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WEAKLY HOMOGENEOUS ORDER TYPES

BY
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An order type α is said to be *weakly homogeneous* ($\aleph_0$-*homogeneous*) if for any $x_1 < x_2$ and $y_1 < y_2$ there exists an order preserving bijection f on α such that $f(x_i) = y_i$ for $i = 1, 2$. The reverse of an order type α is denoted, as usual, by α^*. We say that α is *order invertible* if $\alpha^* \leq \alpha$. J. Q. Longyear [5] has asked whether for a weakly homogeneous order type α such that no (non-trivial) interval of α is order invertible we may deduce that every interval of α contains a copy of $\eta\omega_1$ or $(\eta\omega_1)^*$. We will show this to be false under the assumption of the Continuum Hypothesis (C.H.) by constructing such order types α in λ, the set of reals. We actually prove a more general result which implies, assuming C.H., that an order type β may be embedded in every weakly homogeneous order type α whose intervals are not order invertible iff β is countable.

The proof involves new variations on a type of argument first introduced by B. Dushnik and E. W. Miller [1] (see also S. Ginsburg [2], [3], [4] and B. Rotman [6]). C.H. is needed to ensure that the order types we construct will be weakly homogeneous. The symbol λ will denote the real line and η the rational members. Furthermore the Dedekind completion of a totally ordered set A will be given by A^+. Both components of a Dedekind cut are assumed to be non-void. We note that for totally ordered sets A and B any similarity map of A onto B can be extended to a similarity map of A^+ onto B^+ (B. Rotman [6]).

THEOREM (C.H.). *If C is a set of order types such that $\beta \leq \lambda$, $|\beta| = |C| = |\lambda|$ ($\beta \in C$), then there is a weakly homogeneous order type $\alpha \leq \lambda$ such that no interval of α is order invertible and such that $\beta \nleq \alpha$ ($\beta \in C$).*

Proof. If I is any interval of λ there are only $2^{\aleph_0}$ order inverting injections on I (a monotone function has countably many discontinuities). Since there are $2^{\aleph_0}$ intervals on λ it follows that there are $2^{\aleph_0} = \aleph_1$ order inverting injections f_i ($i < \omega_1$) whose domains are intervals of λ. Similarly if $B \subseteq \lambda$ is an ordered set of type β then there are $2^{\aleph_0}$ order preserving injections of B into λ. Since $|C| = 2^{\aleph_0}$ the order preserving injections of each β ($\beta \in C$) may be gathered into a single sequence of functions g_i ($i < \omega_1$).

Let $A' \subseteq \lambda$ be dense in λ. If A' is denumerable then it is weakly homogeneous. Therefore, if $X = \{x_1, x_2\} \subseteq A'$, $Y = \{y_1, y_2\} \subseteq A'$, $x_1 < x_2$, $y_1 < y_2$ there is an order

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preserving bijection $h=h(X, Y):A'\to A'$ such that $h(x_i)=y_i$ for $i=1, 2$. Since $A'^{+}=\lambda$, $h(X, Y)$ extends to an order preserving bijection $h^{+}(X, Y)$ on λ. We set $H(A')=\{h^{+}(X, Y) \mid X, Y\subseteq A' \text{ and } |X|=|Y|=2\}$. Since A' is denumerable, so is $H(A')$.

Our object is to define by transfinite induction a set A of type α that satisfies the conditions of the theorem. Put $A_0=\eta$, $B_0=\varnothing$, $H_0=H(A_0)$. Let $0<\delta<\omega_1$ and suppose that we have already defined sets A_γ, B_γ, H_γ for $\gamma<\delta$ such that for $\gamma\leq\gamma'<\delta$ we have: (a) $A_\gamma\subseteq A_{\gamma'}$, $B_\gamma\subseteq B_{\gamma'}$, $H_\gamma\subseteq H_{\gamma'}$, (b) A_γ, B_γ, H_γ are countable, (c) $A_\gamma \cap B_\gamma=\varnothing$, and (d) each $h\in H_\gamma$ is an order preserving bijection on λ such that $h(A_\gamma)=A_\gamma$.

The set $S_\delta=f_\delta-\{\langle x, x\rangle \mid x\in\lambda\}$ is uncountable. Thus we can always choose an ordered pair $\langle a_\delta, b_\delta\rangle\in S_\delta$ such that $a_\delta, b_\delta\notin\bigcup_{\gamma<\delta} A_\gamma \cup B_\gamma$. Moreover we will define two sets $T_{1\delta}$, $T_{2\delta}$ and prove that the pair $\langle a_\delta, b_\delta\rangle$ can be chosen in such a way that $a_\delta\notin T_{1\delta}$ and $\langle a_\delta, b_\delta\rangle\notin T_{2\delta}$. Put $H_\delta^-=\bigcup_{\gamma<\delta} H_\gamma$ and $K_\delta=\{h_1'\circ\cdots\circ h_n' \mid n<\omega_0, h_i'\in\{h_i, h_i^{-1}\}, h_i\in H_\delta^-(1\leq i\leq n)\}$. We let $T_{1\delta}=\{k(b)\mid b\in\bigcup_{\gamma<\delta} B_\gamma, k\in K_\delta\}$. $T_{1\delta}$ is countable because both K_δ and $\bigcup_{\gamma<\delta} B_\gamma$ are countable. Since S_δ represents an injection the value x of a pair $\langle x, y\rangle$ in S_δ can occur once at most, thus we can always insist that $a_\delta\notin T_{1\delta}$. Next we let $T_{2\delta}=f_\delta\cap(\bigcup K_\delta)=\{\langle x, y\rangle \mid \langle x, y\rangle\in f_\delta\cap k$ for some $k\in K_\delta\}$. Each $k\in K_\delta$ is a composition of order preserving bijections on λ and therefore is also order preserving. Since f_δ is order reversing it follows that there is at most one pair $\langle x, y\rangle\in f_\delta\cap k$ for each $k\in K_\delta$. Thus $T_{2\delta}$ is countable. Consequently we may choose the pair $\langle a_\delta, b_\delta\rangle\notin T_{2\delta}$.

Set $A_\delta=\bigcup_{\gamma<\delta}A_\gamma\cup\{k(a_\delta)\mid k\in K_\delta\}$. If B is the domain of g_δ choose $c_\delta\in g_\delta(B)-A_\delta$. A choice is always possible since $|B|=\aleph_1$. Put $B_\delta=\bigcup_{\gamma<\delta} B_\gamma\cup\{b_\delta, c_\delta\}$. The denumerable set A_δ is an extension of A_0 and therefore dense in λ. Put $H_\delta=\bigcup_{\gamma<\delta} H_\gamma\cup H(A_\delta)$.

We must now show that the conditions $(a), \ldots, (d)$ are satisfied by A_δ, B_δ, and H_δ. Clearly (a) and (b) are satisfied by definition. To see (c) consider $a\in A_\delta$. If $a\in\bigcup_{\gamma<\delta} A_\gamma$ then $a\notin B_\delta$. If $a\in A_\delta-\bigcup_{\gamma<\delta} A_\gamma$ then $a=k(a_\delta)$ for some $k\in K_\delta$. $a\notin\bigcup_{\gamma<\delta} B_\gamma$ since $a_\delta\notin T_{1\delta}$ and $a\neq b_\delta$ since $\langle a_\delta, b_\delta\rangle\notin T_{2\delta}$. Moreover c_δ is chosen not to be in A_δ. We conclude that $A_\delta\cap B_\delta=\varnothing$. Finally we must show that $h(A_\delta)=A_\delta$ for all $h\in H_\delta$. If $h\in H_\delta-\bigcup_{\gamma<\delta} H_\gamma$ then this is true by definition. If $h\in\bigcup_{\gamma<\delta} H_\gamma$ then we observe that $h(\bigcup_{\gamma<\delta} A_\gamma)=\bigcup_{\gamma<\delta} A_\gamma$. However any $a\in A_\delta-\bigcup_{\gamma<\delta} A_\gamma$ is of the form $k(a_\delta)$ for some $k\in K_\delta$. Thus $h(a)$ and $h^{-1}(a)$ are members of A_δ. We conclude $h(A_\delta)=A_\delta$.

We now let $A=\bigcup_{\delta<\omega_1} A_\delta$. Suppose some interval of A is order invertible. This defines an order inverting injection on the corresponding interval of λ. Hence there exists some $\delta<\omega_1$ such that f_δ represents this injection. a_δ is a member of A while $f_\delta(a_\delta)$ (that is to say b_δ) is not a member of A. By contradiction we conclude that no interval of A is order invertible. Since $c_\delta\notin A$, it follows that $g_\delta(B)\nsubseteq A$ $(\delta<\omega_1)$. Thus A does not contain a subset of type β for any $\beta\in C$. To exhibit the weak homogeneity of α we choose A_δ containing the finite set of points under

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consideration. An order preserving bijection with an appropriate restriction to the weakly homogeneous set A_δ can be found in H_δ. This concludes the proof.

COROLLARY (C.H.). *If β is an order type such that $|\beta|>\aleph_0$ then there exists a weakly homogeneous order type α such that no interval of α is order invertible, $\beta\not\leq\alpha$, and $\beta^*\not\leq\alpha$.*

COROLLARY (C.H.). *An order type β is embeddable in every order type α that is weakly homogeneous and has no order invertible intervals iff $|\beta|=\aleph_0$.*

Proof. Observe that if α has no order invertible intervals then α is dense. Thus we may embed η into any such α.

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