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GRAPH PRODUCTS AND THE ABSENCE OF PROPERTY (AR)

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Abstract

We discuss the internal structure of graph products of right LCM semigroups and prove that there is an abundance of examples without property (AR). Thereby we provide the first examples of right LCM semigroups lacking this seemingly common feature. The results are particularly sharp for right-angled Artin monoids.

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1. Introduction

The starting point of a number of recent breakthroughs in the theory of semigroup C^*-algebras is the seminal work [20, 21], in which a universal C^*-algebra $C^*(S)$ is associated to every left cancellative monoid S. In the last years, a particular line of research focused on left cancellative monoids for which the intersection of two principal right ideals is either empty, or another principal right ideal again. Such monoids are called *right LCM semigroups* (right Least Common Multiple), and they form an intriguing and tractable class of examples in between positive cones in quasi-lattice ordered groups and general left cancellative monoids, see [3, 4, 6] for details.

Inspired by the treatment of the quasi-lattice ordered case in [10], a *boundary quotient* $\mathcal{Q}(S)$ of $C^*(S)$ was introduced for right LCM semigroups S in [5]. Soon thereafter, Starling provided an in-depth analysis of $\mathcal{Q}(S)$ in [24], relying on major advances in the understanding of the connections between inverse semigroups, groupoids and C^*-algebras stemming from [13, 14, 25]. In [6], it was shown that the boundary quotient has a more accessible presentation if the right LCM semigroup has the so-called *accurate refinement property*, henceforth abbreviated to *property (AR)*.

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This property is an analogue of 0-dimensionality for topological spaces in the context of semigroups, and is enjoyed by various examples, see [6, Section 2 and Corollary 3.11].

The presence of property (AR) was found to be useful in the construction of a *boundary quotient diagram* for right LCM semigroups in the spirit of [2], see [23]. This diagram sets the grounds for a unifying approach for the study of equilibrium states (KMS-states) on C^*-algebras in [1], where remarkable results were obtained for C^*-algebras of right LCM semigroups satisfying an admissibility condition which implies property (AR), see Section 2.1. Working with abstract right LCM semigroups as opposed to explicit classes of examples allowed for a unification of the inspiring case studies [2, 8, 17–19], and also for coverage of a substantial number of new examples, most notably, algebraic dynamical systems. Moreover, the techniques in [1, 6, 23] raise several questions on the structure of right LCM semigroups, perhaps most notably:

(a) Are there right LCM semigroups without property (AR)?
(b) Which right LCM semigroups are admissible?

The aim of the present work is to investigate to what extent graph products of right LCM semigroups as considered in [15, 26] provide answers to these two questions. In addition, we also address structural aspects related to the distinguished subsemigroups S^*, S_c and S_{ci}. We apply our results to the classical case of right-angled Artin monoids A_Γ^+ given by a undirected graph Γ, since many graph related phenomena can already be witnessed here. Indeed, the explicit presentation of the boundary quotient in [10, Corollary 8.5] involving only the vertex sets of the finite coconnected components of the graph Γ may be regarded as an indication for a particularly accessible structure of foundation sets. Another motivation comes from the elegant solution to the isomorphism problem for $C^*(A_\Gamma^+)$, see [12].

Since property (AR) is known for various kinds of right LCM semigroups, we were surprised to find that a right-angled Artin monoid A_Γ^+ has property (AR) if and only if all of its finitely generated direct summands are free, see Corollary 4.6. In terms of the Γ, this means that all finite coconnected components Γ_i do not contain any edges. The result follows from more general graph product considerations in Corollary 4.5 that rely on Theorem 4.3, where we show that graph products over infinite coconnected graphs have no foundation set other than the obvious ones containing an invertible element, while the analogous statement holds in the finite case for accurate foundation sets.

The characterisation of property (AR) for right-angled Artin monoids A_Γ^+ in Corollary 4.6 allows us to determine when A_Γ^+ is admissible in the sense of [1]. It turns out that admissibility and the existence of a generalised scale coincide for right-angled Artin monoids, see Corollaries 4.9 and 4.10. If existent, the generalised scale on A_Γ^+ is unique and arises as the product of the unique generalised scales on its nonabelian direct summands $A_{\Gamma_i}^+$, see Proposition 4.8 and Corollary 4.9.

Thus, we are led to the conclusion that graph products of right LCM semigroups mostly lack property (AR), and are therefore not admissible in the sense of [1]. While this rules out the possibility of applying [1] to graph products of right LCM semigroups in great generality, we obtain a fairly detailed description of the behaviour of graph products with respect to the subsemigroups S_c and S_{ci}, see Theorem 3.4. These results show that the graph product represents a useful tool to construct new, and potentially very interesting examples of right LCM semigroups that are well-behaved to some degree, but demand more sophisticated techniques then those applicable to right LCM semigroups that have property (AR) or even a generalised scale. That is why we feel that this work might stimulate further research in the direction of inverse semigroups and groupoids related to (right LCM) semigroups and their C^*-algebras.

2. Preliminaries

Here, we provide the prerequisites we shall need concerning right LCM semigroups and graph products.

2.1. Right LCM semigroups. A left cancellative semigroup S is called *right LCM* if the intersection of two principal right ideals in S is either empty or a principal right ideal. We will appeal to Baumslag–Solitar monoids as our demonstrational class of examples, see Example 2.1, while we refer to [1, Section 5] for a more detailed and extensive discussion. For $s, t \in S$, we say that s and t are *orthogonal* and write $s \perp t$ if $sS \cap tS = \emptyset$. Unless specified otherwise, we will always assume that a right LCM semigroup S has an identity, that is, S is a monoid.

Let us first discuss property (AR). A finite subset $F \subset S$ is called a *foundation set* for S if for every $s \in S$ there is $f \in F$ such that $f \not\perp s$, see [5, Section 5]. A subset $F \subset S$ is *accurate* if $f \perp f'$ for all $f, f' \in F, f \neq f'$, see [6, Definition 2.1]. If F, F' are foundation sets such that $F' \subset FS$, then F' is called a *refinement* of F. We then say that S has the property (AR) if every foundation set for S has an accurate refinement, see [6, Definition 2.3].

We will now discuss core structures of a right LCM semigroup S that became relevant in the course of [1, 23]. The subgroup of invertible elements of S shall be denoted by S^*. It lies inside the *core subsemigroup*

$$S_c := \{a \in S \mid a \not\perp s \text{ for all } s \in S\},$$

which was first considered for right LCM semigroups in [24], but stems from [10, Definition 5.4]. We remark that S_c is again a right LCM semigroup. It induces the *core equivalence relation* $\sim$ on $S \times S$, where s is equivalent to t if there are $a, b \in S_c$ satisfying $sa = tb$. In contrast to S_c, we also consider the subsemigroup S_{ci} of *core irreducible* elements, that is, the collection of all elements $s \in S \setminus S_c$ for which every factorisation $s = ta$ with $t \in S, a \in S_c$ satisfies $a \in S^*$. While S_{ci} does not have an identity by construction, its unitisation $S_{ci}^1 := S_{ci} \cup \{1\}$ and $S'_{ci} := S_{ci} \cup S^*$ do.

A right LCM semigroup S is called *core factorable* if $S = S_{ci}^1 S_c$. We say that $S_{ci} \subset S$ is $\cap$-*closed* if $sS \cap tS = rS$ implies $r \in S_{ci}$ whenever $s, t \in S_{ci}$. To provide some indication why this property is of interest, let us mention that $S_{ci} \subset S$ is $\cap$-closed if and only if S'_{ci} is right LCM and its inclusion into S is a homomorphism of right LCM semigroups, that is, it preserves intersections of principal right ideals, see [1, Proposition 3.3]. Finally, a nontrivial homomorphism $N\colon S \to \mathbb{N}^\times$ is called a *generalised scale* if $|N^{-1}(n)/_\sim| = n$ and every minimal complete set of representatives for $N^{-1}(n)/_\sim$ forms an accurate foundation set for S for all $n \in N(S)$. Every generalised scale N satisfies $\ker N = S_c$ by [1, Proposition 3.6(i)] and the existence of a generalised scale entails vital information on the structure of S. For instance, it implies that the right LCM semigroup has property (AR), see [1, Proposition 3.6(v)].

Finally, we recall from [1, Definition 3.1] that a right LCM semigroup S is called *admissible*, if it is core factorable, $S_{ci} \subset S$ is $\cap$-closed, and S admits a generalised scale N such that $N(S) \subset \mathbb{N}^\times$ is freely generated by its irreducible elements.

Example 2.1. The Baumslag–Solitar monoid

$$BS(c,d)^+ := \langle a, b \mid ab^c = b^d a\rangle$$

for positive integers c, d is a right LCM semigroup, see [22, Theorem 2.11]. We remark that S^* is trivial, and that every element $s \in BS(c,d)^+$ admits a unique normal form $s = b^{j_0}ab^{j_1}ab^{j_2}\cdots ab^{j_n}$, where $0 \le j_k \le d-1$ for $0 \le k < n$ while j_n can be an arbitrary nonnegative integer. The family $(b^j a)_{0\le j\le d-1}$ generate a free monoid $\mathbb{F}_d^+$ of rank d, so that two elements $b^{i_0}ab^{i_1}\cdots ab^{i_m}$ and $b^{j_0}ab^{j_1}\cdots ab^{j_n}$ are not orthogonal precisely if $b^{i_0}ab^{i_1}\cdots ab^{i_{m-1}}a$ is an extension of $b^{j_0}ab^{j_1}\cdots ab^{j_{n-1}}a$ or the other way round. In the case where $d = 1$, this is always possible, so that $S = S_c$ and thus S is left reversible. More concretely, $S = \mathbb{N}_c \ltimes \mathbb{N}$ and this semidirect product is rather well-understood. Let us therefore assume that $d > 1$. Then we arrive at $S_c = \langle b\rangle \cong \mathbb{N}$ and

$$S_{ci}^1 = \{s = b^{j_0}ab^{j_1}\cdots ab^{j_n} \mid 0 \le j_k \le d-1 \text{ for } 0 \le k < n,\ j_n = 0\} \cong \mathbb{F}_d^+,$$

that is, an element is core irreducible if and only if it is equal to its *stem* in the sense of [8]. Therefore, the properties of the normal form show:

(a) For $s, t \in S$, $s \sim t$ holds if and only if $sb^i = tb^j$ for some $i, j \in \mathbb{N}$.

(b) The right LCM semigroup S is core factorable and $S_{ci} \subset S$ is $\cap$-closed.

Finally, we remark that the homomorphism $N\colon S \to \mathbb{N}^\times$ defined by $a \mapsto d$ and $b \mapsto 1$ constitutes a generalised scale for which S is admissible. This map is a version of the *height map* from [8, Section 2.2].

2.2. Graph products. Within this work, a *graph* will mean a countable, undirected graph $\Gamma = (V, E)$ without loops or multiple edges. The concept of a *graph product of groups* emerged in [16] as a generalisation of *graph groups* and has been transferred to the setting of monoids in [26]: for a graph $\Gamma = (V, E)$ and a family of monoids $(S_v)_{v\in V}$, the *graph product* is the monoid S_Γ obtained as the quotient of the direct sum $\bigoplus_{v\in V} S_v$ by the congruence generated by the relation (st, ts) if $s \in S_v, t \in S_w$ with $(v, w) \in E$, see [26, Section 2] and [15, Section 1]. Given a graph Γ, its *right-angled Artin monoid* A_Γ^+ is the graph product with $S_v = \mathbb{N}$ for all $v \in V$. These monoids have also been studied under the names of graph monoids, free partially commutative monoids and trace monoids, see for instance [11]. If one switches the vertex monoids from the

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natural numbers to the integers, the resulting graph product is the *right-angled Artin group* A_Γ associated to Γ, see [7] for more.

It was shown in [9] that the graph product is well-behaved with respect to quasi-lattice orders. Invoking a characterisation of the right LCM property via the inverse hull semigroup, Fountain and Kambites showed that this can be generalised to right LCM semigroups, see [15, Theorem 2.6], where we note that we can move back and forth between right cancellative, left LCM semigroups (used in [15]) and left cancellative, right LCM semigroups by passing to the opposite semigroup.

According to [15, Theorem 1.1], which is an adaptation of the corresponding result in [16], every element s in a graph product S_Γ is represented by an essentially unique *reduced expression* $s_{v(1)}s_{v(2)}\cdots s_{v(n)}$, that is, $s_{v(k)} \in S_{v(k)}$, $v(k) \neq v(k+1)$, and whenever there are $1 \leq k < m \leq n$ such that $v(k) = v(m)$, then there exists $k < \ell < m$ such that $(v(k), v(\ell)) \notin E$. The analogous result had been proven in the quasi-lattice ordered case before, see [9, Theorem 2]. The reduced expression is unique in the sense that any two reduced expressions for the same element are *shuffle equivalent*, that is, we can move from one to the other by a finite number of switches of neighbouring factors whose vertices are adjacent in Γ. Thus, there exists a subadditive function $\ell\colon S_\Gamma \to \mathbb{N}$ that assigns the length of any reduced expression to the element in question.

A graph Γ is said to be coconnected if it does not admit a partition $V = V_1 \sqcup V_2$ with $V_i \neq \emptyset$ and $V_1 \times V_2 \subset E$. Equivalently, Γ is coconnected if the opposite graph $\Gamma^{\text{opp}} := (V, V \times V \setminus (E \cup \{(v, v) \mid v \in V\}))$ is connected. The decomposition of Γ into its *coconnected* components is the initial step in the analysis of S_Γ, see for instance [12], where the synonym *co-irreducible* is used. Every graph Γ has a unique decomposition into coconnected components, which we denote by $(\Gamma_i)_{i\in I}$ with $\Gamma_i = (V_i, E_i)$. The original graph can be recovered from $(\Gamma_i)_{i\in I}$ as $V = \bigsqcup_{i\in I} V_i$ and

$$E = \{(v, w) \in V \times V \mid (v, w) \in E_i \text{ or } w \notin V_i \ni v \text{ for some } i \in I\}.$$

It follows from this observation that S_Γ coincides with $\bigoplus_{i\in I} S_{\Gamma_i}$.

A vertex $v \in V$ is called *isolated* if v does not emit any edge and *universal* if v is connected to every other vertex in Γ. We note that the only coconnected graph with a universal vertex v is $V = \{v\}$ and that any graph containing an isolated edge is necessarily coconnected. For convenience, we let V_u denote the set of universal vertices and $I_2 := \{i \in I \mid |V_i| \geq 2\}$.

We will make use of the following notion of a blocking path, that is actually a path in the opposite graph.

Definition 2.2. Let $\Gamma = (V, E)$ be a graph and $C \subset V$. A *blocking path* for C is a finite sequence of vertices $w(1), \ldots, w(n) \in V$ such that:

(a) $w(1) \notin C$, $(w(k), w(k+1)) \notin E$ for all $1 \leq k \leq n-1$; and
(b) for every $u \in C$ there exists $1 \leq k \leq n$ such that $(w(k), u) \notin E$.

It turns out that blocking paths are almost always available whenever the graph is coconnected and we will frequently make use of this elementary observation in the course of this work.

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Lemma 2.3. *If Γ is a coconnected graph with at least two vertices, then every finite proper subset C of V admits a blocking path ending in any prescribed vertex.*

Proof. Let $C = \{v(1), \ldots, v(m)\} \subset V$ be finite and proper, that is, $V \setminus C \neq \emptyset$. If $(v, u) \in E$ for all $v \in C, u \in V \setminus C$, then we would get a contradiction to Γ being coconnected. Thus, there exists $w(1) \in V \setminus C$ such that $(v(k), w(1)) \notin E$ for some $1 \leq k \leq m$. Without loss of generality, we can assume $k = 1$. Since Γ is coconnected, we can choose $w'(k) \in V$ for $2 \leq k \leq m$ such that $(v(k), w'(k)) \notin E$. Again by coconnectedness, there exists a finite path $w(1), \ldots, w(n)$ in Γ^{opp} that visits every $w'(k), 2 \leq k \leq n$. This is a blocking path for C, and since Γ^{opp} is connected, we can attach to this blocking path a path leading to any prescribed vertex without losing the blocking property for C. □

Remark 2.4. Let $\Gamma = (V, E)$ be a graph and $(S_v)_{v \in V}$ a family of right LCM semigroups. Suppose $w(1), \ldots, w(n)$ is a blocking path for some nonempty C and we can choose $s_n, t_n \in S_{w(n)} \setminus S^*_{w(n)}$. Then for all s_0, t_0 whose reduced expressions only contain parts from vertex semigroups of vertices in C and all $s_k, t_k \in S_{w(k)}, 1 \leq k < n$, $\ell(s_0 s_1 \cdots s_n) = \ell(s_0) + n$ and $s_0 s_1 \cdots s_n \perp t_0 t_1 \cdots t_n$, unless $s_k = t_k$ for $0 \leq k < n$ and $s_n \not\perp t_n$. Thus, blocking paths allow for the construction of shuffle inert elements in graph products, which turns out to be quite useful.

3. The internal structure of graph products

In this section, we show that many of the properties of S_Γ that are of interest to us, for example in connection with [1], can be understood from a study of the corresponding graph products for the coconnected components $(\Gamma_i)_{i \in I}$ of Γ. The reason is $S_\Gamma = \bigoplus_{i \in I} S_{\Gamma_i}$ and the following list of straightforward observations, where we write $s = \bigoplus_{i \in I} s_i$ for $s \in \bigoplus_{i \in I} S_i$:

Proposition 3.1. *Let $(S_i)_{i \in I}$ be a family of right LCM semigroups. Then $S := \bigoplus_{i \in I} S_i$ has the following features:*

(i) *The subgroup of invertible elements is $S^* = \bigoplus_{i \in I} S_i^*$, the core is given by $S_c = \bigoplus_{i \in I} (S_i)_c$ and $S'_{ci} = \bigoplus_{i \in I} (S_i)'_{ci}$.*
(ii) *Two elements $s, t \in S$ are core related if and only if s_i and t_i are core related in S_i for all $i \in I$.*
(iii) *The following statements hold for S if and only if their analogues hold for all S_i:*
   (a) *S is core factorable;*
   (b) *$S_{ci} \subset S$ is $\cap$-closed;*
   (c) *$\alpha\colon S_c \curvearrowright S/_\sim$ given by $\alpha_a([s]) := [as]$ is faithful; and*
   (d) *S has finite propagation.*
(iv) *The action $\alpha\colon S_c \curvearrowright S/_\sim, a.[s] := [as]$ is almost free if and only if one of the following conditions holds:*

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(a) *Every monoid S_i is left reversible, that is, $S = S_c$ so that $S/\sim$ is a singleton.*
(b) *There exists a unique $i \in I$ such that S_i is not left reversible, $\alpha_i\colon (S_i)_c \curvearrowright S_i/_\sim$ is almost free, and the monoid S_j is left reversible for all $j \in I\backslash\{i\}$.*

In view of the direct sum decomposition for graph products over the coconnected components, we need to understand the behaviour of the graph product in the case of a coconnected graph with at least two vertices. To do this, we will need to consider a variant of the action α for S^*, that is, $\alpha^*\colon S^* \curvearrowright S/S^*, x.[s] := [xs]$. Also, we will assume that all vertex semigroups $S_v, v \in V$ are nontrivial in order to avoid pathological cases. For instance, if Γ is the union of a complete graph and an isolated vertex v, and S_v is trivial, then the graph product will be the direct sum of the right LCM semigroups attached to the vertices of the complete graph, even though the original graph was larger and coconnected.

Theorem 3.2. *If $\Gamma = (V, E)$ is coconnected, $|V| \geq 2$ and $(S_v)_{v\in V}$ is a family of nontrivial right LCM semigroups, then the following assertions hold:*

(i) *S_Γ^* is the graph product of $(S_v^*)_{v\in V}$, $(S_\Gamma)_c = S_\Gamma^*$, and $(S_\Gamma)_{ci} = S_\Gamma\backslash S_\Gamma^*$.*
(ii) *For $s, t \in S_\Gamma$, $s \sim t$ is equivalent to $s \in tS_\Gamma^*$.*
(iii) *S_Γ is core factorable and $(S_\Gamma)_{ci} \subset S_\Gamma$ is $\cap$-closed.*
(iv) *The action $\alpha\colon S_\Gamma^* \curvearrowright S_\Gamma/_\sim$ is faithful if and only if S_Γ is not a group.*
(v) *The action α is almost free if and only if:*

(a) *$\alpha_v^*\colon S_v^* \curvearrowright S_v/S_v^*$ is almost free for every isolated vertex $v \in V$; and*
(b) *for every connected component $U \subset V$ with $|U| \geq 2$, either S_u is a group for all $u \in U$ or S_u^* is trivial for all $u \in U$.*

Proof. For (i), let $s_{v(1)}s_{v(2)}\cdots s_{v(n)}$ be a reduced expression for $s \in S_\Gamma$. Clearly, s is invertible in S_Γ if and only if $s_{v(k)} \in S_{v(k)}^*$ for all k. The homomorphism from the graph product of $(S_v^*)_{v\in V}$ to S_Γ (resulting from the universal property) is bijective, so that S_Γ^* is the graph product with respect to Γ and $(S_v^*)_{v\in V}$. Now assume that there is $1 \leq m \leq n$ such that $s_{v(k)} \in S_{v(k)}^*$ for $1 \leq k < m$ but $s_{v(m)} \notin S_{v(m)}^*$. Since Γ is coconnected, there is $w \in V$ with $w \neq v(m)$ and $(v(m), w) \notin E$. For every $t \in S_w\backslash\{1\}$,

$$s_{v(m)}s_{v(m+1)}\cdots s_{v(n)} \perp ts_{v(m)}s_{v(m+1)}\cdots s_{v(n)}.$$

By left cancellation, this yields

$$s \perp s_{v(1)}s_{v(2)}\cdots s_{v(m-1)}ts_{v(m)}s_{v(m+1)}\cdots s_{v(n)},$$

so that $s \notin (S_\Gamma)_c$. This proves $(S_\Gamma)_c = S_\Gamma^*$, and the claims $(S_\Gamma)_{ci} = S_\Gamma\backslash S_\Gamma^*$, (ii) and (iii) are immediate consequences of this.

For (iv), we note that α is not faithful if S_v is a group for all $v \in V$, because then $S_\Gamma/_\sim$ is a singleton while $S_\Gamma^* = S_\Gamma$ is nontrivial. So let us assume that there exists $v \in V$ with $S_v \neq S_v^*$. Every $x \in S_\Gamma^*\backslash\{1\}$ has a reduced expression $x_{u(1)}x_{u(2)}\cdots x_{u(m)}$ with $x_{u(k)} \in S_{u(k)}^*\backslash\{1\}$. Since Γ is coconnected and $|V| \geq 2$, there exists a blocking path $w(1), \ldots, w(n)$ for $\{u(m)\}$ with $w(n) = v$, see Lemma 2.3. Choose $s_{w(k)} \in S_{w(k)}\backslash\{1\}$ for

$1 \leq k < n$ and $s_{w(n)} \in S_{w(n)} \backslash S^*_{w(n)}$. Then $s := s_{w(1)} s_{w(2)} \cdots s_{w(n)} \in S_\Gamma$ satisfies $x_{u(m)} s \perp s$. If $1 \leq k \leq m-1$ satisfies $(u(k), u(\ell)) \in E$ for all $k < \ell \leq m$, then $(u(k), u(m)) \in E$ in particular implies $u(k) \neq v(1)$. For the same reason, $(u(k), v(1)) \in E$ implies $u(k) \neq v(2)$, and so on. Thus,

$$x_{u(1)} x_{u(2)} \cdots x_{u(m)} s_{w(1)} s_{w(2)} \cdots s_{w(n)}$$

is a reduced expression for xs and we conclude that orthogonality is not destroyed by $x_{u(1)} x_{u(2)} \cdots x_{u(m-1)}$, that is, $xs \perp s$. In particular, $[xs] \neq [s]$ and therefore α is faithful.

To prove (v), we first observe that (a) is necessary for α to be almost free: if $v \in V$ is isolated, then $[xs] = [s]$ for $x \in S^*_v \backslash \{1\}$ and $[s] \in S_\Gamma$ implies $s \in S_v$. Suppose next that (b) does not hold, that is, there exists a connected component $U \subset V$ of Γ with $|U| \geq 2$ such that there are $u, v \in U$ with $S_v \neq S^*_v$ and $S^*_u \neq \{1\}$. If $u = v$, then we can pick $w \in U \backslash \{v\}$ with $(v, w) \in E$. If there is $x \in S^*_w \neq \{1\}$, then $[xs] = [sx] = [s]$ for all $s \in S_v$, and since S_v / S^*_v is infinite, α fails to be almost free for x. On the other hand, S_w is nontrivial, so $S^*_w = \{1\}$ implies that S_w / S^*_w is infinite and then almost freeness fails for every $x \in S^*_v \neq \{1\}$.

Now suppose $u \neq v$. As U is connected, we can find a path from u to v inside U, say $v(0) := u, v(1), \ldots, v(n) := v$ with $(v(k), v(k+1)) \in E$ for all $0 \leq k < n$. Then there exists $0 \leq k < n$ such that $S^*_{v(k)} \neq \{1\}$ and $S_{v(k+1)} \neq S^*_{v(k+1)}$, and we can apply the above argument to deduce that α is not almost free. We have thus proven that almost freeness of α implies both (a) and (b).

Conversely, assume that (a) and (b) hold. If S_Γ is a group, then there is nothing to show, so we may suppose that $S_\Gamma \neq S^*_\Gamma$. Let $x \in S^*_\Gamma \backslash \{1\}$ be presented by a reduced expression $x_{u(1)} x_{u(2)} \cdots x_{u(m)}$ with $x_{u(k)} \in S^*_{u(k)} \backslash \{1\}$. Fix $s \in S_\Gamma \backslash S^*_\Gamma$ with reduced expression $s_{v(1)} \cdots s_{v(n)}$, $s_{v(k)} \in S_{v(k)}$. Let $1 \leq j \leq n$ be the smallest number such that $s_{v(j)} \notin S^*_{v(j)}$. By (b), j is invariant under shuffling and we know that $v(j)$ does not belong to the connected component of any $u(k)$ that emits an edge. Therefore, $xs \perp s$ and then $[xs] \neq [s]$, unless $j = m = 1$ and $u(1) = v(1) = v$ for some isolated vertex $v \in V$. In this case, (a) says that there are only finitely many fixed points for x in S_v / S^*_v. Thus, we conclude that α is almost free if (and only if) (a) and (b) hold. □

Remark 3.3. The graph product S_Γ for a coconnected graph Γ with $|V| \geq 2$ has finite propagation if S^*_v is a finite group for all $v \in V$.

Let us now summarise what Proposition 3.1 and Theorem 3.2 imply for graph products of right LCM semigroups.

Theorem 3.4. *Let $\Gamma = (V, E)$ be a graph and $(S_v)_{v \in V}$ a family of nontrivial right LCM semigroups. Then the following assertions hold:*

(i) *The subgroup of units is given by $S^*_\Gamma = \bigoplus_{v \in V_u} S^*_v \oplus \bigoplus_{i \in I_2} S^*_{\Gamma_i}$.*

(ii) *The core is given by $(S_\Gamma)_c = \bigoplus_{v \in V_u} (S_v)_c \oplus \bigoplus_{i \in I_2} S^*_{\Gamma_i}$.*

(iii) *The core irreducibles are given by $(S_\Gamma)'_{ci} = \bigoplus_{v \in V_u} (S_v)'_{ci} \oplus \bigoplus_{i \in I_2} S_{\Gamma_i}$.*

(iv) *Two elements $s, t \in S_\Gamma$ are core related if and only if $s_v \sim_v t_v$ for all $v \in V_u$ and $s_i \in t_i S^*_{\Gamma_i}$ for all $i \in I_2$.*

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(v) *S_Γ is core factorable if and only if S_v is core factorable for every $v \in V_u$.*

(vi) *$(S_\Gamma)_{ci} \subset S_\Gamma$ is $\cap$-closed if and only if $(S_v)_{ci} \subset S_v$ is $\cap$-closed for every $v \in V_u$.*

(vii) *The action $\alpha\colon (S_\Gamma)_c \curvearrowright S_\Gamma/_\sim$ is faithful if and only if the vertex action $\alpha_v\colon (S_v)_c \curvearrowright S_v/_\sim$ is faithful for every $v \in V_u$ and for every $i \in I_2$ there exists $w \in V_i$ such that S_w is not a group.*

(viii) *The action $\alpha\colon (S_\Gamma)_c \curvearrowright S_\Gamma/_\sim$ is almost free if and only if one of the following conditions holds:*

  (a) *$(S_v)_c = \{1\}$ for all $v \in V_u$ and $S_w^* = \{1\}$ for all $w \in V \backslash V_u$.*

  (b) *$(S_v)_c \neq \{1\}$ for a unique $v \in V_u$ with $\alpha_v\colon (S_v)_c \to S_v/_\sim$ almost free, while $S_w = (S_w)_c$ for all $w \in V_u \backslash \{v\}$ and $S_{w'} = S_{w'}^*$ for all $w' \in V \backslash V_u$.*

  (c) *$S_{\Gamma_i}^* \neq \{1\}$ for a unique $i \in I_2$ with $\alpha_i\colon S_{\Gamma_i}^* \to S_{\Gamma_i}/_\sim$ almost free, while $S_w = (S_w)_c$ for all $w \in V_u$ and $S_{\Gamma_j} = S_{\Gamma_j}^*$ for all $j \in I_2 \backslash \{i\}$.*

(ix) *S_Γ has finite propagation if S_v has finite propagation for every $v \in V_u$ and S_w^* is a finite group for all $w \in V \backslash V_u$.*

The conditions for almost freeness in Theorem 3.4 correspond to the conditions $(S_\Gamma)_c = \{1\}$, $(S_\Gamma)_c = (S_v)_c$ and $S_\Gamma/_\sim \cong = S_{\Gamma_i}/_\sim$, respectively. Hence, they are quite restrictive and we view this as an indication that finite propagation might be much more useful for graph products than almost freeness of α, see [1, Theorem 4.2(2)] for details. When applied to right-angled Artin monoids, Theorem 3.4 takes a simpler form:

Corollary 3.5. *For a graph $\Gamma = (V, E)$, the right-angled Artin monid A_Γ^+ satisfies:*

(i) *$(A_\Gamma^+)^* = \{1\}, (A_\Gamma^+)_c = \bigoplus_{v \in V_u} \mathbb{N}$ and $(A_\Gamma^+)_{ci}^1 = \bigoplus_{i \in I_2} A_{\Gamma_i}^+$.*

(ii) *Two elements $s, t \in A_\Gamma^+$ are core related if and only if $s_i = t_i$ for all $i \in I_2$.*

(iii) *A_Γ^+ is core factorable, $(A_\Gamma^+)_{ci} \subset A_\Gamma^+$ is $\cap$-closed and A_Γ^+ has finite propagation.*

(iv) *The action $\alpha\colon (A_\Gamma^+)_c \curvearrowright A_\Gamma^+/_\sim$ is faithful if and only if Γ has no universal vertex.*

(v) *The action $\alpha\colon (A_\Gamma^+)_c \curvearrowright A_\Gamma^+/_\sim$ is almost free if and only if Γ has no universal vertex or all vertices are universal.*

4. The absence of property (AR)

In this section, we will prove that for many graph products of right LCM semigroups S_Γ, the only accurate foundation sets are given by elements of S_Γ^*. In particular, we obtain an abundance of right LCM semigroups that lack property (AR). Again, the starting point is a basic observation for direct sums of right LCM semigroups, which allows us to boil the analysis down to the coconnected case:

Proposition 4.1. *Let $(S_i)_{i \in I}$ be a family of right LCM semigroups. If $\bigoplus_{i \in I} S_i$ has property (AR), then S_i has property (AR) for all $i \in I$.*

Proof. Fix $i \in I$ and let $S := \bigoplus_{i \in I} S_i$. Every foundation set F for S_i is a foundation set for S. Suppose that F has an accurate refinement F_a in S. For $s \in S_\Gamma$, we let $s = s_i + \hat{s}_i$

with $s_i \in S_i$ and $\hat{s}_i \in \bigoplus_{j \in I \setminus \{i\}} S_j$. If $s \in F_a$, then $\{f_i \in S_i \mid f \in F_a : \hat{f}_i \not\perp \hat{s}_i\}$ is an accurate refinement for F inside S_i. $\square$

Corollary 4.2. *If a graph product S_Γ has property (AR), then S_{Γ_i} has property (AR) for each coconnected component Γ_i of Γ.*

Theorem 4.3. *Let $\Gamma = (V, E)$ be a coconnected graph with at least two vertices and suppose $(S_v)_{v \in V}$ is a family of nontrivial right LCM semigroups.*

(i) *If Γ is infinite, then every foundation set for S_Γ contains an invertible element. In particular, S_Γ has property (AR) and $C^*(S_\Gamma) = \mathcal{Q}(S_\Gamma)$.*

(ii) *If Γ is finite and $E \neq \emptyset$, then the accurate foundation sets for S_Γ correspond to S_Γ^*. In particular, S_Γ has property (AR) if and only if S_Γ does not admit a foundation set without invertible elements.*

Proof. Both (i) and (ii) hold for trivial reasons if S_Γ is a group, so we can assume that there exists $w \in V$ with $S_w \neq S_w^*$. Let $F \subset S_\Gamma$ be a finite subset without invertible elements. For every $f \in F$, we choose a reduced expression $f = f_{v(1)} \cdots f_{v(m_f)}$ with $m_f \in \mathbb{N}^\times$ and $f_{v(k)} \in S_{v(k)}$.

Suppose first that Γ is infinite. As $f \in S_\Gamma \setminus S_\Gamma^*$, there is a smallest number $1 \leq k_f \leq m_f$ such that $f_{v(k_f)} \notin S_{v(k_f)}^*$. Then,

$$C := \{v \in V \mid f_{v(k)} \in S_v \text{ for some } f \in F, 1 \leq k \leq k_f\}$$

is a finite set of vertices so that Lemma 2.3 grants us a blocking path $w(1), \ldots, w(n)$ for C ending in w. If we choose any $s_k \in S_{w(k)} \setminus \{1\}$ for $1 \leq k < n$ and $s_n \in S_w \setminus S_w^*$, then $s_1 \cdots s_n \perp f$ for all $f \in F$ as $s_1 \cdots s_n \perp f_{v(1)} \cdots f_{v(k_f)}$ by construction, see Remark 2.4. Therefore, F is not a foundation set. We conclude that every foundation set for S_Γ contains an invertible element x, which clearly gives an accurate refinement $\{x\}$. So S_Γ has property (AR), but if the only accurate foundation sets come from invertible elements, then the boundary relation $\sum_{f \in F} e_{fS_\Gamma} = 1$ becomes trivial so that $C^*(S) = \mathcal{Q}(S)$.

Now let Γ be finite, $E \neq \emptyset$ and assume F to be accurate as well. We need to show that F is not a foundation set. Without loss of generality, we can require that $f_{v(m_f)}$ is not invertible for all $f \in F$ because invertible ends do not play a role when it comes to intersections of right ideals. Since F does not contain any invertibles, $\ell(f) \geq 1$ for all $f \in F$. Let $L := \max_{f \in F} \ell(F)$ and choose $f \in F$ with $\ell(f) = L$. Then, $f = st_v$ for some $v \in V, t_v \in S_v \setminus \{1\}$, and $s \in S_\Gamma$ with $\ell(s) = L - 1$. We will first show that v is isolated and then use this together with $E \neq \emptyset$ to conclude that F cannot be a foundation set.

If $(v, u) \in E$ for some $u \in V$, we employ Lemma 2.3 to obtain a blocking path $w(1), \ldots, w(n)$ for $C := \{u\} \cup N_u$ and set $w(0) := u$. Next, choose $b_k \in S_{w(k)} \setminus S_{w(k)}^*$ for each $1 \leq k \leq n$ and let $r \in S_u \setminus \{1\}$. It then follows that $srb \perp f$ for $b := b_1 \cdots b_n$. Moreover, $\ell(srb) \geq m + 1$. This could be assumed by extending the path $w(0), \ldots, w(n)$ in Γ^{opp}, but actually holds true in any case. It then follows that whenever $f' \in F$ satisfies $f' \not\perp srb$, $srb \in f'S_\Gamma$. If $sr \in f'\Gamma$, then $f' \not\perp f \neq f'$ so that F would not be accurate. The blocking path then forces $f' = srb_1 \cdots b_k$ for some $1 \leq k \leq n$. However, we then

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get $f' \perp sr'b$ for every $r' \in S_u \backslash \{r\}$. Since S_u is a left cancellative semigroup that is not a group, it is infinite. Thus, there is $r \in S_u \backslash \{1\}$ such that $srb \perp f'$ for all $f' \in F$.

We deduce from this that F cannot be a foundation set if there exists $f \in F$ with $\ell(f) = L$ that does not end in a part from an isolated vertex. In particular, if Γ does not have any isolated vertices, no accurate finite subset F without invertible elements is a foundation set. Now suppose Γ has an isolated vertex $\tilde{v}$ and let

$$F' := \{f \in F \mid f_{v(k)} \in S_v \text{ for some } k \Rightarrow v \text{ is not isolated}\},$$

that is, the subset of F consisting of those elements whose reduced expressions do not contain any part coming from an isolated vertex. As $E \neq \emptyset$ and the vertex semigroups are all nontrivial, the finite accurate set F' is also nonempty.

Suppose first that there is $\tilde{f} \in F'$ with $\tilde{f} \in S_v \backslash S_v^*$ for some $v \in V$. Since F' is accurate and $(v, u) \in E$ for some $u \in V$, $s \notin f' S_\Gamma$ for all $s \in S_u$ and $f' \in F'$. Thus, $str \perp f'$ for all $f' \in F'$ whenever $s \in S_u, t \in S_{\tilde{v}}$ and $r \in S_w \backslash S_w^*$, compare Remark 2.4. For $f \in F \backslash F'$, $strtr \perp f$ unless $f \in strtS_\Gamma$ because $\tilde{v}$ is isolated and r is not invertible. Since F is finite while $S_w \backslash S_w^*$ is infinite, we conclude that there are $s \in S_u, t \in S_{\tilde{v}}$ and $r \in S_w \backslash S_w^*$ such that $strtr \perp f$ for all $f \in F$. So F is not a foundation set.

On the other hand, if $\ell(f) \geq 2$ for every $f \in F'$, we pick a vertex v that emits an edge. Then, $s \notin fS_\Gamma$ for all $s \in S_v, f \in F'$; thus, $str \perp f$ for all $f \in F'$ whenever $s \in S_v, t \in S_{\tilde{v}}$ and $r \in S_w \backslash S_w^*$. As in the previous case, there are s, t, r such that $strtr \perp f$ for all $f \in F$ and thus F is not a foundation set.

Finally, if F is a foundation set for S_Γ with $F \cap S_\Gamma^* = \emptyset$, then every refinement F' of F satisfies $F' \cap S_\Gamma^* = \emptyset$ as well and thus can never be accurate. On the other hand, every foundation set F with $x \in F \cap S_\Gamma^*$ has an accurate refinement $\{x\}$. □

We point out that the assumptions in Theorem 4.3 are modest means to avoid the pathological cases: $S_\Gamma = S_v$, the free product $S_\Gamma = *_{v \in V} S_v$ and the graph product of groups.

Remark 4.4. By Theorem 4.3(i), foundation sets of S_Γ are governed by parts from the finite coconnected components in the following sense: let F be a foundation set for S_Γ such that no property subset of F is a foundation set. If $s = s_{v(1)} \cdots s_{v(n)} \in F$ with $s_{v(k)} \in S_{v(k)}$, then $s_{v(k)} \notin S_{v(k)}^*$ implies that $v(k) \in V_i$ for some finite coconnected component $\Gamma_i = (V_i, E_i)$ of Γ.

Corollary 4.5. *Let Γ be a graph and $(S_v)_{v \in V}$ a family of nontrivial right LCM semigroups. If there is $i \in I_2$ for which $\Gamma_i = (V_i, E_i)$ is finite with $E_i \neq \emptyset$, S_v is not a group for some $v \in V_i$ and there exists a foundation set F for S_{Γ_i} without invertible elements, then S_Γ does not have property (AR).*

Proof. The claim follows from combining Theorem 4.3 with Corollary 4.2. □

The previous results apply nicely to right-angled Artin monoids.

Corollary 4.6. *For graph Γ, the following conditions are equivalent:*

(1) *Every finite coconnected component Γ_i of Γ is edge-free.*
(2) *Every finitely generated direct summand of A_Γ^+ is free.*
(3) *The right-angled Artin monoid A_Γ^+ has property (AR).*

Proof. The equivalence of (1) and (2) is clear from the direct sum description of A_Γ^+ in Section 2.2. From Remark 4.4, we infer that it suffices to obtain accurate refinements of foundation sets F for A_Γ^+ with $F \subset \bigoplus_{v\in V_u} S_v \oplus \bigoplus_{i\in I_2:|V_i|<\infty} A_{\Gamma_i}^+$. But if (2) holds, then the latter is just a direct sum of finitely generated free monoids, and clearly admits accurate refinements. So (2) implies (3). Finally, if (3) is valid and $\Gamma_i = (V_i, E_i)$ is a coconnected component of Γ with $2 \leq |V_i| < \infty$, then $\{a_v \mid v \in V_i\}$ is a foundation set for $A_{\Gamma_i}^+$ without invertible elements, so Corollary 4.5 forces $E_i = \emptyset$, that is, (1) holds. □

By Corollary 4.6, there exist countably many mutually nonisomorphic, finitely generated right LCM semigroups without property (AR). As a final part of this section, we address the existence of a generalised scale for right-angled Artin monoids associated to finite graphs. The existence of a generalised scale turned out to be relevant for a standardised approach to study KMS-states on the semigroup C^*-algebra $C^*(A_\Gamma^+)$, see [1]. We first note that free monoids have a generalised scale only if they are finitely generated and nonabelian, in which case it is unique:

Proposition 4.7. *The free monoid $\mathbb{F}_m^+$ in $2 \leq m < \infty$ generators admits a unique generalised scale $N\colon \mathbb{F}_m^+ \to \mathbb{N}^\times$ given by $N(w) = m^{\ell(w)}$, where ℓ denotes the word length of $w \in \mathbb{F}_m^+$.*

Proof. The map N is a generalised scale. On the other hand, let $\tilde{N}$ be a generalised scale on $\mathbb{F}_m^+ = \langle a_1, \ldots, a_m\rangle$ and fix $1 \leq i \leq m$. Then, $\tilde{N}(a_i) > 1$ as a_i is not part of $(\mathbb{F}_m^+)_c = \{1\}$. By definition of $\tilde{N}$ and since $\sim$ is trivial, the set $\tilde{N}^{-1}(\tilde{N}(a_i))$ is an accurate foundation set for $\mathbb{F}_m^+$ of cardinality $\tilde{N}(a_i)$ that contains a_i. If there was $1 \leq j \leq m$, $j \neq i$ such that $\tilde{N}(a_j) \neq \tilde{N}(a_i)$, then the foundation set property would give a $w \in \tilde{N}^{-1}(\tilde{N}(a_i))$ such that $w \in a_j a_i \mathbb{F}_m^+$. As this forces $\tilde{N}(w) \geq \tilde{N}(a_i)\tilde{N}(a_j) > \tilde{N}(a_i) = \tilde{N}(w)$, we arrive at a contradiction. Thus, $\tilde{N}(a_j) = \tilde{N}(a_i)$ for all $j \neq i$. But as $\{a_1, \ldots, a_m\}$ is an accurate foundation set for $\mathbb{F}_m^+$, we conclude that $\tilde{N}(a_i) = m$ for all $1 \leq i \leq m$. □

We call $m \in (\mathbb{N}_{\geq 2})^I$ *rationally independent* if for all distinct $k, k' \in \bigoplus_{i\in I} \mathbb{N}$, the supernatural numbers $\prod_{i\in I} m_i^{k_i}$ and $\prod_{i\in I} m_i^{k_i'}$ are distinct.

Proposition 4.8. *Let M be a free abelian monoid, I a nonempty set and $m \in (\mathbb{N}_{\geq 2})^I$. Then, $S := M \oplus \bigoplus_{i\in I} \mathbb{F}_{m_i}^+$ admits a generalised scale $N\colon S \to \mathbb{N}^\times$ if and only if m is rationally independent. In this case, N restricts to the unique generalised scales N_i on $\mathbb{F}_{m_i}^+$ and is therefore unique.*

Proof. As $M = S_c = \ker N$ for any generalised scale N on S, see [1, Proposition 3.6(i)], we can focus on $(\mathbb{F}_{m_i}^+)_{i\in I}$. Recall that $\mathbb{F}_{m_i}^+$ is the free monoid in m_i generators, which we denote by $a_{i,1}, \ldots, a_{i,m_i}$. The strategy is to prove that:

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(a) any generalised scale N on S restricts to N_i on $\mathbb{F}^+_{m_i}$; and
(b) the homomorphism $N\colon S \to \mathbb{N}^\times$ arising from $(N_i)_{i\in I}$ is a generalised scale if and only if m is rationally independent.

For (a), suppose S admits a generalised scale N and fix $i \in I$ as well as $1 \leq k \leq m_i$. Then, $N(a_{i,k}) > 1$ and there are $w_1, \dots, w_{N(a_{i,k})-1} \in S$ such that $\{a_{i,k}, w_1, \dots, w_{N(a_{i,k})-1}\}$ is an accurate foundation set for S contained in $N^{-1}(N(a_{i,k}))$. Let us decompose w_ℓ as

$$w_\ell = \hat{w}_\ell \oplus \check{w}_\ell \in \mathbb{F}^+_{m_i} \oplus \Big(M_n \oplus \bigoplus_{j\in I\setminus\{i\}} \mathbb{F}^+_{m_j}\Big).$$

Then, $\{a_{i,k}, \hat{w}_1, \dots, \hat{w}_{N(a_{i,k})-1}\}$ is a foundation set for $\mathbb{F}^+_{m_i}$ with $a_{i,k} \perp \hat{w}_\ell$ and $N(\hat{w}_\ell) \leq N(a_{i,k})$ for all ℓ. This forces

$$\{a_{i,k}, \hat{w}_1, \dots, \hat{w}_{N(a_{i,k})-1}\} \supset \{a_{i,1}, \dots, a_{i,m_i}\},$$

and thus $N(a_{i,\ell}) \leq N(a_{i,k})$ for all $1 \leq \ell \leq m_i$, just like in the proof of Proposition 4.7. As k was arbitrary, we deduce $N(a_{i,k}) = m_i = N_i(a_{i,k})$ for all i, k.

In view of (a), the question behind the main claim becomes: under which condition is the homomorphism $N\colon S \to \mathbb{N}^\times$ arising from the family of generalised scales $(N_i)_{i\in I}$ itself a generalised scale? If m is rationally independent, then every $k \in N(S)$ has a factorisation $k = \prod_{i\in I} m_i^{k_i}$ with uniquely determined $k_i \in \mathbb{N}$. This implies

$$N^{-1}(k) = \Big\{t \oplus \bigoplus_{i\in I} w_i \mid t \in M, w_i \in \mathbb{F}^+_{m_i} \text{ with } \ell_i(w_i) = k_i\Big\}.$$

Therefore, $|N^{-1}(k)/_\sim| = k$ and any transversal of $N^{-1}(k)/_\sim$ is an accurate foundation set for S, that is, N is a generalised scale. On the other hand, if there are $k, k' \in \bigoplus_{i\in I} \mathbb{N}, k \neq k'$ such that $K := \prod_{i\in I} m_i^{k_i} = \prod_{i\in I} m_i^{k'_i}$, then both k and k' yield a set of K mutually orthogonal elements $s_1, \dots, s_K \in S$ and $t_1, \dots, t_K \in S$, respectively, with $N(s_j) = K = N(t_j)$ for all j. Since there is $i \in I$ with $k_i \neq k'_i$, the ith components of s_j and $t_{j'}$ have different length for all j, j'. Thus, $s_j \nsim t_{j'}$ for all j, j' and $|N^{-1}(K)/_\sim| \geq 2K$. Therefore, N is not a generalised scale in this case. □

We can now state our conclusions for right-angled Artin monoids.

Corollary 4.9. *For every graph Γ, the right-angled Artin monoid A^+_Γ admits a generalised scale N if and only if $V_u \neq V$, all coconnected components $\Gamma_i = (V_i, E_i)$ are finite and edge-free and $(|V_i|)_{i\in I_2}$ is rationally independent. In this case, N is unique.*

Proof. The condition $V_u \neq V$ is equivalent to saying that A^+_Γ is nonabelian, that is, $I_2 \neq \emptyset$. So if all coconnected components $\Gamma_i = (V_i, E_i)$ are finite and edge-free, then $A^+_\Gamma \cong \bigoplus_{v\in V_u} \mathbb{N} \oplus \bigoplus_{i\in I_2} \mathbb{F}^+_{|V_i|}$. Hence, Proposition 4.8 implies that A^+_Γ has a (unique) generalised scale N if and only if $(|V_i|)_{i\in I_2}$ is rationally independent.

Conversely, suppose A^+_Γ admits a generalised scale N. Since N is a nontrivial homomorphism with $\ker N = \bigoplus_{v\in V_u} \mathbb{N}$, we need to have $V_u \neq V$ so that the set I_2 is nonempty. Moreover, A^+_Γ has property (AR) by [1, Proposition 3.6], so Corollary 4.6

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implies that all finite coconnected components Γ_i of Γ are edge-free. If there was an infinite coconnected component $\Gamma_i = (V_i, E_i)$, then $1 < N(a_v) < \infty$ for all $v \in V_i$ and the defining property of a generalised scale would yield an accurate foundation set of the form $\{a_v, f_1, \ldots, f_{N(a_v)-1}\}$ for suitable $f_k \in A_\Gamma^+$. However, this contradicts Remark 4.4 and we conclude that Γ_i is finite for all $i \in I_2$. But then A_Γ^+ is covered by Proposition 4.8 and it follows that $(|V_i|)_{i\in I_2}$ is rationally independent. □

Corollary 4.10. *For every graph Γ, the right-angled Artin monoid A_Γ^+ is admissible if and only if it admits a generalised scale.*

Proof. According to Corollary 3.5(iii), the monoid A_Γ^+ is core factorable and $(A_\Gamma^+)_{ci} \subset A_\Gamma^+$ is $\cap$-closed, no matter what Γ is. By Corollary 4.9, the conditions characterising the existence of a generalised scale N include rational independence of $\bigoplus_{i\in I_2} |V_i|$. This feature allows us to deduce $\mathrm{Irr}(N(A_\Gamma^+)) = \{|V_i| \mid i \in I_2\}$ and that this set freely generates $N(A_\Gamma^+)$, which is the last extra condition for admissibility. □

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