The result given above for unrestricted x suggests my second proof, which is considerably simpler. From the asymptotic expansion of the Gamma-function it follows that the function

$$-\frac{\Gamma(x)\,\Gamma(-x-\frac{1}{2})}{8\Gamma(\frac{3}{2}+x)\,\Gamma(1-x)}$$

satisfies the conditions (*Cf. Modern Analysis*, § 7·4) which admit of its being expressed as a sum of partial fractions. Since the residues of the function at the poles $x=-n$, $x=n-\frac{1}{2}$ are respectively

$$\tfrac{1}{2}A_n, \quad -\tfrac{1}{2}A_n,$$

we obtain anew the result

$$-\frac{\Gamma(x)\,\Gamma(-x-\frac{1}{2})}{8\,\Gamma(\frac{3}{2}+x)\,\Gamma(1-x)}=\sum_{n=0}^{\infty}\frac{A_n}{2x+2n}-\sum_{n=0}^{\infty}\frac{A_n}{2x-2n+1}$$

for unrestricted x.

It is, of course, possible to write down any number of generalisations of this result, the simplest perhaps being the expression for

$$\frac{\Gamma(x)\,\Gamma(-x-a)}{\Gamma(1+a+x)\,\Gamma(1-x)}$$

as a sum of partial fractions, where a is a suitably restricted constant.

On the configuration known as a double-six of lines

By H. W. Richmond, King's College, Cambridge.

In geometry of three dimensions it is well known that, when two quadrics Q_1, Q_2 are given, if one set of four points exists having the properties that each point lies on Q_1, and each two points are conjugate with respect to Q_2, an infinity of such sets of points can be found. The quadrics Q_1, Q_2 stand in a special relation to one another[1], expressed by the vanishing of the coefficient of λ in the discriminant of $Q_2+\lambda Q_1$, an invariant of Q_1, Q_2. Two quadrics Q_1, Q_2 are thus related if the equation of Q_1 contains no squares

[1] See Salmon, *Analytic Geometry of Three Dimensions*, Rogers' revised edition, Vol. 1, p. 204; or Sommerville, *Analytical Geometry of Three Dimensions*, p. 309.

https://doi.org/10.1017/S1757748900002462 Published online by Cambridge University Press

of the coordinates, and that of Q_2 contains no products of two coordinates; for then the vertices of the tetrahedron of reference form such a set of four points.

Even better known are the analogous properties in plane geometry concerning sets of three points and two conics: they need not be quoted. Similar results hold when more than four homogeneous variables are used; they may be enunciated as theorems in algebra, or in geometry of more than three dimensions. With six homogeneous coordinates, if in the first paragraph we replace "four" by "six," and the word "tetrahedron" by "simplex," we have a theorem of geometry of five dimensions.

But under certain conditions the six coordinates lend themselves to a simple interpretation in three-dimensional space. Points being represented by homogeneous coordinates (x_0, x_1, x_2, x_3), Plücker represented a line by six homogeneous coordinates $p_{01}, p_{02}, p_{03}, p_{23}, p_{31}, p_{12}$ satisfying a quadratic relation

$$p_{01}\, p_{23} + p_{02}\, p_{31} + p_{03}\, p_{12} \equiv 0: \qquad (Q_1)$$

since this contains no squared terms, it will serve for Q_1. Again, if S is any quadric surface in the space whose equation is expressed as

$$a_0\, x_0^2 + a_1\, x_1^2 + a_2\, x_2^2 + a_3\, x_3^2 = 0,$$

the condition that a line should touch S is

$$a_0\, a_1\, p_{01}^2 + a_0\, a_2\, p_{02}^2 + a_0\, a_3\, p_{03}^2 + a_2\, a_3\, p_{23}^2 + a_3\, a_1\, p_{31}^2 + a_1\, a_2\, p_{12}^2 = 0, \quad (Q_2)$$

or

$$\Sigma\, a_m\, a_n\, p_{mn}^2 = 0:$$

which, containing only squares of the coordinates (p), will serve for Q_2. The condition

$$\Sigma\, a_m\, a_n\, p_{mn}\, p'_{mn} = 0,$$

(which asserts that two sets of six coordinates p_{mn} and p'_{mn} are conjugate with respect to Q_2) bears the interpretation that either line meets the polar of the other with respect to S. Hence we have the theorem.

In connection with any quadric surface in space of three dimensions, there exist sets of six lines having the property that any one of the lines intersects the polar line of any other.

The quadric being given, a set of six lines may be built up as follows:—the first line (a_1) is chosen at random; its polar line (b_1) is then known. The second line (a_2) may be any line that meets b_1;

its polar line (b_2), which must meet a_1, is then known. For a_3 choose any line that meets b_1 and b_2; its polar line (b_3) will meet a_1 and a_2 and will be known. For a_4 choose any line that meets b_1, b_2, b_3; its polar line (b_4) will meet a_1, a_2, a_3 and will be known. But at this point, since only two lines meet b_1, b_2, b_3, b_4, one of them must be a_5 and the other a_6; their polars b_5 and b_6 are the two lines which meet a_1, a_2, a_3, a_4: and *at this point*, since all but one of the conditions binding the lines (a) have been satisfied, the force of the invariant relation between Q_1 and Q_2 becomes effective and proves the last condition, viz. that a_5 meets the polar line of a_6 and vice versa. The twelve lines a and b thus constitute a double-six of lines. Moreover any double-six of lines may be thus obtained.

The evaluation of certain continued fractions

By C. G. Darwin, University of Cambridge.

1. If the approximate numerical value of e is expressed as a continued fraction the result is

$$e = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1}\cdots \tag{1}$$

and it was in finding the proof that the sequence extends correctly to infinity that the following work was done. First the continued fraction may be simplified by setting down the difference equations for numerator and denominator as usual, and eliminating two out of every successive three equations. A difference equation is thus formed between the first, fourth, seventh, tenth .... convergents $\left(\text{counting the first as } 2 + \frac{1}{1}\right)$, and this equation will generate another continued fraction. After a little rearrangement of the first two members it appears that (1) implies

$$\frac{e-1}{e+1} = \frac{1}{2+}\,\frac{1}{6+}\,\frac{1}{10+}\cdots \tag{2}$$

2. We therefore consider the continued fraction

$$F\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \frac{\alpha}{\gamma+}\,\frac{\alpha+\beta}{\gamma+\delta+}\,\frac{\alpha+2\beta}{\gamma+2\delta+}\cdots \tag{3}$$

https://doi.org/10.1017/S1757748900002462 Published online by Cambridge University Press