UNIFIED FIELD THEORY

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Introduction. In a recent unified theory originated by Einstein and Straus [1], the gravitational and electromagnetic fields are represented by a single non-symmetric tensor g_{ij} which is a function of four coordinates $x^r (r = 1, 2, 3, 4)$. In addition a non-symmetric linear connection $\Gamma_{jk}{}^i$ is assumed for the space and a Hamiltonian function is defined in terms of g_{ij} and $\Gamma_{jk}{}^i$. By means of a variational principle in which the g_{ij} and $\Gamma_{jk}{}^i$ are allowed to vary independently the field equations are obtained and can be written

$$g_{ik,a} - g_{sk}\,\Gamma_{ia}{}^s - g_{is}\,\Gamma_{ak}{}^s = 0, \tag{0.1}$$

$$\Gamma_{ia}{}^a - \Gamma_{ai}{}^a = 0, \tag{0.2}$$

$$R_{\underline{ik}} = 0, \tag{0.3}$$

$$R_{\underset{\vee}{ik},a} + R_{\underset{\vee}{ka},i} + R_{\underset{\vee}{ai},k} = 0. \tag{0.4}$$

In the above equations the comma in $g_{ik,a}$ or $R_{ik,a}$ denotes partial differentiation with respect to x^a. Further R_{ik} stands for the Ricci tensor based on the linear connection $\Gamma_{jk}{}^i$. The symbols $R_{\underline{ik}}$, $R_{\underset{\vee}{ik}}$ stand, respectively, for the symmetric and skew-symmetric parts of the tensor R_{ik} and hence

$$R_{\underline{ik}} = \tfrac{1}{2}(R_{ik} + R_{ki}), \tag{0.5}$$

$$R_{\underset{\vee}{ik}} = \tfrac{1}{2}(R_{ik} - R_{ki}). \tag{0.6}$$

The same notation is used throughout to denote the symmetric and skew-symmetric parts of other quantities entering into the new theory.

In the linearized field equations corresponding to the rigorous field equations (0.1)-(0.4) it has been found that the linearized field equations for the skew-symmetric part of the field are weaker than Maxwell's equations. It was pointed out that this in itself did not constitute a justified objection to the new theory as it was not known whether there were rigorous solutions of the field equations which were regular in all space and which would correspond to the solutions one could obtain for the linearized equations. For this reason it became important to determine rigorous solutions of equations (0.1)-(0.4).

Recently Papapetrou[1] has discussed the static spherically symmetric form of these equations and has discovered two rigorous solutions. The second

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[1]In [2] the field equations contain a cosmological constant λ which is zero in the Einstein field equations. When using Papapetrou's results we shall always take $\lambda = 0$.

solution is a very special case. In discussing his solutions Papapetrou points out that neither solution approaches asymptotically the corresponding solution obtained by means of the General Theory of Relativity.

In the present paper we shall generalize Papapetrou's second solution and shall in addition discuss some of the difficulties presented by the new Unified Theory.

1. Papapetrou's second case. Papapetrou took the static spherically symmetric tensor g_{ik} to have, in spherical polar coordinates, the form

$$g_{ik} = \begin{bmatrix} -\alpha & 0 & 0 & w \\ 0 & -\beta & r^2 v \sin\theta & 0 \\ 0 & -r^2 v \sin\theta & -\beta \sin^2\theta & 0 \\ -w & 0 & 0 & \gamma \end{bmatrix},$$

where α, β, γ, v, w are undetermined functions of r. For the case $v = 0$, $w \neq 0$, the general solution of the field equations was found to be [2]

$$\alpha = (1 - 2m/r)^{-1}, \ \beta = r^2,$$

$$\gamma = (1 + l^4/r^4)\,(1 - 2m/r), \ v = 0, \ w = \pm\, l^2/r^2,$$

where m, l are constants of integration. For the second case $v \neq 0$, $w = 0$ Papapetrou was unable to find the general solution but found a special case

$$\gamma = \alpha^{-1} = (1 - 2m/r), \ \beta = r^2, \ v = -c, \ w = 0,$$

where m, c are constants of integration. We shall now proceed to find the general solution corresponding to this second case $v \neq 0$, $w = 0$.

For the case $v \neq 0$, $w = 0$ Papapetrou has shown that the field equations reduce to

$$(1.1) \qquad f = vr^2, \quad A = (\beta\beta' + ff')/(f^2 + \beta^2), \ B = (f\beta' - \beta f')/(f^2 + \beta^2),$$

$$(1.2) \qquad A' + \tfrac{1}{2}(A^2 + B^2) - \tfrac{1}{2}A\,[(\alpha'/\alpha) + (\gamma'/\gamma)] = 0,$$

$$(1.3) \qquad \gamma'' - \tfrac{1}{2}\gamma'[(\alpha'/\alpha) + (\gamma'/\gamma)] + A\gamma' = 0,$$

$$(1.4) \qquad \beta'' - f'B - \tfrac{1}{2}\beta'[(\alpha'/\alpha) - (\gamma'/\gamma)] + 2\alpha(2\beta f c - \beta^2 + f^2)/(f^2 + \beta^2) = 0,$$

$$(1.5) \qquad f'' + \beta' B - \tfrac{1}{2}f'[(\alpha'/\alpha) - (\gamma'/\gamma)] - 2\alpha(2\beta f + c\beta^2 - cf^2)/(f^2 + \beta^2) = 0,$$

where the prime notation indicates differentiation with respect to r and c is an arbitrary constant of integration. In the above equations (1.1) is simply a definition of the symbols A, B and f while the remaining equations are the field equations for this particular case.

Since $A = \dfrac{d}{dr} \log\, (f^2 + \beta^2)^{\frac{1}{2}}$ equation (1.3) can be integrated to give

$$(1.6) \qquad \gamma' = 2m\,[\alpha\gamma/(f^2 + \beta^2)]^{\frac{1}{2}},$$

where m is an arbitrary constant of integration. It has been taken in this form as it will later be identified with the mass of the spherical body. We shall throughout the remainder of this section deal only with the case $m \neq 0$. When this is so $\gamma' \neq 0$ and γ is not a constant.

Due to the tensorial character of g_{ik} one of α, β, γ can be chosen arbitrarily.[2] We shall find that the general solution is most easily obtained if we allow γ to be the variable that has this arbitrary character.

Concentrating our attention on equations (1.4) and (1.5) we find it advantageous to replace these equations by two equivalent equations. Multiplying (1.4) by $\beta/(f^2+\beta^2)$ and (1.5) by $f/(f^2+\beta^2)$ and adding the results we find

$$(1.7)\quad (\beta\beta''+ff'')/(f^2+\beta^2)+B^2-\tfrac{1}{2}A[(\alpha'/\alpha)-(\gamma'/\gamma)]+2\alpha(cf-\beta)/(f^2+\beta^2)=0.$$

Since

$$A'=(\beta\beta''+ff'')/(f^2+\beta^2)+B^2-A^2,$$

(1.7) can be written

$$(1.8)\quad A'+A^2-\tfrac{1}{2}A[(\alpha'/\alpha)-(\gamma'/\gamma)]+2\alpha(cf-\beta)/(f^2+\beta^2)=0.$$

Similarly by multiplying (1.4) by $f/(f^2+\beta^2)$ and (1.5) by $\beta/(f^2+\beta^2)$ and subtracting the results we can obtain the equation

$$(1.9)\quad B'+AB-\tfrac{1}{2}B[(\alpha'/\alpha)-(\gamma'/\gamma)]+2\alpha(c\beta+f)/(f^2+\beta^2)=0.$$

Thus equations (1.8) and (1.9) are equivalent to (1.4) and (1.5).

If we let $i=(-1)^{\frac{1}{2}}$ and introduce the complex variable $q=k+iu$ by means of the equation

$$(1.10)\qquad f+i\beta=e^q,$$

we find that

$$(1.11)\qquad A+iB=q',$$

and hence $A=k'$, $B=u'$. Multiplying (1.9) by i and adding (1.8) one obtains the equation

$$(1.12)\quad q''+[A-\tfrac{1}{2}\{(\alpha'/\alpha)-(\gamma'/\gamma)\}]q'+2\alpha(c+i)e^q/(f^2+\beta^2)=0.$$

Thus the single equation (1.12) in the complex variable q is equivalent to the two real equations (1.8) and (1.9).

Since m was assumed to be non-zero we can solve (1.6) for α to obtain

$$(1.13)\qquad \alpha=(\gamma')^2(f^2+\beta^2)/4m^2\gamma.$$

Substituting in (1.12) for α gives

$$(1.14)\quad q''-q'[(\gamma''/\gamma')-(\gamma'/\gamma)]+\gamma'^2(c+i)e^q/2m^2\gamma=0.$$

[2]Since we are excluding the case γ = constant our phrase "chosen arbitrarily" excludes this choice of γ for which the statement is not true. Certain differentiability conditions are also implied by the field equations.

From the fact that $q' = \frac{dq}{d\gamma}\gamma'$ and $q'' = \frac{d^2q}{d\gamma^2}\gamma'^2 + \frac{dq}{d\gamma}\gamma''$, equation (1.14) can be written

$$\frac{d^2q}{d\gamma^2} + \left(\frac{dq}{d\gamma}/\gamma\right) + (c+i)e^q/2m^2\gamma = 0. \tag{1.15}$$

The substitution

$$q = y - \log\gamma, \quad x = \log\gamma, \tag{1.16}$$

reduces this equation to

$$\frac{d^2y}{dx^2} + [(c+i)e^y/2m^2] = 0. \tag{1.17}$$

Equation (1.17) is easily integrated once to give

$$\left(\frac{dy}{dx}\right)^2 + [(c+i)e^y/m^2] = h, \tag{1.18}$$

where h is an arbitrary complex constant of integration. In (1.18) we can separate the variables and then integrate to find

$$e^y = [m^2h \operatorname{sech}^2(\tfrac{1}{2}h^{\frac{1}{2}}x + a)]/(c+i), \tag{1.19}$$

where a is a second complex constant of integration. Returning to our original variable we find

$$e^q = 4m^2h/[(e^a\gamma^{\frac{1}{2}h^{\frac{1}{2}}} + e^{-a}\gamma^{-\frac{1}{2}h^{\frac{1}{2}}})^2\gamma(c+i)]. \tag{1.20}$$

Thus far we have found the general solution of equations (1.3), (1.4) and (1.5) and so far no use has been made of equation (1.2). From the tensorial character of our equations and the arbitrary character of γ we know that one of the equations (1.2), (1.3), (1.4) and (1.5) is redundant. It has however been shown by Papapetrou that this redundant equation is (1.5). We shall see that in order for our solution (1.20) to satisfy (1.2) the number of arbitrary constants in the solution is reduced by one.

If we transform equation (1.18) back to the variables q and γ by means of $y = q + \log\gamma$, $x = \log\gamma$ we find

$$\left(\gamma\frac{dq}{d\gamma} + 1\right)^2 + [(c+i)e^q\gamma/m^2] = h. \tag{1.21}$$

Since $\frac{dq}{d\gamma} = q'/\gamma'$ this equation can be written

$$(q')^2 + (2\gamma'q'/\gamma) + [(c+i)e^q\gamma'^2/m^2\gamma] = (h-1)\gamma'^2/\gamma^2. \tag{1.22}$$

Substituting $q' = A + iB$ and $e^q = f + i\beta$ we can by equating real and imaginary parts of (1.22) obtain the equations

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$$(1.23) \qquad A^2 - B^2 + [2\gamma' A/\gamma] + (cf - \beta)\gamma'^2/m^2\gamma = (h_0 - 1)\gamma'^2/\gamma^2,$$

$$(1.24) \qquad AB + (\gamma' B/\gamma) + [(c\beta + f)\gamma'^2/2m^2\gamma] = h_1\gamma'^2/2\gamma^2,$$

where the complex constant h has been written $h = h_0 + ih_1$. If we multiply (1.23) by $\frac{1}{2}$ and subtract the result from (1.8) we have

$$(1.25) \qquad A' + \tfrac{1}{2}(A^2+B^2) - \tfrac{1}{2}A[(\alpha'/\alpha) + (\gamma'/\gamma)] = (1 - h_0)\gamma'^2/2\gamma^2.$$

Hence (1.2) will be satisfied only if $h_0 = 1$.

Thus for the case $m \neq 0$ the general solution of the field equations is

$$(1.26) \qquad f + i\beta = 4m^2h/[(e^a\gamma^{\frac{1}{2}h^{\frac{1}{2}}} + e^{-a}\gamma^{-\frac{1}{2}h^{\frac{1}{2}}})^2\gamma(c + i)]$$

$$(1.27) \qquad \alpha = \gamma'^2\,(f^2 + \beta^2)/4m^2\gamma$$

where γ can be chosen to be any arbitrary function of r that we please and f, β are obtained by equating real and imaginary parts of (1.26). It is well to note that m, c are real arbitrary constants, that h has the form $h = 1 + ih_1$, and a is an arbitrary complex constant of integration.

Finally it is of interest to see that Papapetrou's special solutions result from the choice $\gamma = 1 - 2m/r$, $h_1 = 0$ and $e^{2a} = -1$.

We should at this point go on to see how the boundary conditions at infinity allow us to evaluate the arbitrary constants of our solution. However since there is a difficulty in choosing suitable boundary conditions, which we would like to present in some detail, we shall postpone this discussion to a later section.

2. Case $m = 0$. When the constant m is taken to be zero, equation (1.6) becomes $\gamma' = 0$. Thus γ is a constant and can be taken equal to one without loss of generality. Equation (1.12) is still valid and hence for $\gamma' = 0$ becomes

$$(2.1) \qquad q'' + (A - \tfrac{1}{2}\,\alpha'/\alpha)\,q' + 2\alpha(c + i)e^q/(f^2 + \beta^2) = 0.$$

Multiplying by $(f^2 + \beta^2)q'/\alpha$ we can immediately integrate once with respect to r to give

$$(2.2) \qquad q'^2 + 4\alpha(c + i)e^q/(f^2 + \beta^2) = 4h\alpha/(f^2 + \beta^2),$$

where h is an arbitrary complex constant of integration. From the tensorial character of g_{ik} we know that we can make any transformation of the form $r = r(x)$ without destroying the relationship $\gamma = 1$. Thus if we make the transformation

$$(2.3) \qquad x = \int [\alpha/(f^2 + \beta^2)]^{\frac{1}{2}}\,dr$$

then $\alpha/(f^2 + \beta^2) = \left(\frac{dx}{dr}\right)^2$ and (2.2) can be written

$$(2.4) \qquad \left(\frac{dq}{dx}\right)^2 + 4(c + i)e^q = 4h.$$

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The solution of this equation is

(2.5) $$e^q = [h\,\text{sech}^2(h^{\frac{1}{2}}x + a)]/(c + i)$$

if $h \neq 0$, and is

(2.6) $$e^q = (i - c)/[(c^2 + 1)(x + a)^2]$$

if $h = 0$. In either case a is an arbitrary complex constant of integration. At this stage we have not ensured that equation (1.2) is satisfied. By an analysis similar to that used in the previous section we find that this will be so only if the constant h has the form $h = ih_0$. Thus the case $m = 0$ leads to the two possibilities

(2.7) $$\begin{cases} f + i\beta = [h\,\text{sech}^2(h^{\frac{1}{2}}x + a)]/(c + i), \\ \gamma = 1, \\ \alpha = (f^2 + \beta^2)\left(\dfrac{dx}{dr}\right)^2, \end{cases}$$

(2.8) $$\begin{cases} f + i\beta = (i - c)/[(c^2 + 1)(x + a)^2], \\ \gamma = 1, \\ \alpha = (f^2 + \beta^2)\left(\dfrac{dx}{dr}\right)^2, \end{cases}$$

where in each case x can be any arbitrary function of r.

We shall again leave the discussion of the implications of the boundary conditions to a later section.

3. The metric of space-time. In the General Theory of Relativity we assume at the outset a four dimensional Riemannian space which of course implies the existence of a metric tensor which determines the properties of space-time. When the equations of motion of a particle are considered, the derivatives of the metric tensor a_{ik} enter in such a way that the components a_{ik} appear as gravitational potentials. This dual character of the metric tensor arises quite naturally and leads to no ambiguity. In the new theory the point of view has been altered. We assume at the outset certain field quantities g_{ij}, $\Gamma_{jk}{}^i$ and then derive field equations which will determine these field quantities. If we interpret the tensor g_{ij} as a representation of the combined gravitational and electromagnetic fields the question arises as to how the results of the new theory compare with the corresponding results of the General Theory of Relativity. Before this question can be answered we must in some way introduce a metric for space-time so that corresponding results can be compared.

It is natural to assume that at any point in space the symmetric metric tensor a_{ij} will be completely determined by our field quantities. This implies that the components a_{ij} will be certain functions of g_{ij}, $\Gamma_{jk}{}^i$. We denote this functional relationship by

(3.1) $$a_{ij} = f_{ij}(g_{rs}, \Gamma_{rs}{}^p).$$

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The field equations determine the quantities $\Gamma_{rs}{}^{p}$ in terms of g_{rs} and their first derivatives. Thus the above assumption is equivalent to saying that the metric tensor becomes completely determined at any point in space by a knowledge of the g_{rs} and their first derivatives.

The functional relationship of (3.1) is not quite arbitrary in that the components a_{ik} must be the components of a tensor. It is not too difficult to show that the allowable functions f_{ij} must satisfy certain partial differential equations in order for this to be true. Since however the field quantities g_{rs}, $\Gamma_{rs}{}^{p}$ determine the tensors $g_{\underline{rs}}$, $g_{\underset{\vee}{rs}}$, $g^{\underline{rs}}$, $g^{\underset{\vee}{rs}}$, $\Gamma_{\underset{\vee}{rs}}{}^{p}$ we can construct an infinity of tensors of the form (3.1).

The field equations of the Unified Theory reduce to those of General Relativity if $g_{\underset{\vee}{ij}} = 0$. Hence we shall make the requirement that

$$f_{ij}(g_{\underline{rs}}, \Gamma_{\underline{rs}}) = g_{\underline{rs}}, \quad \text{if } g_{\underset{\vee}{ij}} = 0.$$

At this stage of the theory there seems to be little to guide us in a suitable choice of metric tensor. However when one considers the equations of motion a strong argument can be advanced for a particular choice for the metric tensor.

Since the linear connection $\Gamma_{jk}{}^{i}$ has been assumed to be the linear connection by means of which we define the parallel displacement of a vector it seems natural to require that the equations of motion of a free particle can be put into the form

$$\frac{d^2x^i}{ds^2} + \Gamma_{jk}{}^{i}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{3.2}$$

where s is a suitable parameter along the trajectory of motion. Because of the symmetry of $\dfrac{dx^j}{ds}\dfrac{dx^k}{ds}$ in the indices j, k the skew-symmetric part of the second term will cancel out and the equations of motion have the form

$$\frac{d^2x^i}{ds^2} + \Gamma_{\underline{jk}}{}^{i}\frac{dx^j}{ds}\frac{dx}{ds} = 0. \tag{3.3}$$

Multiplying (3.3) by $g_{\underline{im}}\dfrac{dx^m}{ds}$ and summing with respect to i we obtain

$$g_{\underline{im}}\frac{dx^m}{ds}\frac{d^2x^i}{ds^2} + g_{\underline{im}}\Gamma_{\underline{jk}}{}^{i}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{dx^m}{ds} = 0. \tag{3.4}$$

This can be put into the form

$$\frac{d}{ds}\left(g_{\underline{im}}\frac{dx^m}{ds}\frac{dx^i}{ds}\right) - g_{\underline{jk}/m}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{dx^m}{ds} = 0, \tag{3.5}$$

where $g_{jk/m}$ means the covariant derivative of $g_{\underline{jk}}$ with respect to the symmetric linear connection $\Gamma_{\underline{jk}}{}^{i}$. Thus

$$g_{\underline{im}}\frac{dx^i}{ds}\frac{dx^m}{ds} = \text{constant} \tag{3.6}$$

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will be an integral of (3.5) providing we can show that

$$g_{\underline{jk}/m} + g_{\underline{km}/j} + g_{\underline{mj}/k} = 0. \tag{3.7}$$

These relations, we shall show, are an immediate consequence of equations (1) given in the introduction of our paper. From equations (1) we have

$$\begin{aligned} g_{\underline{ik},a} &= \tfrac{1}{2}(g_{sk}\Gamma_{ia}{}^{s} + g_{is}\Gamma_{ak}{}^{s} + g_{si}\Gamma_{ka}{}^{s} + g_{ks}\Gamma_{ai}{}^{s}) \\ &= g_{\underline{sk}}\Gamma_{\underline{ia}}{}^{s} + g_{\underset{\vee}{sk}}\Gamma_{\underset{\vee}{ia}}{}^{s} + g_{\underline{is}}\Gamma_{\underline{ak}}{}^{s} + g_{\underset{\vee}{is}}\Gamma_{\underset{\vee}{ak}}{}^{s}, \end{aligned}$$

and hence

$$g_{\underline{ik}/a} = g_{\underset{\vee}{sk}}\Gamma_{\underset{\vee}{ia}}{}^{s} + g_{\underset{\vee}{is}}\Gamma_{\underset{\vee}{ak}}{}^{s}. \tag{3.8}$$

Applying two cyclic permutations to the indices i, k, a, and adding the results to (3.8) we immediately find

$$g_{\underline{ik}/a} + g_{\underline{ka}/i} + g_{\underline{ai}/k} = 0. \tag{3.9}$$

Equations (3.9) are of course equivalent to (3.7).

Since the equations of motion (3.2) always have the quadratic expression (3.6) as an integral it seems natural to assume that the metric of space time is given by $g_{\underline{ij}}\, dx^i\, dx^j$ and hence our choice of metric tensor would have to be $a_{ij} = g_{\underline{ij}}$ even though $g_{\underset{\vee}{ij}} \neq 0$. Papapetrou has used the requirement $a_{ij} = g_{\underline{ij}}$ in connection with his solution of the field equations and has found the results of the new theory do not agree with those given by the General Theory of Relativity. He attributes this difficulty to the uncertainty in the physical identification of the tensor $g_{\underset{\vee}{ij}}$. While this may be true we feel that other possibilities exist. For example it might be that equations (3.2) are not the true equations of motion and that other equations will replace them. In this case it might be that the requirement $a_{ij} = g_{\underline{ij}}$ is an approximation and that the true metric will involve all our fundamental field quantities. In order to show the possibilities that exist we will examine the physical consequences when a different choice of metric is made.

Let us define two covariant vectors h_i, u_i by means of

$$h_i = g_{\underline{ab}}\, g^{\underset{\vee}{sa}}\Gamma_{\underset{\vee}{si}}{}^{b}, \tag{3.10}$$

$$u_i = h_i/(g^{rs}h_r h_s)^{\frac{1}{2}} = h_i/(g^{\underline{rs}}h_r h_s)^{\frac{1}{2}}. \tag{3.11}$$

In (3.3) if h_i turns out to be a zero vector (i.e. $h_i = 0$) we simply take $u_i = 0$. There is of course a possibility that $g^{rs}h_r h_s \equiv 0$ with $h_i \neq 0$. We shall discuss this possibility at the end of this section. Finally we note that $g^{rs}h_r h_s$ could be negative and hence u_i would become an imaginary tensor. However (3.3) is only an intermediate step in our calculations and we shall see that this difficulty is removed with our final choice of metric.

We define a third covariant vector q_i by means of

$$q_i = (g_{im}\, g^{\underline{mn}}\, u_n)/[1 + \tfrac{1}{2}g_{\underset{\vee}{rs}}\, g^{\underset{\vee}{sr}}]^{\frac{1}{2}},$$

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and then choose the metric

$$a_{ij} = g_{\underline{ij}} + q_i q_j. \tag{3.12}$$

Referring back to Papapetrou's exact solution as given in §**1** we have that the non-zero components of g_{ij} are

$$g_{11} = -[1 - (2m/r)]^{-1},\ g_{22} = -r^2,\ g_{33} = -r^2 \sin^2\theta,$$

$$g_{44} = [1 + (l^4/r^4)][1 - (2m/r)],\ g_{14} = -g_{41} = \pm l^2/r^2.$$

From these we can calculate the non-zero components of g^{ij} to be $g^{11} = -g_{44}$, $g^{22} = -1/r^2$, $g^{33} = -1/r^2\sin^2\theta$, $g^{44} = -g_{11}$, $g^{14} = -g^{41} = -g_{14}$. The non-vanishing components of $\Gamma_{\check{jk}}{}^i$ are

$$\Gamma_{\check{42}}{}^2 = -\Gamma_{\check{24}}{}^2 = -w/rg_{11} = \Gamma_{\check{43}}{}^3 = -\Gamma_{\check{34}}{}^3,\ \Gamma_{\check{14}}{}^1 = -\Gamma_{\check{41}}{}^1 = -2w/rg_{11}.$$

From these we can compute the components of $h_\imath$ to be $[-l^4/r^5, 0, 0, 0]$. Hence the components of u_i are $[(g^{11})^{-\frac{1}{2}}, 0, 0, 0]$ and of q_i are $[0, 0, 0, g_{41}(g^{11})^{\frac{1}{2}}/(1 + g_{14}{}^2)^{\frac{1}{2}}]$. This finally gives the metric

$$a_{ij} = g_{\underline{ij}},\quad \text{if } i, j \text{ are not both equal to 4,}$$

$$a_{44} = g_{44} + g_{41}{}^2 g^{11}/(1 + g_{14}{}^2).$$

Substituting the values of the g's we find that the non-vanishing components of the metric tensor a_{ij} are given by

$$a_{11} = -(1 - 2m/r)^{-1},\quad a_{22} = -r^2,\quad a_{33} = -r^2\sin^2\theta,\quad a_{44} = 1 - 2m/r.$$

This is of course the Schwarzschild solution of General Relativity.

We are not advocating the choice of metric (3.4) because it has been constructed in a very artificial manner. We use it to illustrate the importance of the choice of metric and to discuss several important points. If we assume that the metric of (3.4) is the true metric then we have seen the line element corresponding to Papapetrou's solution of the field equations is the Schwarzschild line element for a spherical mass with zero charge. Thus under this particular choice of metric we would have to say that Papapetrou's solution of the field equations is still a solution which corresponds to a pure gravitational field even though a second constant of integration l appears in the solution. This constant completely disappears when the components of the metric tensor are evaluated. Since $g_{\check{ij}} \neq 0$ in Papapetrou's solution our choice of metric also implies that $g_{\check{ij}}$ cannot be interpreted in terms of the electromagnetic field alone or else there exist electromagnetic fields which do not influence our measurements of space-time. This latter conclusion seems hardly likely and hence our example would seem to strengthen Papapetrou's conclusion that the physical interpretation of $g_{\check{ij}}$ is an open question. Finally we might point out that the disappearance of a constant of integration by

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choice of metric may be connected with the fact that the linearized equations of the Unified Theory are weaker than Maxwell's equations. It might be possible that a choice of metric exists which make these weaker equations equivalent to Maxwell's equations.

If we agree that Papapetrou's choice of metric $a_{ij} = g_{\underline{ij}}$ is at best an approximation to the true metric then of course the accuracy of this approximation must be discussed. It is not difficult to construct metrics in which this approximation is valid only up to and including terms of the order $1/r$. Since it is the terms of order $1/r^2$ which measure the electromagnetic effect on space-time we see, for such metrics, that Papapetrou's approximation is equivalent to assuming a zero electromagnetic field. This then would be the reason that Papapetrou's solution does not behave asymptotically in the same manner as the solution in General Relativity corresponding to a point charge in which the terms of order $1/r^2$ are retained. It is very easy to construct metrics which show the same asymptotic behaviour as the General Relativity solutions up to and including terms of order $1/r^2$. However our construction is still very artificial and we shall not include this work in this paper.

In our derivation of the metric (3.4) we left in abeyance the possibility that $g^{rs}h_r h_s = 0$ with $h_r \neq 0$. For the static case, in which there exists a coordinate system in which the g_{rs} are all independent of the time-like coordinate x^4, it is possible to show, under suitable restrictions, that $g^{rs}h_r h_s = 0$ implies $h_r = 0$. We have not studied the non-static case in detail because we doubt very much that (3.4) will provide a suitable choice of metric. This section has been used only to show that a problem exists in the choice of a metric and that some logical physical reason should be advanced for the choice of metric for our new theory.

To conclude this section of the paper we would like to anticipate one criticism that might be made. It might be argued that the analogy from General Relativity would allow us to assume the dual nature of the tensor g_{ij}. By this I mean that the metric in space should be determined as a function of this tensor alone and would be independent of $\Gamma_{jk}{}^{i}$. Although this may be true it still does not destroy the point that we have been trying to make in this section. Out of such a tensor an unlimited number of metric tensors can be constructed and we must still advance some reason for a particular choice. As an example we might choose $a_{ij} = g^{\underline{mn}}\, g_{im}\, g_{jn}$. This particular metric turns out to be completely equivalent to Papapetrou's metric $a_{ij} = g_{\underline{ij}}$ for Papapetrou's particular spherically symmetric solution.

4. The boundary conditions. Since our field equations reduce to those of General Relativity if $\check{g}_{ij} = 0$ it is natural to assume that when $\check{g}_{ij} = 0$ our field is purely gravitational. Thus as boundary conditions it is natural to assume, in the general case, that at large distances from matter or charge there will exist a coordinate system in which the components of the metric tensor approach the scheme given by

$$g_{ij} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.1}$$

We shall denote the scheme of (4.1) by γ_{ij} and as is usual we shall call this the Galilean tensor. In any other coordinate system the components of γ_{ij} are of course obtained by the tensor law of transformation. We notice of course that γ_{ij} provides an exact solution of our field equations in which $\Gamma_{jk}{}^{i} = 0$ in the coordinate system used for (4.1). This tensor is taken as the mathematical representation of the absence of both gravitational and electromagnetic fields. If we use the transformation

$$x^1 = r \sin \theta \cos \phi, \; x^2 = r \sin \theta \sin \phi, \; x^3 = r \cos \theta, \; x^4 = x^4,$$

the components $\bar{\gamma}_{ij}$ of the Galilean tensor are given by

$$\bar{\gamma}_{ij} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.2}$$

It would be in keeping with the principle of relativity if the condition that $g_{ij} \to \gamma_{ij}$ in one coordinate system implied that this was true in every coordinate system. Unfortunately this is not true and in fact it was used as a criticism of General Relativity when that theory was first proposed. For the General Theory of Relativity this difficulty was, in a sense, resolved for spherically symmetric solutions of the field equations, by means of Birkhoff's Theorem. Since the approach of a tensor to its Galilean values is not an invariant condition we must then single out a particular coordinate system if this condition is to be used as a boundary condition. We shall show by using our second solution of the field equations that this singling out of a special coordinate system presents a real difficulty in our new theory.

Papapetrou [2, p. 70] has shown that the general spherically symmetric form of $\underset{\vee}{g}_{ij}$ in Cartesian coordinates is

$$\underset{\vee}{g}_{ij} = \begin{bmatrix} 0 & \frac{z}{r}v & -\frac{y}{r}v & \frac{x}{r}w \\ -\frac{z}{r}v & 0 & \frac{x}{r}v & \frac{y}{r}w \\ \frac{y}{r}v & -\frac{x}{r}v & 0 & \frac{z}{r}w \\ -\frac{x}{r}w & -\frac{y}{r}w & -\frac{z}{r}w & 0 \end{bmatrix}, \tag{4.3}$$

where v, w are functions of r alone. Hence $\underset{\vee}{g}_{ij} \to 0$ as $r \to \infty$ implies that $v \to 0$ and $w \to 0$ as $r \to \infty$. In spherical polar coordinates the components $\underset{\vee}{g}_{ij}$ of this tensor are given by

$$(4.4) \qquad \underset{\vee}{\bar{g}_{ij}} = \begin{bmatrix} 0 & 0 & 0 & w \\ 0 & 0 & r^2 v \sin\theta & 0 \\ 0 & -r^2 v \sin\theta & 0 & 0 \\ -w & 0 & 0 & 0 \end{bmatrix}.$$

Hence the same conditions in this coordinate system imply $r^2v \to 0$, $w \to 0$ as $r \to \infty$. This of course is a much stronger condition than the corresponding condition in Cartesian coordinates. We shall now use our second solution to show that these conditions imply different solutions of the field equation.

Returning to the solution given by (1.26), (1.27) our complete boundary conditions are

$$\alpha \to 1,\ \beta \to r^2,\ \gamma \to 1,\ f = vr^2 \to 0 \text{ as } r \to \infty$$

or

$$\alpha \to 1,\ \beta \to r^2,\ \gamma \to 1,\ v \to 0 \qquad \text{as } r \to \infty,$$

depending on the coordinate system used. Since $\beta \to \infty$ as $\gamma \to 1$ we must have from (1.26) that $e^a + e^{-a} = 0$. Thus $e^{2a} = -1$ and (1.26) can be written

$$(4.5) \qquad f + i\beta = -4m^2 h\, \gamma^{h^{\frac{1}{2}}-1}/(\gamma^{h^{\frac{1}{2}}} - 1)^2\,(c + i).$$

Moreover if we let $\gamma = 1 - x$ and expand (4.5) in terms of x we find

$$f + i\beta = \frac{4m^2(i - c)}{(c^2 + 1)x^2}\left[1 - (h - 1)\frac{x^2}{12} + \ldots\right].$$

Remembering that $h = 1 + ih_1$, we can equate real and imaginary parts to find

$$\beta = \frac{4m^2}{(c^2 + 1)x^2} + \frac{m^2}{3(c^2 + 1)}\, ch_1 + O(x),$$

$$f = -\frac{4m^2 c}{(c^2 + 1)x^2} + \frac{h_1 m^2}{3(c^2 + 1)} + O(x),$$

where we mean by $O(x)$ terms of the order x. As $\gamma \to 1$, $x \to 0$. Hence $f \to 0$ as $x \to 0$ only if $c = 0$ and $h_1 = 0$. In this solution $m = 0$ is not possible. Since $h_1 = 0$ implies $h = 1$ we find (4.5) becomes

$$(4.6) \qquad f + i\beta = +4m^2 i/(\gamma - 1)^2.$$

From the fact that the right side is a pure imaginary we can conclude that the strong boundary conditions result in $f \equiv 0$ and hence $\underset{\vee}{g_{ij}} \equiv 0$ and our resulting solution degenerates into the Schwarzschild solution.

If we use the weaker boundary condition that $v = f/r^2 \to 0$ as $r \to \infty$, we still find that $c = 0$ but we no longer have the condition that $h_1 = 0$. Thus in our final solution two arbitrary constants m, h_1 remain which can be interpreted as being determined by the mass and charge of the particle. Thus we see that the requirement that the tensor g_{ij} approach its Galilean values as

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$r \to \infty$ implies different solutions in the two coordinate systems. As a matter of preference I feel the stronger boundary conditions will prove correct and that the solution we have obtained degenerates into the Schwarzschild solution for a pure gravitational field. I feel that the physical problem of a charged particle will only be solved when the general field equations are solved under the more general conditions $vw \neq 0$. The main reason for this belief is that we have shown that the solution resulting from the assumption $v = 0$, $w \neq 0$ can be interpreted under proper choice of metric, as being equivalent to the assumption $v = w = 0$. Similarly, under the strong boundary conditions, we have shown that the solution resulting from the assumption $w = 0$, $v \neq 0$ also degenerates to the case $v = w = 0$. For this reason it is possible that either of the restrictions imposed by Papapetrou, namely $v = 0$, $w \neq 0$, or $v \neq 0$, $w = 0$, may be equivalent to destroying the electromagnetic field.

For our solution of the field equations corresponding to the case $m = 0$ we can by similar analysis to that used in the present section show that the strong boundary conditions reduce this solution to that for zero mass and zero charge.

5. Conclusion. At the present stage our theory is still far from complete. A proper choice of metric has not been made nor have the equations of motion of a particle been defined. It seems necessary, therefore, to study the physical significance of our field quantities so that the present theory can be completed in a logical manner. When this is done it seems likely that the difficulties raised in the present paper will be removed.

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