\Lambda)$ is trivial.

Lemma 7.2.8. *We have* $\mathcal{P}(\Lambda) = \operatorname{proj}\Lambda$ *if and only if* $\operatorname{Hom}(X, \Lambda) \neq 0$ *for every finitely presented* Λ^{op}*-module* $X \neq 0$.

Proof Write $P^* = \operatorname{Hom}(P, \Lambda)$ for $P \in \operatorname{proj}\Lambda$. We have $\mathcal{P}(\Lambda) = \operatorname{proj}\Lambda$ if and only if every monomorphism $P \to Q$ in $\operatorname{proj}\Lambda$ splits. Such a monomorphism $P \to Q$ splits if and only if $Q^* \to P^*$ is an epimorphism. It remains to observe that $\operatorname{Hom}(X, \Lambda) = 0$ for $X = \operatorname{Coker}(Q^* \to P^*)$. □

We continue with an elementary characterisation of projective tilting objects; so all objects need to have finite projective dimension.

Lemma 7.2.9. *Let $\mathcal{A}$ be an exact category. Then a projective object P is a tilting object if and only if every object $A \in \mathcal{A}$ admits a finite resolution*

$$0 \longrightarrow P_r \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \qquad (P_i \in \operatorname{add} P).$$

Proof If P is a tilting object, then $P^{\perp} = \operatorname{Res}(\operatorname{add} P)$, by Proposition 7.1.10. Now use that $P^{\perp} = \mathcal{A}$ since P is projective. The other direction is clear since $\operatorname{Res}(\operatorname{add} P) \subseteq \operatorname{Thick}(P)$. □

Proposition 7.2.10. *Let $\mathcal{A}$ be an exact category and $P \in \mathcal{A}$ a projective tilting object. Then an object $T \in \mathcal{A}$ is a tilting object if and only if*

(1) $\operatorname{Ext}^n(T, T) = 0$ *for all* $n \neq 0$, *and*
(2) *there is an exact sequence*

$$0 \longrightarrow P \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^r \longrightarrow 0 \qquad (T^i \in \operatorname{add} T).$$

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Proof If T is a tilting object, then $P \in {}^{\perp}T = \mathrm{Cores}(\mathrm{add}\, T)$, by Proposition 7.1.10. Conversely, if $P \in \mathrm{Cores}(\mathrm{add}\, T)$, then $\mathcal{A} = \mathrm{Thick}(P) \subseteq \mathrm{Thick}(T) \subseteq \mathcal{A}$. □

When Λ is a ring, then a Λ-module T is called a *tilting module* (of finite projective dimension) if it is a tilting object of the exact category $\mathcal{P}(\Lambda)$. This means $\mathrm{Ext}^n(T,T) = 0$ for all $n \neq 0$, and $\mathrm{Thick}(T) = \mathrm{Thick}(\Lambda)$. More concretely, it follows from the above proposition that a Λ-module T is a tilting module if and only if

(T1) there is an exact sequence

$$0 \longrightarrow P_r \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow T \longrightarrow 0 \qquad (P_i \in \mathrm{proj}\, \Lambda),$$

(T2) $\mathrm{Ext}^n(T,T) = 0$ for all $n \neq 0$, and
(T3) there is an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^s \longrightarrow 0 \qquad (T^i \in \mathrm{add}\, T).$$

Now let Λ be an Artin k-algebra, and write $D = \mathrm{Hom}_k(-, E)$ for the Matlis duality given by an injective k-module E.

Example 7.2.11. The algebra Λ is Gorenstein if and only if $D(\Lambda)_\Lambda$ is a tilting module. In fact, the finite injective dimension of Λ_Λ corresponds to (T3), while the finite injective dimension of ${}_\Lambda\Lambda$ corresponds to (T1).

Example 7.2.12. Let Λ be an algebra such that every module of finite projective dimension is projective, so $\mathcal{P}(\Lambda) = \mathrm{proj}\, \Lambda$. This holds if and only if $\mathrm{Hom}(S, \Lambda) \neq 0$ for every simple Λ^{op}-module S, so for example when Λ is self-injective or local; see Lemma 7.2.8. Then a Λ-module T is tilting if and only if $\mathrm{add}\, T = \mathrm{proj}\, \Lambda$.

Any tilting object gives rise to a derived equivalence by Theorem 7.2.3, and the following result makes this more precise for modules over Artin algebras. For a generalisation involving tilting complexes, see Theorem 9.2.4.

Proposition 7.2.13. *Let Λ and Γ be Artin algebras of finite global dimension. Suppose that T_Λ is a tilting module and $\Gamma \cong \mathrm{End}_\Lambda(T)$. Then we have an adjoint pair of triangle equivalences*

$$\mathbf{D}^b(\mathrm{mod}\, \Lambda) \underset{\mathrm{RHom}_\Lambda(T,-)}{\overset{-\otimes^L_\Gamma T}{\rightleftarrows}} \mathbf{D}^b(\mathrm{mod}\, \Gamma).$$

Proof The pair of adjoint functors is taken from Proposition 4.3.15 and provides equivalences by Theorem 7.2.3, keeping in mind that $\mathbf{D}^{\mathrm{perf}}(\Gamma) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{mod}\, \Gamma)$. □

Remark 7.2.14. It suffices to assume that Λ has finite global dimension. Then for any tilting module T the algebra $\operatorname{End}_\Lambda(T)$ has finite global dimension, by Theorem 9.3.11.

Tilting Objects and Cotorsion Pairs

In Proposition 7.2.1 we have seen that each tilting object T yields a cotorsion pair $({}^\perp T, T^\perp)$. Now we wish to characterise the cotorsion pairs of an exact category that are induced by tilting objects. Let us keep the assumption that the category admits a projective tilting object.

Lemma 7.2.15. *Let $\mathcal{A}$ be an exact category and suppose $P \in \mathcal{A}$ is a projective tilting object. For a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ the following are equivalent.*

(1) *There is an exact sequence*

$$0 \longrightarrow P \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow \cdots \longrightarrow Y^r \longrightarrow 0 \qquad (Y^i \in \mathcal{Y}).$$

(2) $\operatorname{gl.dim} \mathcal{X} < \infty$.

(3) *There exists a tilting object T in $\mathcal{A}$ such that* $(\mathcal{X}, \mathcal{Y}) = ({}^\perp T, T^\perp)$.

Proof (1) $\Rightarrow$ (2): Set $Z^i := \operatorname{Ker}(Y^i \to Y^{i+1})$. Then we have for any $X \in \mathcal{X}$

$$\operatorname{Ext}^1(X, Z^i) \cong \operatorname{Ext}^{i+1}(X, P)$$

and therefore $\operatorname{Ext}^i(X, P) = 0$ for all $i > r$. This implies $\operatorname{Ext}^i(X, -) = 0$ for all $i > r$ since every object in $\mathcal{A}$ has a finite projective resolution; see Lemma 7.2.9.

(2) $\Rightarrow$ (3): Suppose that $\operatorname{gl.dim} \mathcal{X} = r$. We apply successively Remark 7.1.9 and obtain an exact sequence

$$0 \longrightarrow P \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^r \longrightarrow 0 \qquad (T^i \in \mathcal{X} \cap \mathcal{Y})$$

that terminates since $\operatorname{Ext}^{r+1}(X, P) = 0$. Thus $T = T^0 \oplus \cdots \oplus T^r$ is a tilting object by Proposition 7.2.10. We have

$${}^\perp T = {}^\perp(T^\perp) \subseteq {}^\perp\mathcal{Y} = \mathcal{X} \subseteq {}^\perp T$$

where the first equality holds by Proposition 7.2.1. Therefore ${}^\perp T = \mathcal{X}$. Analogously, $T^\perp = \mathcal{Y}$.

(3) $\Rightarrow$ (1): Apply Proposition 7.2.10. □

We call tilting objects T and T' *equivalent* if $\operatorname{add} T = \operatorname{add} T'$.

Proposition 7.2.16. *Let $\mathcal{A}$ be an exact category and P a projective tilting object. Then the assignment $T \mapsto T^\perp = \operatorname{Res}(\operatorname{add} T)$ gives a bijection between the equivalence classes of tilting objects of $\mathcal{A}$ and full additive subcategories $\mathcal{Y} \subseteq \mathcal{A}$ satisfying the following.*

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(1) *Each object $A \in \mathcal{A}$ fits into an admissible exact sequence*

$$0 \longrightarrow A \longrightarrow Y^A \longrightarrow X^A \longrightarrow 0$$

with $X^A \in {}^{\perp}\mathcal{Y}$ and $Y^A \in \mathcal{Y}$.

(2) *There is an exact sequence*

$$0 \longrightarrow P \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow \cdots \longrightarrow Y^r \longrightarrow 0 \qquad (Y^i \in \mathcal{Y}).$$

(3) *The subcategory $\mathcal{Y} \subseteq \mathcal{A}$ is closed under direct summands.*

Proof We have a correspondence between tilting objects and cotorsion pairs by Proposition 7.2.1. Combining this with Lemma 7.2.15 the assertion follows, once we observe that a subcategory $\mathcal{Y} \subseteq \mathcal{A}$ satisfying (1)–(3) gives rise to a cotorsion pair $({}^{\perp}\mathcal{Y}, \mathcal{Y})$ by Lemma 7.1.14. □

Let us consider the case $\mathcal{A} = \operatorname{mod} \Lambda$ when Λ is right coherent.

Proposition 7.2.17. *Let Λ be a right coherent ring of finite global dimension. Then the asssignment $T \mapsto ({}^{\perp}T, T^{\perp})$ gives a bijection between the equivalence classes of tilting objects of* $\operatorname{mod} \Lambda$ *and the cotorsion pairs for* $\operatorname{mod} \Lambda$.

Proof The assertion follows from Proposition 7.2.1 and Lemma 7.2.15. The inverse maps sends $(\mathcal{X}, \mathcal{Y})$ to $\mathcal{X} \cap \mathcal{Y} = \operatorname{add} T$. □

Finite Global Dimension

We consider an exact category $\mathcal{A}$ and make some additional assumptions:

(1) $\mathcal{A}$ is a Krull–Schmidt category, and
(2) $\mathcal{A}$ admits a projective tilting object.

Then the correspondence in Proposition 7.2.16 can be reformulated as follows.

Theorem 7.2.18. *The assignment $T \mapsto T^{\perp} = \operatorname{Res}(\operatorname{add} T)$ gives a bijection between the equivalence classes of tilting objects of $\mathcal{A}$ and full additive subcategories $\mathcal{Y} \subseteq \mathcal{A}$ that are covariantly finite and coresolving with* $\operatorname{gl.dim} {}^{\perp}\mathcal{Y} < \infty$. *The inverse map sends a subcategory $\mathcal{Y} \subseteq \mathcal{A}$ to an object T satisfying* $\operatorname{add} T = {}^{\perp}\mathcal{Y} \cap \mathcal{Y}$.

Clearly, the condition $\operatorname{gl.dim} {}^{\perp}\mathcal{Y} < \infty$ is obsolete when $\operatorname{gl.dim} \mathcal{A} < \infty$.

Proof We apply the correspondence of Corollary 7.1.15 between cotorsion pairs and covariantly finite and coresolving subcategories. The cotorsion pairs corresponding to tilting objects are characterised in Lemma 7.2.15. Given a tilting cotorsion pair $(\mathcal{X}, \mathcal{Y})$, the tilting object T is determined by the equality $\operatorname{add} T = \mathcal{X} \cap \mathcal{Y}$; see Proposition 7.2.1. □

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For a ring Λ we consider the category $\mathcal{P}(\Lambda)$ of modules having a finite projective resolution via finitely generated projective modules. Then gl.dim $\mathcal{P}(\Lambda)$ is called the *finitistic dimension* of Λ; it is conjectured to be finite when Λ is an Artin algebra, and this has been established for many classes of algebras.

Corollary 7.2.19. *Let Λ be an Artin algebra and suppose that Λ is Gorenstein.*

(1) *The assignment $T \mapsto T^{\perp} = \operatorname{Res}(\operatorname{add} T)$ gives a bijection between the equivalence classes of tilting objects of $\mathcal{P}(\Lambda)$ and full additive subcategories $\mathcal{Y} \subseteq \mathcal{P}(\Lambda)$ that are covariantly finite and coresolving.*
(2) *The assignment $T \mapsto {}^{\perp}T = \operatorname{Cores}(\operatorname{add} T)$ gives a bijection between the equivalence classes of tilting objects of $\mathcal{P}(\Lambda)$ and full additive subcategories $\mathcal{X} \subseteq \mathcal{P}(\Lambda)$ that are contravariantly finite and resolving.*

Proof Let Λ be Gorenstein of dimension d. This means that the injective dimensions of Λ_Λ and ${}_\Lambda\Lambda$ equal d. Then the projective dimension of every injective Λ-module is bounded by d. Thus gl.dim $\mathcal{P}(\Lambda) = d < \infty$ (Lemma 6.2.2) and $\mathcal{P}(\Lambda)$ has enough injective objects. Then (1) follows from Theorem 7.2.18, and (2) follows from (1) with Corollary 7.1.16. □

There are examples of rings such that the finitistic dimension is infinite. That means an exact category with a projective tilting object need not be of finite global dimension.

Example 7.2.20. Let k be a field and fix a partition $\mathbb{N} = \bigcup_i I_i$ into finite sets of unbounded cardinality. Consider the ring Λ which is obtained from localising the polynomial ring $A = k[x_0, x_1, x_2, \dots]$ at the complement of the union of the infinite set of prime ideals $\bigcup_i \mathfrak{p}_i$, where $\mathfrak{p}_i$ denotes the ideal generated by $\{x_n \mid n \in I_i\}$. Then this is an example of a commutative noetherian ring of infinite Krull dimension [146, Appendix, Example 1]. In fact, the height of $\mathfrak{p}_i$ in A is card I_i; so there is no bound for the height of a prime ideal in Λ. Moreover, there is no bound on the length of a regular sequence in Λ. It remains to note that for any commutative noetherian ring the supremum of the lengths of the regular sequences equals the finitistic dimension [12, Theorem 1.6].

APR Tilting Modules

Let Λ be an Artin algebra. We exhibit a particular class of tilting modules which have projective dimension one.

Proposition 7.2.21. *Let $e \in \Lambda$ be an idempotent such that no direct summand of $(1-e)\Lambda$ is isomorphic to $e\Lambda$. Suppose that $e\Lambda$ is a simple and non-injective Λ-module. Then $T = (1-e)\Lambda \oplus \operatorname{Tr} D(e\Lambda)$ is a tilting module.*

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Proof Set $S = e\Lambda$ and denote by $P_1, \dots, P_n$ a representative set of indecomposable projective Λ-modules which are not isomorphic to S. We need to check conditions (T1)–(T3) for T. The almost split sequence starting at S is of the form

$$0 \longrightarrow S \longrightarrow \bigoplus_i P_i^{d_i} \longrightarrow \operatorname{Tr} DS \longrightarrow 0 \tag{7.2.22}$$

for some $d_i \geq 0$. This follows from the fact that for each indecomposable summand X of the middle term, the morphism $\phi\colon X \to \operatorname{Tr} DS$ yields a morphism $D \operatorname{Tr} \phi\colon D \operatorname{Tr} X \to S$, which is non-zero when $D \operatorname{Tr} X \neq 0$. Thus X is projective since S is simple projective. This sequence gives immediately (T1) and (T3). Condition (T2) is deduced from the Auslander-Reiten formula, so $D \operatorname{Ext}^1(T,T) \cong \overline{\operatorname{Hom}}(T, D \operatorname{Tr} T) = 0$, since $D \operatorname{Tr} T \cong S$ is simple projective. □

Let us consider some specific algebras over a field k.

Example 7.2.23. Denote by Λ the k-algebra given by the following quiver with a commutativity relation:

$$\begin{array}{ccccc} & & 2 & & \\ & {\scriptstyle\alpha}\nearrow & & \searrow{\scriptstyle\beta} & \\ 1 & & & & 4 \\ & {\scriptstyle\gamma}\searrow & & \nearrow{\scriptstyle\delta} & \\ & & 3 & & \end{array} \qquad \text{with} \qquad \beta\alpha = \delta\gamma$$

Let $P_i = e_i\Lambda$ denote the indecomposable projective module corresponding to the vertex i. Then P_1 is simple, so

$$T = \operatorname{Tr} DP_1 \oplus P_2 \oplus P_3 \oplus P_4$$

is a tilting module with $\Gamma = \operatorname{End}(T)$ isomorphic to the path algebra of the quiver of Dynkin type D_4:

$$\begin{array}{cccc} 2 & & & \\ & \searrow & & \\ & & 1 \longrightarrow 4 & \\ & \nearrow & & \\ 3 & & & \end{array}$$

We may also get back from Γ to Λ, because the Γ-module

$$T' = P_1 \oplus \operatorname{Tr} DP_2 \oplus \operatorname{Tr} DP_3 \oplus \operatorname{Tr} DP_4$$

is tilting and $\operatorname{End}(T') \cong \Lambda$.

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Now let $\Lambda = kQ$ be the path algebra given by a finite quiver Q without oriented cycles. Suppose the vertex $i \in Q_0$ is a *source*, so no arrow ends in i, but at least one arrow starts at i. Denote by $Q(i)$ the quiver which is obtained from Q by reversing the orientation of each arrow starting at i.

Observe that the algebra Λ is hereditary. Thus the following illustrates the tilting process for hereditary abelian categories discussed in Theorem 5.1.2.

Proposition 7.2.24. *Let $i \in Q_0$ be a source of Q. Then the indecomposable projective module P_i is simple, and therefore*

$$T(i) = \operatorname{Tr} DP_i \oplus \Big(\bigoplus_{i \neq j \in Q_0} P_j \Big)$$

is a tilting module with $\operatorname{End}(T(i)) \cong kQ(i)$.

Proof The module $T(i)$ is tilting by Proposition 7.2.21. To compute its endomorphism algebra, we note that the almost split sequence (7.2.22) starting at P_i is of the form

$$0 \longrightarrow P_i \longrightarrow \bigoplus_{i \to j} P_j \longrightarrow \operatorname{Tr} DP_i \longrightarrow 0$$

where $i \to j$ runs through all arrows in Q starting at i. □

We denote by $|Q|$ the *underlying diagram* of Q which is obtained by forgetting the orientation of each arrow.

Corollary 7.2.25. *Let Q and Q' be acyclic quivers such that $|Q| = |Q'|$. Then there is a triangle equivalence* $\mathbf{D}^b(\operatorname{mod} kQ) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} kQ')$.

Proof For a sequence $i_1, \ldots, i_n$ of vertices in Q one defines recursively

$$Q(i_1, \ldots, i_n) = Q(i_1, \ldots, i_{n-1})(i_n).$$

Because the quivers are acyclic, it is not difficult to construct from the assumption $|Q| = |Q'|$ a sequence $i_1, \ldots, i_n$ of vertices such that $Q' = Q(i_1, \ldots, i_n)$. Then we obtain a sequence of n tilting modules from Proposition 7.2.24. These yield triangle equivalences connecting $\mathbf{D}^b(\operatorname{mod} kQ)$ and $\mathbf{D}^b(\operatorname{mod} kQ')$, by applying Theorem 5.1.2 or Proposition 7.2.13, and keeping in mind that a path algebra is hereditary. □

Tilting Objects for Quivers of Type A_n

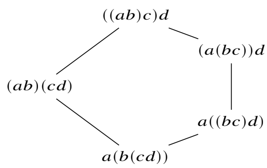
We describe the lattice of tilting objects for the category of representations of a quiver of type A_n; it is isomorphic to the Tamari lattice of order n.

The Tamari Lattice

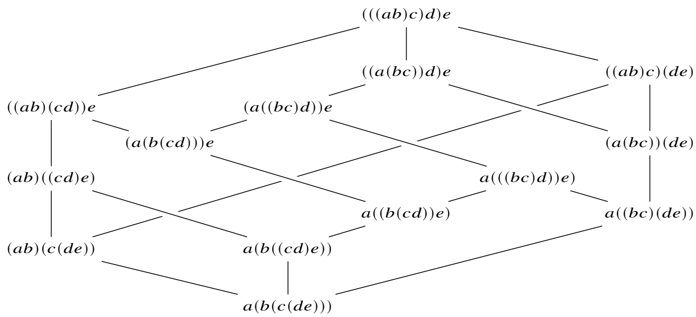
Fix an integer $n \geq 1$. The *Tamari lattice* of order n is a partially ordered set and is denoted by T_n. The elements consist of the meaningful bracketings of a string of $n+1$ letters. The partial order is given by applying the rule $(xy)z \to x(yz)$ from left to right. For example, when $n = 3$, we have

$$((ab)c)d \geq (a(bc))d \geq a((bc)d) \geq a(b(cd)).$$

Here is the Hasse diagram of the lattice T_3:



And here is the Hasse diagram of the lattice T_4:



The cardinality of the Tamari lattice T_n equals the *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Let $\mathcal{I}(n)$ denote the set of intervals $[i, j] = \{i, i+1, \ldots, j\}$ in $\mathbb{Z}$ with $0 \leq i < j \leq n$. For a pair of intervals I, J we set

$$I \perp J \quad :\Longleftrightarrow \quad I \subseteq J \quad \text{or} \quad J \subseteq I \quad \text{or} \quad I \cap J = \varnothing.$$

Let $\mathcal{T}(n)$ denote the set of all subsets $X \subseteq \mathcal{I}(n)$ of cardinality n such that $I \perp J$ for all $I, J \in X$.

Lemma 7.2.26. *Sending an interval* $[i, j]$ *to the bracketing*

$$x_0 \ldots (x_i \ldots x_j) \ldots x_n$$

of the string $x_0 \ldots x_n$ *induces a bijection* $\mathcal{T}(n) \xrightarrow{\sim} T_n$. □

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Representations of Type A_n

Fix an integer $n \geq 1$ and a field k. We consider the quiver of type A_n with linear orientation

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n$$

and denote by Λ_n its path algebra over k. For each $j \in \{1, \ldots, n\}$ let P_j denote the indecomposable projective Λ_n-module having as a k-basis all paths ending in the vertex j, and for each interval $I = [i, j]$ in $\mathbb{Z}$ with $0 \leq i < j \leq n$ we set $M_I := P_j / \operatorname{rad}^{j-i} P_j$.

Lemma 7.2.27. *The following holds for the modules M_I.*

(1) *The set $\{M_I \mid I \in \mathcal{I}(n)\}$ is a complete set of isomorphism classes of indecomposable Λ_n-modules.*
(2) $\operatorname{Ext}^1(M_I, M_J) = 0 = \operatorname{Ext}^1(M_J, M_I)$ *if and only if $I \perp J$.*
(3) *There is an epimorphism $M_I \to M_J$ if and only if $J \subseteq I$ and* $\sup J = \sup I$. □

A Λ_n-module T is a *basic tilting module* if T has precisely n pairwise non-isomorphic indecomposable direct summands and $\operatorname{Ext}^1(T, T) = 0$. Observe that the isomorphism classes of basic tilting modules correspond bijectively to the equivalence classes of tilting objects in $\operatorname{mod} \Lambda_n$, since $T \cong T'$ if and only if $\operatorname{add} T = \operatorname{add} T'$.

Write $T \geq T'$ if there is an epimorphism $T^r \to T'$ for some positive integer r. This induces a partial order on the isomorphism classes of basic tilting modules, and we have

$$T \geq T' \quad \Longleftrightarrow \quad T^{\perp} \supseteq T'^{\perp}$$

since an object M is in $T^{\perp}$ if and only if there is an epimorphism $T^r \to M$ for some positive integer r.

Proposition 7.2.28. *The assignment $X \mapsto \bigoplus_{I \in X} M_I$ induces a bijection between $\mathcal{T}(n)$ and the set of isomorphism classes of basic tilting modules over Λ_n. Composition with the bijection $T_n \xrightarrow{\sim} \mathcal{T}(n)$ yields a lattice isomorphism.* □

Notes

Tilting theory has a rich history [5]. The notion of a tilting module over a finite dimensional algebra was introduced by Brenner and Butler [41], using the conditions of Proposition 7.2.10 and assuming projective dimension at most one. A generalisation of the Coxeter functors arising in the work of Bernšteĭn,

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Gel'fand and Ponomarev [36] motivated the study of tilting modules; see also the contribution of Auslander, Platzeck and Reiten leading to the notion of APR tilting [14]. The original definition of a tilting module was later generalised in various directions.

According to Brenner and Butler, the term 'tilting' was chosen for the following reasons. Given a tilting object $T \in \mathcal{A}$, the functor $\mathrm{RHom}(T, -)$ swaps the components of the torsion pair $(\mathcal{T}, \mathcal{F})$ for $\mathcal{A}$ (Theorem 5.1.2). Inside the Grothendieck group $K_0(\mathcal{A}) \cong \mathbb{Z}^n$, the functor $\mathrm{RHom}(T, -)$ tilts the axes given by the standard basis vectors (Corollary 7.2.4). Moreover, the word 'tilting' inflicts well.

For representations of finite dimensional algebras, the link between tilting and derived categories was first established by Happel [101]. He proved that any tilting module induces a derived equivalence. A predecessor is a theorem of Beilinson that identifies a tilting object in the category of coherent sheaves on the projective n-space [25].

For modules over Artin algebras, the correspondence $T \mapsto T^{\perp}$ between tilting objects and covariantly finite and coresolving subcategories (Theorem 7.2.18) is due to Auslander and Reiten [16].

Gabriel noticed that the Catalan number C_n counts the tilting modules of the equioriented quiver of type A_n [81]. The connection with the Tamari lattice was pointed out in [43].

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