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# ON TOPOLOGICAL SEQUENCE ENTROPY OF PIECEWISE MONOTONIC MAPPINGS

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In this paper a full classification of piecewise monotonic maps from the point of view of the topological sequence entropy is established.

## 1. INTRODUCTION

Let $f : [0,1] \to [0,1]$ be a continuous map. $f$ is said to be *chaotic in the sense of Li-Yorke* if there are two points $x, y \in [0,1]$, $x \neq y$, such that

$$\liminf_{n\to\infty} \left| f^n(x) - f^n(y) \right| = 0,$$
$$\limsup_{n\to\infty} \left| f^n(x) - f^n(y) \right| > 0.$$

A useful tool to distinguish between chaotic and non-chaotic maps is the topological sequence entropy (see definition below), as stated in the following result (see [3]).

THEOREM 1.1. *Let $f : [0,1] \to [0,1]$ be continuous. Then $f$ is chaotic if and only if there exists an increasing sequence of positive integers $A$ such that $h_A(f) > 0$.*

Theorem 1.1 does not establish apparently any difference between two essentially different types of chaotic maps: chaotic maps of type $2^\infty$ and of type greater than $2^\infty$. On the other hand, for chaotic maps of type greater than $2^\infty$ there is a universal sequence, $A = (i)_{i=0}^{\infty}$, for which $h_A(f) > 0$ [6, Theorem 4.19]. For chaotic maps of type $2^\infty$ there is no such universal sequence (see [5]). However, we shall show that at least for piecewise monotonic maps a universal property characterises chaotic maps of type $2^\infty$: they have bounded topological sequence entropy. This property allows us to distinguish among piecewise monotonic chaotic maps of type $2^\infty$ and of type greater than $2^\infty$ from the topological sequence entropy point of view.

Let us introduce some definitions (see [6]). Recall that a point $x \in [0,1]$ is *periodic* if there exists a positive integer $n$ for which $f^n(x) = x$. The smallest integer satisfying this condition is called the *period* of $x$. For $r \in \mathbb{N}$, $f$ is said to be of *type* $2^r$ if it has

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periodic points of period $1, 2, \dots, 2^r$, but no other periods. $f$ is said to be of *type* $2^\infty$ if it has periodic points of period $2^n$ for all positive integers $n$, but no other periods. $f$ is said to be of type greater than $2^\infty$ if it has a periodic point with period $n = 2^p q$, where $q \in \{2k+1 : k \in \mathbb{N}\}$ and $p \in \mathbb{N} \cup \{0\}$.

Let $x \in [0,1]$. The set of limit points of the sequence $\left(f^n(x)\right)_{i=0}^{\infty}$, $\omega(x,f)$, is called the *ω-limit set* of $x$. Thus $y \in \omega(x,f)$ if and only if there is a sequence of positive integers $\{n_i\}_{i=1}^{\infty}$, with $\lim_{i\to\infty} n_i = \infty$, such that $\lim_{i\to\infty} f^{n_i}(x) = y$. $\omega(x,f)$ is closed and strongly invariant $\Big(f\big(\omega(x,f) = \omega(x,f)\big)\Big)$. Recall that a continuous map $f$ is *piecewise monotonic* if there exists a partition of $[0,1]$ into intervals such that $f$ is monotonic in each interval of the partition.

Now we introduce the notion of *topological sequence entropy* (see [**4**]). Let $A = (a_i)_{i=1}^{\infty}$ be an increasing sequence of positive integers and let $Y \subset [0,1]$. Fix $\varepsilon > 0$. A set $E \subset Y$ is said to be $(A, n, \varepsilon, Y, f)$–separated if for any $x, y \in E$, $x \neq y$, there is an $i \in \{1, 2, \dots, n\}$ such that $\left|f^{a_i}(x) - f^{a_i}(y)\right| > \varepsilon$. Denote by $s_n(A, \varepsilon, Y, f)$ the cardinality of any maximal $(A, n, \varepsilon, Y, f)$–separated set in $Y$. Define

$$s(A, \varepsilon, Y, f) := \limsup_{n\to\infty} \frac{1}{n} \log s_n(A, \varepsilon, Y, f).$$
$$h_A(f, Y) := \lim_{\varepsilon\to 0} s(A, \varepsilon, Y, f).$$

Define the *topological entropy of* $f$ as

$$h_A(f) := h_A\big(f, [0,1]\big).$$

When $A = (i)_{i=1}^{\infty}$, $h_A(f) = h(f)$ is the standard topological entropy (see [**1**]).

Let

$$h_\infty(f) := \sup_A h_A(f).$$

According to [**4**], we say that $f$ is *null* if $h_\infty(f) = 0$, $f$ is *bounded* if $h_\infty(f) < \infty$ and $f$ is *unbounded* if $h_\infty(f) = \infty$.

## 2. Preliminary Results

Before proving our main results, we need some additional notation and known results concerning $\omega$–limit sets of interval maps of type $2^\infty$. For any infinite $\omega$–limit set $\omega(x,f)$ of a map of type $2^\infty$ there exists a sequence of closed intervals $J^0 \supset J^1 \supset J^2 \supset \dots \supset J^n \supset \dots$ satisfying the following ([**7**]):

(C1) $f^{2^i}(J^i) = J^i$ and $f^j(J^i) \cap f^k(J^i) = \emptyset$ for $0 \leqslant j < k < 2^i$ and for all $i = 0, 1, \dots$ .

(C2) $\omega(x,f) \subset \bigcap_{i\geqslant 0} \bigcup_{j=0}^{2^i-1} f^j(J^i)$.

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Let $\mathrm{Orb}_f(J^i) = \bigcup_{j=0}^{2^i-1} f^j(J^i)$ for all $i \in \mathbb{N}$. The set $\bigcap_{i\geqslant 0} \mathrm{Orb}_f(J^i)$ is called a *solenoid*. For any $i \in \mathbb{N}$, let $J^i$ be a periodic interval of period $2^i$ with the following additional condition: if $I^i$ is an interval satisfying the conditions (C1)–(C2) and $I^i \cap J^i \neq \emptyset$, then $J^i = I^i$. In this case we say that $\bigcap_{i\geqslant 0} \mathrm{Orb}_f(J^i)$ is a *maximal solenoid*.

In order to handle easily each interval $f^j(J^i)$ with $i \geqslant 0$ and $0 \leqslant j \leqslant 2^i - 1$ we shall write them as follows. Consider the set of infinite sequences $\{0,1\}^\infty = \{(\alpha_i)_{i=1}^\infty : \alpha_i \in \{0,1\}\}$, and the sets $\{0,1\}^n = \{(\theta_i)_{i=1}^n : \theta_i \in \{0,1\}\}$ for any $n \in \mathbb{N}$. For each $n$ and $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in \{0,1\}^n$, let $\tau(\theta) := \sum_{i=1}^n \theta_i 2^{i-1}$. Clearly $\tau(\theta) \neq \tau(\vartheta)$ for any $\theta, \vartheta \in \{0,1\}^n$, $\theta \neq \vartheta$. For all $\alpha \in \{0,1\}^\infty$, let $\alpha|_n = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{0,1\}^n$. Let $\mathbf{0} = (0,0,\dots) \in \{0,1\}^\infty$ and $\mathbf{1} = (1,1,\dots) \in \{0,1\}^\infty$.

Fix $n \in \mathbb{N}$ and let $K_{\mathbf{0}|_n} = J^n$ ($K = J^0$). Put $K_\theta := f^{2^n - \tau(\theta)}(J^n)$ for any $\theta \in \{0,1\}^n \backslash \{\mathbf{0}|_n\}$. Since $\tau(\theta) \neq \tau(\vartheta)$ for any $\theta, \vartheta \in \{0,1\}^n$, $\theta \neq \vartheta$, it is clear that $K_\vartheta \cap K_\theta = \emptyset$. It is easy to check the properties summarised in the following lemma.

LEMMA 2.1. *Let $f$ be of type $2^\infty$. Let $\omega(x,f)$ be an infinite $\omega$-limit set of $f$. Then there exists a family of pairwise disjoint compact intervals (possibly degenerate) $\{K_\alpha : \alpha \in \{0,1\}^\infty\}$ satisfying the following conditions:*

(a) $\omega(x,f) \subset \bigcup_{\alpha \in \{0,1\}^\infty} K_\alpha$.

(b) $K_{\alpha|_n} \subset K_{\alpha|_m}$ *if* $n \geqslant m$, $n, m \in \mathbb{N}$, $\alpha \in \{0,1\}^\infty$. *Also* $K_\alpha = \bigcap_{n\geqslant 1} K_{\alpha|_n}$.

(c) *The set of nondegerate intervals $K_\alpha$ is at most countable. Moreover, if $K_\alpha$ in nondegenerate, then $f^j(K_\alpha) \cap f^i(K_\alpha) = \emptyset$ if $j \neq i$ ($K_\alpha$ is a* wandering interval*).*

(d) *Let $\alpha \in \{0,1\}^\infty \setminus \{\mathbf{0}\}$ and let $j$ be the first integer such that $\alpha_j = 1$. Then $f(K_\alpha) = K_\beta$ where $\beta_i = 1$ if $i < j$, $\beta_i = 0$ and $\beta_i = \alpha_i$ if $i > j$. Additionally $f(K_{\mathbf{0}}) = K_{\mathbf{1}}$.*

PROOF: The proof is easy and we omit it. □

The following lemma will be useful in what follows and was proved in [2].

LEMMA 2.2. *Fix $\varepsilon > 0$, and let $\mathcal{A}_\varepsilon = \{K_\alpha : |K_\alpha| \geqslant \varepsilon\}$. Then there exists a positive integer $n_0$ for which the following two conditions hold:*

(a) *If $K^-_{\alpha|_{n_0}}$ and $K^+_{\alpha|_{n_0}}$ denote the left and right side components of $K_\alpha \setminus K_{\alpha|_{n_0}}$ and $K_\alpha \in \mathcal{A}_\varepsilon$ then $\max\{|K^-_{\alpha|_{n_0}}|, |K^+_{\alpha|_{n_0}}|\} < \varepsilon$.*

(b) *If $\theta \in \{0,1\}^{n_0}$ and $K_\theta \neq K_{\alpha|_{n_0}}$ for all $K_\alpha \in \mathcal{A}_\varepsilon$, then $|K_\theta| < \varepsilon$.*

Let $\mathcal{S}$ be the set containing all the increasing sequences of positive integers. Define the *shift map* $\sigma : \mathcal{S} \to \mathcal{S}$ by $\sigma((a_i)_{i=1}^\infty) = (a_{i+1})_{i=1}^\infty$ for all $(a_i)_{i=1}^\infty \in \mathcal{S}$.

LEMMA 2.3. *Let $A = (a_i)_{i=1}^\infty \in \mathcal{S}$. Let $f : [0,1] \to [0,1]$ be continuous. Let $Y \subset [0,1]$ be an invariant set such that $f|_Y$ is surjective to $Y$. Let $Z \subset [0,1]$ satisfy*

*$f^k(Z) \subset Y$ for some positive integer $k$. Then for all $\varepsilon > 0$ it follows that*

$$s_n(A, 2\varepsilon, Z, f) \leqslant s_{n_0}(A, \varepsilon, Z, f) s_n\big(\sigma^{n_0}(A), \varepsilon, Y, f\big),$$

*where $n_0$ is the first integer such that $f^{a_{n_0+1}}(Z) \subset Y$.*

PROOF: Let $E_1$ and $E_2$ be an $(A, n_0, \varepsilon, Z, f)$–separated set and an $(\sigma^{n_0}(A), n - n_0, \varepsilon, Y, f)$–separated set with maximal cardinalities. Let $z \in Z$ and choose $x_z \in Y$ such that $f^k(x_z) = f^k(z)$ (this can be done because $f|_Y$ is surjective). Then for any $z \in Z$ one can assign a pair $(x_1, x_2) \in E_1 \times E_2$ such that $\left|f^{a_i}(z) - f^{a_i}(x_1)\right| < \varepsilon$ if $1 \leqslant i \leqslant n_0$ and $\left|f^{a_i}(z) - f^{a_i}(x_2)\right| = \left|f^{a_i}(x_z) - f^{a_i}(x_2)\right| < \varepsilon$ if $n_0 < i \leqslant n$. Two different points $z_1$ and $z_2$ of an $(A, n_0, 2\varepsilon, Z, f)$–separated set have different pairs associated. Conversely if they are not associated to the same pair, they cannot belong to the same $(A, n_0, 2\varepsilon, Z, f)$–separated set. Then

$$\begin{aligned} s_n(A, 2\varepsilon, Z, f) &\leqslant s_{n_0}\big(A, \varepsilon, Z, f\big) s_{n-n_0}(\sigma^{n_0}(A), \varepsilon, Y, f) \\ &\leqslant s_{n_0}(A, \varepsilon, Z, f) s_n\big(\sigma^{n_0}(A), \varepsilon, Y, f\big), \end{aligned}$$

and this concludes the proof. ∎

LEMMA 2.4. *Let $A = (a_i)_{i=1}^{\infty} \in \mathcal{S}$. Let $f : [0,1] \to [0,1]$ be of type $2^\infty$ with a finite number of maximal solenoids $S^i$, with $1 \leqslant i \leqslant k$. Let $K^i_{0|_j}$ be periodic intervals of period $2^j$, $j \in \mathbb{N}$, such that $S_i \subset \mathrm{Orb}_f(K^i_{0|_j})$. For all $n \in \mathbb{N}$ let*

$$L_n = \left\{x \in [0,1] : f^{a_r}(x) \notin \bigcup_{i=1}^{k} \mathrm{Orb}_f(K^i_{0|_j}) \text{ for } 1 \leqslant r \leqslant n\right\}.$$

*Then for any $\varepsilon > 0$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log s_n(A, \varepsilon, L_n, f) = 0.$$

PROOF: Assume the contrary for some $\varepsilon > 0$ and define the following map $\widehat{f} : [0,1] \to [0,1]$. Let $\widehat{f}(x) = f(x)$ if $x \in [0,1] \setminus \bigcup_{i=1}^{k} \mathrm{Orb}_f(K^i_{0|_j})$, and define $\widehat{f}$ on $\bigcup_{i=1}^{k} \mathrm{Orb}_f(K^i_{0|_j})$ to be continuous and linear in each subinterval. Hence, $(\widehat{f}|_{K^i_\theta})^{2^j}$ is monotone for any $i = 1, 2, \dots, k$ and $\theta \in \{0,1\}^j$. So $(\widehat{f}|_{K^i_\theta})^{2^j}$ has periodic points of period at most two. Then any $\omega(x, \widehat{f})$ is finite, that is, it is a periodic orbit. From [**6**, p.73], $\widehat{f}$ is of type $2^l$ for some $l \in \mathbb{N}$, and therefore it is non–chaotic. On the other hand, it is obvious that $s_n(A, \varepsilon, L_n, f) = s_n(A, \varepsilon, L_n, \widehat{f})$. Since $s_n(A, \varepsilon, L_n, \widehat{f}) \leqslant s_n\big(A, \varepsilon, [0,1], \widehat{f}\big)$ we conclude that

$$\begin{aligned} h_A(\widehat{f}) &\geqslant \limsup_{n\to\infty} \frac{1}{n} \log s_n\big(A, \varepsilon, [0,1], \widehat{f}\big) \\ &\geqslant \limsup_{n\to\infty} \frac{1}{n} \log s_n(A, \varepsilon, L_n, f) > 0, \end{aligned}$$

and this leads to a contradiction because $\widehat{f}$ is non–chaotic and, according to Theorem 1.1, its topological sequence entropy must be zero. ∎

## 3. Main Results.

Our main goal is the following theorem.

**Theorem 3.1.** *Let $f$ be a continuous map having a finite number of maximal solenoids. Then*

(a) *$f$ is non chaotic if and only if $h_\infty(f) = 0$ ($f$ is null).*

(b) *$f$ is chaotic of type $2^\infty$ if and only if $h_\infty(f) = \log 2$ ($f$ is bounded and non-null).*

(c) *$f$ is chaotic of type greater than $2^\infty$ if and only if $h_\infty(f) = \infty$ ($f$ is unbounded).*

Proof: Part (a) follows from Theorem 1.1. For part (c), let $A = (2^i)_{i=1}^\infty$. Then

$$K(A) = \lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{n}\operatorname{Card}\bigcup_{i=1}^{n}\{2^i, 2^i+1, \dots, 2^i+k\} = \infty.$$

By [**4**], $h_A(f) = K(A)h(f)$. Since $f$ is of type greater than $2^\infty$, $h(f) > 0$ (see, for example, [**6**]). Then $h_A(f) = \infty$.

Part (b). Let $S^i = \bigcap_{n\geqslant 1} \operatorname{Orb}_f(K^i_{0|_n})$ be the maximal solenoids of $f$ with $1 \leqslant i \leqslant k$. By Lemma 2.2, for any $\varepsilon > 0$ we can find a positive integer $n_\varepsilon$ such that if $\mathcal{A}^i_\varepsilon = \{K^i_\alpha : |K^i_\alpha| \geqslant \varepsilon\}$, $i = 1, 2, \dots, k$, then:

(a) $\operatorname{Orb}_f(K^i_{0|_n}) \cap \operatorname{Orb}_f(K^j_{0|_n}) = \emptyset$ if $i \neq j$.

(b) If $K^{-,i}_{\alpha|_{n_\varepsilon}}$ and $K^{+,i}_{\alpha|_{n_\varepsilon}}$ denotes the left and right side components of $K^i_\alpha \setminus K^i_{\alpha|_{n_\varepsilon}}$ and $K^i_\alpha \in \mathcal{A}^i_\varepsilon$ then $\max\{|K^{-,i}_{\alpha|_{n_\varepsilon}}|, |K^{+,i}_{\alpha|_{n_\varepsilon}}|\} < \varepsilon$.

(c) If $\theta \in \{0,1\}^{n_\varepsilon}$ and $K^i_\theta \neq K^i_{\alpha|_{n_\varepsilon}}$ for all $K^i_\alpha \in \mathcal{A}^i_\varepsilon$, then $|K^i_\theta| < \varepsilon$.

We claim that for any positive integer $n$ the number $s_n(A, \varepsilon, [0,1], f) \leqslant q(n)2^n$, where $\limsup_{n\to\infty}(\log q(n))/n = 0$. This will give us

$$\begin{aligned} s(A, \varepsilon, [0,1], f) &= \limsup_{n\to\infty} \frac{1}{n} \log s_n(A, \varepsilon, [0,1], f) \\ &\leqslant \limsup_{n\to\infty} \frac{1}{n} \log q(n)2^n = \log 2 \end{aligned}$$

for all $\varepsilon > 0$, and this will imply that $h_A(f) \leqslant \log 2$ for any increasing sequence $A$. Following the proof of [**3**, Theorem 2.2], there is a sequence $B$ such that $h_B(f) \geqslant \log 2$. Thus, we shall obtain the result.

Let $L_0 = \bigcup_{i=1}^{k} \operatorname{Orb}_f(K^i_{0|_{n_\varepsilon}})$, and let $L_n$ be the set defined in Lemma 2.4 for any $n \in \mathbb{N}$. First, we estimate $s_n(A, \varepsilon, K^i_\theta, f)$ for some $1 \leqslant i \leqslant k$ and $\theta \in \{0,1\}^{n_\varepsilon}$. Let $\theta_j \in \{0,1\}^{n_\varepsilon}$ be such that $f^{a_j}(K^i_\theta) = K^i_{\theta_j}$ for $1 \leqslant j \leqslant n$. We assign to each $x \in K^i_\theta$ a code $(C_1, C_2, \dots, C_n)$ as follows: if $f^{a_j}(x) \in K^i_{\theta_j}$ with $|K^i_{\theta_j}| < \varepsilon$, then we put $C_j = C$. If $\theta_j = \alpha|_{n_\varepsilon}$ for some

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$K_\alpha^i \in \mathcal{A}_\varepsilon^i$, then we put $C_j = L$ if $f^{a_j}(x) \in K_{\theta_j}^{-,i}$ and $C_j = R$ if $f^{a_j}(x) \in K_{\theta_j}^{+,i}$. Divide all the intervals $K_\alpha^i \in \mathcal{A}_\varepsilon^i$ into pairwise disjoint intervals, $K_\alpha^{i,1}$, $K_\alpha^{i,2}$, ... ,$K_\alpha^{i,r_\alpha}$, each of length smaller than $\varepsilon$. If $f^{a_j}(x) \in K_\alpha^{i,s}$ we put $C_j = K_\alpha^{i,s}$. It is clear that

$$s_n(A, \varepsilon, K_\theta^i, f) \leqslant \operatorname{Card}\{(C_1, C_2, \dots, C_n) : C_i \in \{C, L, R, K_\alpha^{i,j}\}\}. \tag{1}$$

We are going to obtain an upper bound for $s_n(A, \varepsilon, K_\theta^i, f)$. Let $r_i = \operatorname{Card}(\mathcal{A}_\varepsilon^i)$ and let $R_i = \max\{r_\alpha : K_\alpha^i \in \mathcal{A}_\varepsilon^i\}$. Notice that any $K_\alpha^i \in \mathcal{A}_\varepsilon^i$ is a wandering interval. So, by Lemma 2.1, if $x \in K_\theta^i$, then $f^{a_j}(x)$ belongs to at most $r_i$ intervals $K_\alpha^i \in \mathcal{A}_\varepsilon^i$. Moreover, if for some $x \in K_\theta^i$ and $f^{a_j}(x) \in K_\alpha^i \in \mathcal{A}_\varepsilon^i$, then $f^{a_s}(x) \notin K_\alpha^i$ for all $s \neq j$. Hence, the number of codes is at most

$$\left(\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{r_i}\right)(R_i)^{r_i} 2^n. \tag{2}$$

By (1)

$$s_n(A, \varepsilon, K_\theta^i, f) \leqslant \left(\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{r_i}\right)(R_i)^{r_i} 2^n. \tag{3}$$

Finally, if $R = \max\{R_i : 1 \leqslant i \leqslant k\}$, $r = \max\{r_i : 1 \leqslant i \leqslant k\}$ and

$$p(n) = \left(\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{r}\right),$$

then

$$s_n(A, \varepsilon, L_0, f) \leqslant kR^r p(n) 2^{n+n_\varepsilon}. \tag{4}$$

By Lemma 2.3, since $f|_{L_0}$ is surjective, we obtain for $1 \leqslant i \leqslant n$ that

$$s_n(A, 2\varepsilon, L_i, f) \leqslant s_i(A, \varepsilon, L_i, f) s_n(\sigma^i(A), \varepsilon, L_0, f).$$

By (4)

$$s_n(A, 2\varepsilon, L_i, f) \leqslant s_i(A, \varepsilon, L_i, f) kR^r p(n) 2^{n+n_\varepsilon}.$$

Hence

$$\begin{aligned}
s_n(A, 2\varepsilon, [0,1], f) &\leqslant s_n\Big(A, 2\varepsilon, [0,1] \setminus \bigcup_{i=0}^n L_i, f\Big) + \sum_{i=0}^n s_n(A, 2\varepsilon, L_i, f) \\
&\leqslant s_n\Big(A, \varepsilon, [0,1] \setminus \bigcup_{i=0}^n L_i, f\Big) + \sum_{i=0}^n s_n(A, 2\varepsilon, L_i, f) \\
&\leqslant s_n\Big(A, \varepsilon, [0,1] \setminus \bigcup_{i=0}^n L_i, f\Big) \\
&\quad + \Big(1 + \sum_{i=1}^n s_i(A, \varepsilon, L_i, f)\Big) kR^r p(n) 2^{n+n_\varepsilon} \\
&\leqslant (2+n) Q(n) kR^r p(n) 2^{n+n_\varepsilon},
\end{aligned}$$

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where

$$Q(n) = \max\Big\{\{s_i(A,\varepsilon,L_i,f) : i = 1,2,\dots,n\} \cup \Big\{s_n\Big(A,\varepsilon,[0,1]\setminus\bigcup_{i=0}^{n} L_i, f\Big)\Big\}\Big\}.$$

By Lemma 2.4, $\limsup_{n\to\infty}(\log Q(n))/n = 0$, and this concludes the proof. □

**Theorem 3.2.** *Let $f : [0,1] \to [0,1]$ be piecewise monotonic. Then:*

(a) *$f$ is non-chaotic if and only if $h_\infty(f) = 0$ ($f$ is null).*

(b) *$f$ is chaotic of type $2^\infty$ if and only if $h_\infty(f) = \log 2$ ($f$ is bounded and non-null).*

(c) *$f$ is chaotic of type greater than $2^\infty$ if and only if $h_\infty(f) = \infty$ ($f$ is unbounded).*

Proof: Let $I_1, I_2, \dots, I_k$ be maximal subintervals of $[0,1]$ such that $f|_{I_i}$ is monotone and $[0,1] = \bigcup_{i=1}^{k} I_i$ for some $k \in \mathbb{N}$. By Theorem 3.1 it suffices to prove that the number of maximal solenoids is smaller than $k$. Let $\omega(x,f)$ be an infinite $\omega$-limit set contained in a solenoid $\bigcap_{n\geqslant 1}\bigcup_{\theta\in\{0,1\}^n} K_\theta$. Then for any positive integer $n$ there must exist an interval $K_\theta$ with $\theta \in \{0,1\}^n$ for which $f|_{K_\theta}$ is not monotone. Assume the contrary: let $n \in \mathbb{N}$ be such that $f|_{K_\theta}$ is monotonic for any $\theta \in \{0,1\}^n$. Then $(f|_{K_\theta})^{2^n}$ has at most periodic points of period 2 and this leads to a contradiction because $\bigcup_{\theta\in\{0,1\}^n} K_\theta$ could not contain an infinite $\omega$-limit set of $f$. So the number of maximal solenoids is smaller than $k$ and the proof is complete. □

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