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INFINITELY DETERMINED MAPGERMS

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1. Introduction. In differential analysis, it is very useful to have the local behavior of a differentiable map be determined by the derivatives of the map at a point. Hence we have the theories of finite and infinitely determined germs. Let m_n^p be the space of germs of C^∞ maps $f: (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ and G a group operating on m_n^p. A germ f is called *finitely G-determined* if there exists an integer k such that every germ having the same k-jet as f is G-equivalent to (i.e., in the same G-orbit as) f. A germ f is called *∞-G-determined* if every germ having the same formal power series as f is G-equivalent to f.

In this paper we give necessary and sufficient conditions for a germ to be ∞-determined with respect to one of the groups C, R, K or L which appear frequently in Singularity Theory. In another paper ([**13**]) we consider ∞-A-determined mapgerms. Also in this paper we show that a germ f is ∞-K-determined if and only if it is finitely v-determined (meaning that every representative of some k-jet of f has homeomorphic zero set), we prove a version of Glaeser's Theorem on closedness of pullback rings for ∞-L-determined germs, and discuss the relationship between ∞-K- or C-determination and properness.

In [**14**] we employ ∞-K-determination in studying the question: when can one find conditions on the derivatives of a map at a point which will guarantee that the zero set of the map is nonsingular at that point? The Implicit Function Theorem gives one such condition: that the first derivative be surjective. However, we show that, for singular mapgerms whose zero sets are of dimension at least two, any such condition must involve all the derivatives. Thus finite jets are inadequate for this study. The mapgerms for which such conditions on ∞-jets exist are the ∞-K-determined germs.

R is the group of germs of origin preserving diffeomorphisms acting on mapgerms by composition on the right; L is similar but acts by composition on the left; C is the group of germs of diffeomorphisms at 0 on $\mathbf{R}^{n+p}$ leaving $\mathbf{R}^n$ fixed and commuting with orthogonal projection onto $\mathbf{R}^n$ and acting on a mapgerm by mapping its graph onto the graph of another mapgerm; K is the semidirect product of R and C and A is the direct product of R and L. For more detailed descriptions of these group actions, see [**8**]. Let E_n be the ring of C^∞ real-valued germs at 0 in $\mathbf{R}^n$ and m_n its maximal ideal; m_n^p will sometimes denote p-tuples of elements of m_n and

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sometimes the p-th power of m_n, which should be clear by context. For $f \in m_n^p$, let TCf denote $(f^*m_p)E_n^p$, TRf denote dfE_n^n, TKf denote $(f^*m_p)E_n^p + dfE_n^n$, TLf denote $f^*m_p^p$ and TAf denote $f^*m_p^p + dfE_n^n$ (for interpretations of these see [**8**]).

THEOREM 1.1. *The following are equivalent:*
(1) *f is finitely G-determined;*
(2) *$TGf \supset m_n^kE_n^p$ for some positive integer k;*
(3) *(if f is analytic) for each representative $\tilde{f}$ of the complexification of f, there is a deleted neighborhood of the origin in $\mathbf{C}^n$ on which $\tilde{f}$ is infinitesimally G-stable (defined below).*

By saying $\tilde{f}$ is infinitesimally G-stable we mean: $\tilde{f}$ is never 0 (if $G = C$); $\tilde{f}$ is a submersion (if $G = R$); $\tilde{f}$ has no critical zeros (points at which $\tilde{f}$ is zero but not a submersion) (if $G = K$); $\tilde{f}$ is an embedding (if $G = L$); $\tilde{f}$ is infinitesimally stable in the usual sense (if $G = A$). For the theories of stability and infinitesimal stability see [**4**]. Such theories can be developed for each of the groups C, R, K and L. However we do not need this here; we use the term infinitesimally G-stable merely to unify our results and for intuition. The result (1) if and only if (2) is due to Mather (see Theorem 3.5 of [**8**]). The result (1) if and only if (3) in the cases $G = R$, C or K is a special case of Tougeron's Proposition VIII.4.2 in [**11**]; the case $G = A$ is due to Mather, $G = L$ to Gaffney, and both proofs first appeared in [**3**].

We prove an analogous theorem for ∞-determined germs. Note that finite-determinedness depends upon what complex singularities occur nearby; ∞-determinedness only depends upon what real singularities occur nearby. Thus (3) is replaced by: (3a) (if f is analytic) for each representative $\tilde{f}$ of f, there is a deleted neighborhood of the origin in $\mathbf{R}^n$ on which $\tilde{f}$ is infinitesimally G-stable.

Every finitely determined germ is equivalent to its Taylor polynomial of some degree, so the restriction of condition (3) to analytic germs is no real problem. However, it is not known whether ∞-determined germs need be equivalent to analytic germs, hence the analyticity in (3a) is a real restriction. One cannot simply drop the word analytic; for example, any function in m_n^∞ which is positive except at 0 would satisfy (3a) for $G = C$, but would not be ∞-C-determined. Thus it is necessary to restrict the rate at which $\tilde{f}$ approaches being unstable as we approach 0.

The germ g is said to satisfy a *Lojasiewicz inequality of order r at a set X* if there are constants $c > 0$ and $r \geqq 0$ such that $|g(x)| \geqq cd(x, X)^r$ as germs; one also says in this case that the ideal $(g^*m_p)E_n$ is Lojasiewicz at X (this does not depend upon the generators $g_1, \ldots, g_p$ chosen). Suppose positive real numbers b_i converge to zero. A sequence of real numbers a_i is *flat* along b_i if, for each $r > 0$, there is an N such that $i \geqq N$ implies $|a_i| \leqq b_i^r$. A sequence of vectors, matrices or ∞-jets is flat along b_i if each entry is, and is flat along x_i in $\mathbf{R}^n$ if it is flat along $|x_i|$. Note that an ideal $(g^*m_p)E_n$ is not Lojasiewicz at X if and only if there is a sequence $x_i \to 0$ such that $g(x_i)$ is flat along $d(x_i, X)$.

One defines G-stable jets analogously with infinitesimally G-stable mappings. See [**4**] for the theory of jet spaces. We will write $\mathbf{R}^{pr} \times J^k(n, p)^r$ for the space of r-tuples of k-jets of elements of E_n^p; if f maps U into $\mathbf{R}^p$, we define a map $(j^kf)^r$ into the above jet space in the obvious way. Let Uns_R denote the set of critical jets, Uns_C the set of jets with value 0 and Uns_K the set of critical jets with value 0; let Uns_L be $V_1 \cup V_2$, where V_1 is the set of pairs of jets, at least one of which is not the jet of an immersion, and V_2 is the set of pairs of jets having the same value. Let A_G denote the ideal of all polynomials on the jet space which vanish on Uns_G. Let $I_G(f)$ denote $(((j^kf)^r)^*A_G)E_{nr}$, where $k = 0$ if $G = C$ and otherwise $k = 1$, and $r = 2$ if $G = L$ and otherwise $r = 1$. So $I_C(f) = (f^*m_p)E_n$, $I_R(f) = Jf$, the ideal generated by the determinants of the p by p submatrices of df, and $I_K(f)$ is the sum of these. Let Δ_n denote the diagonal in $\mathbf{R}^n \times \mathbf{R}^n$. If $G = L$, let D denote $(\mathbf{R}^n \times 0) \cup (0 \times \mathbf{R}^n) \cup \Delta_n$; otherwise $D = \{0\}$.

THEOREM 1.2. *Suppose f is in E_n^p. For G either C, R, K or L, the following conditions are equivalent:*

(1) *f is ∞-G-determined;*

(2) *$TGf \supset m_n^\infty E_n^p$;*

(3) *$I_G(f)$ is Lojasiewicz at D.*

For $G = L$, condition (3) is equivalent to: $f(x) - f(y)$ is Lojasiewicz at Δ_n and $J_n(f)$, the ideal generated by the determinants of the n by n submatrices of df, is Lojasiewicz at 0.

There are several useful restatements of (3):

(3′) $I_G(f) \supset m_D^\infty$ (all germs which are flat on D);

(3″) $d((j^kf)^r(x), Uns_G)$ is Lojasiewicz at D;

(3‴) The graph of $(j^kf)^r$ and $\mathbf{R}^{nr} \times Uns_G$ are regularly situated and $D = ((j^kf)^r)^{-1}(Uns_G)$.

The equivalence (3) and (3′) follows from Proposition V.4.3 of [**11**]. The equivalence of (3″) and (3‴) follows easily from the definition of regularly situated, which can be found in [**11**]. The equivalence of (3) and (3″) follows from the next Lemma.

LEMMA 1.3. *Let p be a polynomial mapping, V its zero set, F a C^∞ germ. Then $p \circ F$ is Lojasiewicz at $D = F^{-1}V$ if and only if $d(F(x), V)$ is.*

Proof. "If" follows from the fact that every polynomial satisfies a Lojasiewicz inequality at its zero set. "Only if" follows from the fact that, for z restricted to a compact set, there is a positive k such that $|p(z)| \leqq kd(z, V)$.

Belickii, in [**1**], proved that (3) implies (1) when G is R or L. His proof in the R case is quite different from ours. His proof in the L case is, in this author's opinion, quite sketchy. Also his result as stated is wrong. Instead of requiring that $|f(x) - f(y)|$ satisfy a Lojasiewicz inequality at Δ_n, he only requires that f be one-to-one and $|f(x)|$ satisfy a Lojasiewicz inequality at 0. A map which satisfies his conditions but is not ∞-L-determined is:

Example 1.4. $f(x) = (x^2, g(x))$, where $g(x)$ is C^∞, positive when x is positive and zero when x is nonpositive.

Kucharz in [**6**] states that (1) is equivalent to (3) when G is R or L, but gives no proofs. He repeats Belickii's mistake. In addition, he states that f is ∞-R-determined if and only if f is finitely C^k-R-determined, $0 \leqq k < \infty$, and that f is ∞-L-determined if and only if f is finitely C^k-L-determined, $1 \leqq k < \infty$. We prove a similar result for ∞-K-determined germs in Section 5 (earlier proofs of this result by Tougeron and Bochnak-Kuo are referred to in [**2**]).

Brodersen in [**2**] proves Theorem 1.2 for $G = R$ and K. His proof, obtained independently from and simultaneously with ours, differs from ours largely in the case "(1) implies (3)." He also expounds upon the relationship between infinite determination and finite C^k determination ($k < \infty$) for these cases.

Sections 2, 3, and 4 are devoted to the proof of Theorem 1.2. Some related results are given in Section 5. Among these results are that ∞-C-determined germs are precisely the proper germs, and ∞-K-determined germs are precisely the germs which are proper on their critical sets. Such germs are quite common.

While finitely L-determined germs only occur when $p \geqq 2n$ and finitely R-determined germs only occur when $p = 1$, ∞-L-determined germs occur whenever $p \geqq n$ and ∞-R-determined germs occur whenever $p \leqq n$. When $p = n$, the ∞-L-determined germs form a subcollection of the ∞-R-determined germs (the latter are covering spaces on a deleted neighborhood of the origin, while the former are diffeomorphisms on a deleted neighborhood; if $p \geqq 3$, these collections are identical). Belickii gives examples of infinitely L- and R-determined germs which are not finitely L- or R-determined. We close this section with some more examples.

Example 1.5. The complex map $f(z) = z^n$, $n > 1$, is, as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$, ∞-R-determined but not ∞-L-determined.

Example 1.6. $f(x, y, z) = (x, y, z^r + (x^s + y^t)z)$ is ∞-L-determined if r is odd and s and t are even.

Example 1.7. $f(x, y) = (x, y^2, y^r + x^n y)$ is ∞-L-determined if r is odd and n is even.

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2. Proof that (2) **implies** (1)**.** We will first do the case $G = K$. Cases $G = C$ and $G = R$ will be omitted as they are entirely analogous to this case.

We are given that

$$dfE_n^n + (f^*m_p)E_n^p \supset m_n^\infty E_n^p.$$

Let $g(x, t) = f(x) + tu(x)$, u flat, let $F = (f, t)$ and $G = (g, t)$, and let f^a, g^a, F^a and G^a denote the germs of these maps at $(0, a)$ (where $f(x, t) = f(x)$ for all t). Via translation we identify the germs at $(0, a)$ with those at $(0, 0)$. Thus $g^a - f^a$ is in $m_n^\infty E_{n+1}^p$. We fix a and henceforth will suppress the superscript a, writing F for F^a, etc.

Let m_T^{k+1} denote the germs which vanish to order k along the t-axis in either $\mathbf{R}^{n+1}$ or $\mathbf{R}^{p+1}$. Then

$$m_n^\infty E_{n+1} = m_T^\infty \quad \text{and} \quad m_p^\infty E_{p+1} = m_T^\infty.$$

This follows from the proof of Lemma V.2.4 of [**11**]. In that proof, let X be the t-axis and observe that the functions α_i can be chosen to be independent of t.

Let dg denote the p by n matrix of partials of g with respect to the x variables. Let Jg denote the ideal in E_{n+1} generated by the determinants of the p by p submatrices of dg.

LEMMA 2.1. *For any g in E_{n+1}^p,*

$$dgE_{n+1}^n + (g^*m_p)E_{n+1}^p \supset (m_T^\infty)^p$$

if and only if

$$Jg + (g^*m_p)E_{n+1} \supset m_T^\infty.$$

Proof. "If": Using the fact that, for any square matrix A, $A(\operatorname{adj} A) = (\det A)I$, one can easily show that

$$dgE_{n+1}^n \supset (Jg)E_{n+1}^n.$$

The conclusion is immediate.

"Only if": Pick u in m_T^∞. By Lemma V.2.4 of [**11**], there exists $u_1, \ldots, u_p$ in m_T^∞ such that $u = u_1 \cdot \ldots \cdot u_p$. Let M be the diagonal p by p matrix with $u_1, \ldots, u_p$ the diagonal entries. By assumption, there is an n by p matrix N with entries in E_{n+1} and a p by p matrix P with entries in $(g^*m_p)E_{n+1}$ such that $dgN = M + P$. Take the determinant of both sides of this equation. By the Cauchy-Binet formula for computing the determinant of a product of matrices, the left hand side is in Jg; the right hand side is u plus something in $(g^*m_p)E_{n+1}$. Thus u is in $Jg + (g^*m_p)E_{n+1}$ as claimed.

Since $(m_T^\infty)^2 = m_T^\infty$, the two equivalent conditions in the Lemma are also equivalent to

$$dg(m_T^\infty)^n + (g^*m_p)(m_T^\infty)^p \supset (m_T^\infty)^p$$

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and to

$$(Jg)m_T^\infty + (g^*m_p)m_T^\infty \supset m_T^\infty.$$

We are given that $dfE_n^n + (f^*m_p)E_n^p \supset (m_n^\infty)^p$. Multiplying both sides by E_{n+1}, we have that

$$dfE_{n+1}^n + (f^*m_p)E_{n+1}^p \supset (m_T^\infty)^p.$$

Applying Lemma 2.1 to f, we see that

$$(Jf)E_{n+1} + (f^*m_p)E_{n+1} \supset m_T^\infty.$$

Thus the ideal on the left hand side satisfies a Lojasiewicz inequality at the t-axis. Since g is infinitely close to f, $Jg + (g^*m_p)E_{n+1}$ also satisfies a Lojasiewicz inequality at the t-axis, hence contains m_T^∞. Thus, by Lemma 2.1,

$$dg(m_T^\infty)^n + (g^*m_p)(m_T^\infty)^p \supset (m_T^\infty)^p.$$

Since $G^*(m_pE_{p+1})E_{n+1} = (g^*m_p)E_{n+1}$, this is equivalent to

$$dgm_n^\infty E_{n+1}^n + G^*(m_pE_{p+1})m_n^\infty E_{p+1}^p \supset m_n^\infty E_{n+1}^p.$$

This is exactly what is needed to make Mather's proof in Section 5.3 of [**8**] work. The conclusion is that $f(x) + tu(x) = K_t \cdot f(x)$, where all the components of K_t (an element of the group K) are flat and depend smoothly on t.

Next we do the case $G = L$.

The condition $f^*E_p^p \supset m_n^\infty E_n^p$ is equivalent to $f^*E_p \supset m_n^\infty$. Suppose $g - f$ is in $m_n^\infty E_n^p$. By Lemma 5.1 of [**3**], f and g are L-equivalent if and only if $g^*E_p = f^*E_p$. Taking Taylor series, we see that

$$g^*E_p \subseteq f^*E_p \subseteq g^*E_p + m_n^\infty.$$

That $g^*E_p = f^*E_p$ follows from Nakayama's Lemma and the following two facts, which we will establish:

(i) $g^*E_p + g^*m_p f^*E_p = f^*E_p$;
(ii) f^*E_p is finitely generated as a g^*E_p module.

Since $g^*E_p \subset f^*E_p$, the left hand side of (i) is clearly contained in the right hand side. Since f, hence g, is ∞-C-determined, $(g^*m_p)m_n^\infty \supset m_n^\infty$. Thus

$$g^*m_p \cdot f^*E_p \supset (g^*m_p)m_n^\infty \supset m_n^\infty.$$

Thus the right hand side of (i) is contained in the left hand side.

By (i), $f^*E_p/(g^*m_p f^*E_p)$ is a one-dimensional real vector space. Furthermore, $f^*E_p = (f, g)^*E_{2p}$ so, by Gaffney's Preparation Theorem (Theorem 2.7 of [**3**]), (ii) holds.

3. Proof that (1) **implies** (3). We assume (3) fails and prove the existence of two representatives g and h of the given formal power series (namely Tf) which, because of the differing geometry of their G-unstable sets, must be G-inequivalent. First we prove two lemmas: a transversality lemma which is used to prove the existence of h having as nice as possible geometry of its G-unstable set; a construction lemma which is used to prove the existence of g having G-unstable points at preassigned locations. These lemmas will be proved in a more general form than is needed for this paper.

Let N and P be C^∞ manifolds, X a closed subset of N and $h: X \to J^\infty(N, P)$ a section ($J^\infty(N, P)$ is the limit of $\ldots \to J^{k+1}(N, P) \to J^k(N, P) \to \ldots \to J^0(N, P) = N \times P$). Let

$$A = \{ f \text{ in } C^\infty(N, P) : j^\infty f | X = h\};$$

A is a closed subset of $C^\infty(N, P)$ in the (weak or strong) C^∞ topology. Hence A is a Baire space (see Theorem 4.4 and the following discussion in Chapter 2 of [**5**]), i.e., every residual set (= countable intersection of open, dense sets) is dense.

Now suppose X is closed in N. Let

$$\pi: {}_sJ^k(N, P) \to N^{(s)} \times P^s$$

denote the s-fold k-jet bundle and let $X^{(s)}$ be the set of s-tuples with at least one component in X; $X^{(s)}$ is closed in $N^{(s)}$. Suppose W is an immersed submanifold of $\pi_1^{-1}(N^{(s)} - X^{(s)})$. Let

$$A^t = \{ f \in A : {}_sj^kf | (N^{(s)} - X^{(s)}) \pitchfork W\}.$$

LEMMA 3.1. (Multijet Transversal Extension Theorem.) *A^t is residual in A.*

Proof. W can be covered by a countable collection of compact, codimension 0 submanifolds M_i (with boundary); furthermore, the M_i may be chosen so that, for each i, there exist relatively compact, mutually disjoint coordinate patches $U_{i,1}, \ldots, U_{i,s}$ whose closures are contained in $N - X$ and $V_{i,1}, \ldots, V_{i,s}$ in P such that

$$\pi(M_i) \subset U_{i,1} \times \ldots \times U_{i,s} \times V_{i,1} \times \ldots \times V_{i,s}.$$

Let

$$B_i = \{ f \in C(N, P) : {}_sj^kf \pitchfork W \text{ on } M_i\}$$

and let $A_i = B_i \cap A$; A_i is open in A since B_i is open (see II.4.14 of [**4**]). Pick any neighborhood U of f. Exactly as in the proof of the Multijet Transversality Theorem ([**9**] or [**4**]), there is a g in $B_i \cap U$ which agrees with f outside $U_{i,1} \cup \ldots \cup U_{i,s}$; thus g is in $A_i \cap U$. Hence A_i is dense in A; $A^t = \cap A_i$ is residual.

COROLLARY 3.2. *If $\{W_i\}$ is a countable collection of immersed submanifolds of $\pi_1^{-1}(N^{(s)} - X^{(s)})$, then*

$$\{f \in A: j^k f \,|\, (N^{(s)} - X^{(s)}) \pitchfork W_i \text{ for all } i\}$$

is residual.

LEMMA 3.3. *Suppose there exist w_i in $\mathbf{R}^{pr} \times J^k(n, p)^r (k \leqq \infty)$, $x_i = (x_i^1, \ldots, x_i^r)$ in $\mathbf{R}^{nr}$ converging to 0, and f in E_n^p such that $q_i = w_i - (j^k f)^r(x_i)$ is flat along $|x_i^s| \neq 0$ for each s and along $|x_i^s - x_i^t| \neq 0$ for each $s \neq t$. Then there is a g such that $Tg = Tf$ and $(j^k g)^r(x_i) = w_i$ holds for a subsequence of $\{x_i\}$.*

Proof. If k is finite, then we transform each w_i into an ∞-jet in such a way that all the terms of order greater than k of q_i are zero. Thus we will assume $k = \infty$.

Let Q be the Taylor field given by q_i^s at x_i^s and by the zero series at 0. We want to show that Q is a C^∞ Whitney field. By IV.1.5 and IV.1.6 of [**11**], it is enough to show, for each m and each multi-index K, $|K| \leqq m$, that

$$(R_y^m Q)^K(x) = o(|x - y|^{m-|K|}),$$

where

$$\text{(i)} \quad (R_y^m Q)^K(x) = Q^K(x) - \sum_{|L| \leqq m - |K|} Q^{K+L}(y)(x - y)^L / L!.$$

$Q^K(x_i^s) = q_i^{s,K}$ is flat along $|x_i^s|$ and along $|x_i^s - x_i^t|$. Passing to a subsequence if necessary (and renumbering so the subsequence is also labeled x_i), we may assume that $|x_i^s| \leqq 2|x_i^s - x_j^t|$ for all s and t and $j > i$, and that $|q_i^{s,K}|$ is a decreasing function of i for each s and K, $|K| \leqq m$. For each l, there is an N such that, for all $j > i \geqq N$ and all K, $|K| \leqq m$,

$$\text{(ii)} \quad |q_j^{s,K}| \leqq |q_i^{s,K}| \leqq |x_i^s - x_i^t|^l \quad \text{for all } s \neq t,$$

and

$$\text{(iii)} \quad |q_j^{s,K}| \leqq |q_i^{s,K}| \leqq |x_i^s|^l \leqq 2^l |x_i^s - x_j^t|^l \quad \text{for all } s, t.$$

For all x and y in $\{x_i^s\colon \text{all } s \text{ and all } i \geqq N\} \cup \{0\}$, $|Q^K(x)|$ and $|Q^{K+L}(y)|$ are, by (ii) and (iii), no greater than $2^l |x - y|^l$. Thus

$$|(R_y^m Q)^K(x)| \leqq C|x - y|^l,$$

where C depends only on m and l. Let $l = m + 1$. By Whitney's Extension Theorem (IV.3.1 of [**11**]), there is a $C^\infty q$ with $(j^\infty q)^r(x_i) = q_i$ and $j^\infty q(\mathbf{0}) = \mathbf{0}$. Let $g = f + q$.

Now we proceed to the proof that (1) implies (3). We assume (3) does not hold for f. We will prove that two representatives of Tf exist which cannot be G-equivalent.

Case C. First note that if two germs are C-equivalent, then their zero sets are the same. Suppose f is identically zero; we can certainly find a g with $Tg = 0$ which is not identically zero.

Suppose f isn't identically zero. We are given that f doesn't satisfy a Lojasiewicz inequality at 0. Then there is a sequence x_i converging to 0 so that $f(x_i)$ is flat along x_i, but $f(x_i) \neq 0$. By Lemma 3.3, there is a g with $Tg = Tf$ such that $g(x_i) = 0$.

Case R. If two germs are R-equivalent, then their critical values are identical. By assumption, $d(Uns_R, j^1 f)$ doesn't satisfy a Lojasiewicz inequality at 0. Thus there is a sequence x_i converging to 0 with $d(Uns_R, j^1 f(x_i))$ flat along x_i. By Sard's Theorem, we can find a sequence y_i converging to 0, y_i not a critical value of f, such that $f(x_i) - y_i$ is flat along x_i. By Lemma 3.3, there is a g with $Tg = Tf$ such that $y_i = g(x_i)$ is a critical value of g.

Case K. If two germs are K-equivalent, their sets of critical zeros are diffeomorphic. Note Uns_K is the union of the sets $0 \times S_r$, where S_r is the set of 1-jets of rank r, $r < p$. By the Transversal Extension Theorem, there is an h with $Th = Tf$ and $j^1 h \pitchfork 0 \times S_r$ on a deleted neighborhood of 0 for all $r < p$. Since cod $(Uns_K) = n + 1$, h has no critical zeros on a deleted neighborhood of 0. Since $d(Uns_K, j^1 f(x_i))$ is flat along some sequence x_i converging to 0, there is a g with $Tg = Tf$ and x_i critical zeros of g. Thus g and h aren't K-equivalent.

Case L. If f and g are L-equivalent, then their nonimmersion sets are identical as are their double point pairs (i.e., $f(x_1) = f(x_2)$ if and only if $g(x_1) = g(x_2)$). Uns_L in $\mathbf{R}^{2p} \times J^1(n, p)^2$ is the union of V_1 (the set of those pairs of jets, at least one of which has rank less than n) and V_2 ($= \Delta_p \times J^1(n, p)^2$, where Δ_p is the diagonal in $\mathbf{R}^p \times \mathbf{R}^p$). There is a sequence $x_i = (x_i^1, x_i^2)$ converging to 0 with $d(z_i, Uns_L)$, $z_i = (z_i^1, z_i^2) = (j^1 f(x_i^1), j^1 f(x_i^2))$, flat along $d(x_i, D)$. Recall that $D = (0 \times \mathbf{R}^n) \cup (\mathbf{R}^n \times 0) \cup \Delta_n$.

Note that

$$d(x_i, D) = \min (|x_i^1|, |x_i^2|, d(x_i, \Delta_n)).$$

The closest point in Δ_n to x_i is (w_i, w_i), where $w_i = (x_i^1 + x_i^2)/2$, so

$$d(x_i, \Delta_n) = |x_i^1 - x_i^2|/2^{1/2}.$$

Since V_1 and V_2 are algebraic varieties, they are regularly situated. Thus, passing to a subsequence (and renumbering) if necessary, $d(z_i, V_1)$ or $d(z_i, V_2)$ is flat along $d(x_i, D)$. Thus either

(a) $d(z_i, V_1)$ is flat along $|x_i^1|$ and $|x_i^2|$ or

(b) $|f(x_i^1) - f(x_i^2)|$ is flat along $|x_i^1|$, $|x_i^2|$ and $|x_i^1 - x_i^2|$.

Assume (a) holds. Without loss of generality we assume $d(z_i^1, S)$ is flat along $|x_i^1|$, where S is the set of 1-jets of maps of rank less than n. Suppose $n \leqq p$. There is an h with $Th = Tf$ and j^1h transverse on a deleted neighborhood U of 0 to the stratification of J^1 by rank. Either (i) the nonimmersion set of h is nonempty but nowhere dense in U, or (ii) the nonimmersion set is empty in U.

If (i), then we can find a sequence u_i in U converging to 0 such that $d(j^1h(u_i), S)$ is flat along u_i and such that h is an immersion at u_i. By Lemma 3.3, there is a g with $Tg = Th$ such that g is not an immersion at u_i. Thus g and h are not L-equivalent. Suppose (ii). There is a g such that $Tg = Tf$ and g is not an immersion at x_i^1. Thus g and h are not L-equivalent.

Suppose $n > p$. The case f identically zero is trivial. If f is not identically zero, then there are points arbitrarily near 0 with at least two preimages under f. Thus we are in case (b).

Assume (b) holds. Then there is a sequence (x_i^1, x_i^2) not in D converging to 0 such that $|f(x_i^1) - f(x_i^2)|$ is flat along $|x_i^1|$, $|x_i^2|$ and $|x_i^1 - x_i^2|$. We can pick x_i^3 arbitrarily near x_i^2 so that $f(x_i^1)$ is unequal to $f(x_i^3)$ and $|f(x_i^1) - f(x_i^3)|$ is flat along $|x_i^1|$, $|x_i^3|$ and $|x_i^1 - x_i^3|$. Then by Lemma 3.3 there is a g with $Tg = Tf$ such that $g(x_i^1) = g(x_i^3)$. Thus g is not L-equivalent to f.

4. Proof that (3) implies (2).

Cases K, R, and C. For the case K, apply Lemma 2.1 to $f(x)$ in place of $g(x, t)$. As we stated before, similar lemmas hold for R and C.

Case L. Assume (3) holds, i.e.:

(i) $d(j^kf(x), S) \geqq C|x|^r$ for some $C, r > 0$ (where S is the set of k-jets of maps of rank $< n$), and

(ii) $|f(x) - f(y)| \geqq K|x - y|^l$ for some $K, l > 0$.

We are to establish (2), which is equivalent to $f^*m_p \supset m_n^\infty$. Thus we are given $a \in m_n^\infty$. Since $M = \operatorname{im}(f) - \{0\}$ is a manifold, there is a germ along M of a submersion q onto M whose fibers are the normal spaces $N_{f(x)}$ to $TM_{f(x)}$. We let $g = f^{-1} \circ q$ and let b be the Taylor field on $\operatorname{im}(f)$ defined by the Taylor series of $a \circ g$ on M and the zero Taylor series at 0. We will prove that b is a Whitney field on $\operatorname{im}(f)$, hence has a smooth extension $b' \in m_p^\infty$ such that $b' \circ f = a$.

Clearly b is a Whitney field on M in $\mathbf{R}^p - \{0\}$. We will prove

(iii) for each multi-index I, $(D^Ib) \circ f$ is flat,

which means that each derivative of $(D^Ib) \circ f$ goes to zero as x goes to zero. It follows from Hestenes' Lemma (see [**11**]) that $(D^Ib) \circ f$ is in E_n.

Then, to show b is a Whitney field one mimics Tougeron's proof of Glaeser's Theorem (see particularly p. 181 of [**11**]); in that proof, replace Lemma 1.5 by (iii) and Lemma 1.6 by (ii).

In order to establish (iii) we construct maps F: $\mathbf{R}^n \times \mathbf{R}^{p-n} \rightarrow \mathbf{R}^p$ so that $g \circ F = p$, the projection of $\mathbf{R}^n \times \mathbf{R}^{p-n}$ onto $\mathbf{R}^n$. To do this, we choose a basis $N_1(x), \ldots, N_{p-n}(x)$ for $N_{f(x)}$ and let

$$F(x, t) = f(x) + t_1N_1(x) + \ldots + t_{p-n}N_{p-n}(x).$$

Since it may not be possible to choose the N_i smoothly for all x in $\mathbf{R}^n - \{0\}$, we must restrict ourselves to certain open sets as follows. Given a fixed orthonormal basis of $\mathbf{R}^p$, choose n of these basis vectors and label them $e_1, \ldots, e_n$, and label the rest $k_1, \ldots, k_{p-n}$. Let I^k be the set of k-jets of immersions, and let O^k be the set of k-jets of those immersions with image transverse to K, the span of $k_1, \ldots, k_{p-n}$. $J^k - O^k$ is an algebraic variety.

Apply the Gram-Schmidt algorithm to $\partial f/\partial x_1, \ldots, \partial f/\partial x_n, k_1, \ldots, \mathrm{k}_{p-n}$ to produce an orthogonal basis; the last $p - n$ vectors, which we label $N_1, \ldots, N_{p-n}$, form a basis for $N_{f(x)}$. (N_i is the projection of k_i onto the normal space to the span of $TM_{f(x)}$ and $N_1, \ldots, N_{i-1}$.) Then $(N_1, \ldots, N_{p-n}) = \Phi(j^1f)$, where Φ is a rational map on J^1, regular on O^1 (which means that the denominators of the rational functions which define Φ do not vanish on O^1). By differentiating the above equation, we get a rational map Φ^k on J^{k+1}, regular on O^{k+1}, assigning to $j^{k+1}f(x)$ the k-jet of $N_1(x), \ldots, N_{p-n}(x)$.

Define F as above. There is a rational map Ψ on J^k, regular on O^k, such that

$$\Psi(j^kf(x)) = j^kF(x, 0).$$

Thus there is a rational map Λ on J^k, regular on O^k, such that

$$\Lambda(j^kf(x)) = j^k(p \circ F^{-1})(f(x)) = j^kg(f(x)).$$

By varying our choice of $k_1, \ldots, k_{p-n}$ from the given basis of $\mathbf{R}^p$, we get a cover $O_1, \ldots, O_s$ of I and rational maps Λ_i regular on O_i. $\Lambda_i = \Lambda_j$ on $O_i \cap O_j$, so we can piece together the Λ_i to give a map Λ regular on I so that the component functions of Λ are flat multipliers at $S = J^k(n, p) - I$ (see p. 80 of [**11**] for the theory of flat multipliers).

Since $(j^kg)(f(x)) = \Lambda(j^kf(x))$ for all nonzero x and $d(j^kf(x), S) \geqq C|x|^r$, it follows that the components of $(j^kg)(f(x))$ are flat multipliers at 0 in $\mathbf{R}^n$. For each multi-index I, $(D^Ib) \circ f = (D^I(a \circ g)) \circ f$ is a sum of terms, each a product of flat functions and flat multipliers, hence is flat. This proves (iii) and Theorem 1.2.

An important theorem of Glaeser (see [**11**]) states that if f from

$U \subset \mathbf{R}^n$ to $V \subset \mathbf{R}^p$ is an analytic map satisfying certain conditions, then $f^*E(V)$ is a closed subring of $E(U)$ in the weak C^∞ topology (of uniform convergence of derivatives on compact sets). One of the requirements is that n be at least p. Little is known in the case $n < p$. However, we have:

COROLLARY 4.1. *Suppose f in E_n^p is an ∞-L-determined germ. Then f has a representative F: $U \rightarrow V$ such that $F^*E(V)$ is a closed subring of $E(U)$ in the weak C^∞ topology. U can be chosen arbitrarily small.*

Proof. Since f is a proper germ, it has a proper representative F. For any neighborhood W of 0 we can find $U \subset W$ and V such that $F^{-1}V = U$. From the proof of "(3) implies (2)" for $G = L$, we see that U and V can be chosen so that $F^*m_0^\infty \supset m_0^\infty$ (m_0^∞ consists of functions in $E(U)$ or $E(V)$ which are flat at 0) and so that condition (ii) of this section holds on all of U.

Condition (ii) implies that F satisfies condition (H) of Theorem 1.1 of [**12**]. The implication of that theorem is that $\overline{F^*E(V)} = \widehat{F^*E(V)}$, where the latter is defined to be $\{a \in E(U)$: for each y in V there exists b in $E(V)$ such that $b \circ F - a$ is flat on $F^{-1}(y)\}$. In our case, this implies

$$\overline{F^*E(V)} \subseteq F^*E(V) + m_0^\infty \subseteq F^*E(V).$$

5. Some other results.

PROPOSITION 5.1. (a) *f is ∞-G-determined if and only if $E_n/I_G(f)$ is Noetherian ($G = R$, C, or K).* (b) *Suppose f is finitely C-determined. Then f is ∞-L-determined if and only if E_n/f^*E_p is a Noetherian $(Tf)^*F_p$ module (F_p the formal power series in p variables).*

Proof. The proof of (a) is essentially the same as that given in [**10**] for the real valued case, with $G = R$. The assumption that f is finitely C-determined gives that F_n is a finite $(Tf)^*F_p$ module. If $f^*E_p \supset m_n^\infty$, then

$$E_n/f^*E_p = (E_n/m_n^\infty)/(f^*E_p/m_n^\infty) = F_n/(Tf)^*F_p$$

which is a finite $(Tf)^*F_p$ module, hence Noetherian. Conversely, for E_n/f^*E_p to be a $(Tf)^*F_p$ module at all, f^*E_p must contain m_n^∞.

PROPOSITION 5.2. *f is ∞-K-determined if and only if f is finitely v-sufficient.*

Proof. Part of Theorem 1 in [**7**] states that an r-jet z is v-sufficient if and only if the following condition holds:

(C^{r+1}) For any C^{r+1} representative f of z, f is a submersion at each point of $f^{-1}(0)$ in a deleted neighborhood of 0.

If f is not ∞-K-determined, then (by the proof of "(1) implies (3)" for the case K) there is a g with $Tg = Tf$ such that g has critical zeros arbitrarily close to 0. Thus $j^r f$ fails to satisfy (C^{r+1}) for any r. Thus f is not finitely v-determined.

Suppose f is ∞-K-determined. Let $J_1, \ldots, J_s$ denote the generators of Jf. Then $F = (f, J_1, \ldots, J_s)$ satisfies a Lojasiewicz inequality of some order k at 0. Let $z = j^{k+1}f(0)$. Let g be any C^{k+2} representative of z, with G defined as F is. Then each component function $G_i - F_i$ is in $O(|x|^{k+1})$, so G also satisfies a Lojasiewicz inequality of order k at 0. Thus g is a submersion at each point of $g^{-1}(0)$ in a deleted neighborhood of 0. Thus z is v-sufficient.

Recall that an analytic germ is *finitely K-determined* if and only if its complexification is finite-to-one on its critical set or, equivalently, the complexification is proper on its critical set. Thus a smooth, finitely K-determined germ is finite-to-one and proper on its critical set. What can be said about ∞-K-determined germs?

Definition 5.3. A germ is *finite-to-one* if it has a finite-to-one representative. A germ (at x) is *proper* if, for every representative f of the germ, there are open neighborhoods U of x and V of $f(x)$ such that $f\,|U: U \to V$ is proper.

LEMMA 5.4. *The germ at x of a map f is proper if, and only if, for every neighborhood U of x there exist neighborhoods $U_1 \subset U$ of x and V of y such that $(f|U_1)^{-1}(V)$ is contained in a compact subset of U_1.*

Proof. Assume f is proper at x. Then there are neighborhoods U_1 of x and V_1 of y such that $U_1 \subset U$ and $(f|U_1): U_1 \to V_1$ is proper. There is a $V \subset V_1$ such that $\bar{V}$ is compact and contained in V_1 (by local compactness). Then $(f|U_1)^{-1}(\bar{V})$ is a compact subset of U_1.

Now suppose we are given $U_2 = (f|U_1)^{-1}(V)$ contained in a compact subset K of U_1. Then $f\,|U_2: U_2 \to V$ is proper.

LEMMA 5.5. *Suppose S is a closed subset of an open set U in $\mathbf{R}^n$, $f: S \to \mathbf{R}^p$ is continuous, and $f^{-1}(0) = 0$. Then the germ of f at 0 is proper.*

Proof. There is an $r > 0$ such that the disk $\bar{D}(0, r)$ is contained in U. $A = \bar{D}(0, r) - D(0, r/2)$ is compact, so $B = A \cap S$ is compact. $f(B)$ is compact and misses 0, hence misses an open neighborhood V of 0. Thus $(f|D(0, r) \cap S)^{-1}(V)$ is contained in $\bar{D}(0, r/2)$. By the preceeding Lemma, f is proper at 0.

Let C denote the critical set of f. If f is ∞-K-determined, then $f^{-1}(0) \cap C = 0$ and, if f is analytic, the converse holds. By Lemma 5.5, $f\,|C$ is proper. Conversely, if f is analytic and $f\,|C$ is proper, then f is ∞-K-determined: for if the dimension of $f^{-1}(0) \cap C$ is greater than 0, then for every small, open U in C, $(f|U)^{-1}(0)$ is noncompact.

Example 5.6. $f(x, y) = (x^2 + y^2, 0)$ is ∞-K-determined at 0. It is not finite-to-one, and it is not proper at any point except 0. Note this shows ∞-K-determination is not an open property.

Similarly, an analytic f is ∞-C-determined if and only if $f^{-1}(0) = 0$ or, equivalently, f is a proper germ. Every smooth ∞-C-determined germ f has $f^{-1}(0) = 0$ and is proper.

Finally we prove a result which is used in [**14**].

PROPOSITION 5.7. *If f is ∞-K-determined and $f^{-1}(0) = 0$, then f is ∞-C-determined.*

Proof. By Corollary V.5.7 of [**11**], $(f^*m_p)E_n$ is closed. By Corollary V.4.4 of [**11**], $(f^*m_p)E_n$ satisfies a Lojasiewicz inequality at its zero set, which is 0. Thus, by our main theorem, f is ∞-C-determined.

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