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THE MEAN CURVATURE EQUATION ON SEMIDIRECT PRODUCTS $\mathbb{R}^2 \rtimes_A \mathbb{R}$: HEIGHT ESTIMATES AND SCHERK-LIKE GRAPHS

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Abstract

In the ambient space of a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, we consider a connected domain $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$. Given a function $u : \Omega \to \mathbb{R}$, its π-graph is $\mathrm{graph}(u) = \{(x, y, u(x, y)) \mid (x, y, 0) \in \Omega\}$. In this paper we study the partial differential equation that u must satisfy so that graph(u) has prescribed mean curvature H. Using techniques from quasilinear elliptic equations we prove that if a π-graph has a nonnegative mean curvature function, then it satisfies some uniform height estimates that depend on Ω and on the supremum the function attains on the boundary of Ω. When trace$(A) > 0$, we prove that the oscillation of a minimal graph, assuming the same constant value n along the boundary, tends to zero when $n \to +\infty$ and goes to $+\infty$ if $n \to -\infty$. Furthermore, we use these estimates, allied with techniques from Killing graphs, to prove the existence of minimal π-graphs assuming the value zero along a piecewise smooth curve γ with endpoints p_1, p_2 and having as boundary $\gamma \cup (\{p_1\} \times [0, +\infty)) \cup (\{p_2\} \times [0, +\infty))$.

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1. Introduction

Let $A \in M_2(\mathbb{R})$ be a two-by-two matrix. The semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is, as a set, the Euclidean 3-space $\mathbb{R}^3$, but endowed with a group operation and with a left-invariant metric that come from the exponential map $z \mapsto e^{Az}$. More details about its construction are given in Section 2, below. Also, the work of Meeks III and Pérez [10] is a good reference on the subject, and provides the basic aspects of the geometry in these spaces.

There are two main difficulties when dealing with minimal π-graphs in semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$. The first is that vertical translations $(x, y, z) \mapsto (x, y, z + t)$ are not isometries of the ambient space. In particular, this affects the mean curvature operator so that the coefficients of its second-order terms depend on the solution, and the

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comparison principle (for example [5, Theorem 10.1] and its generalisations) does not apply. The second is that, unless $\operatorname{trace}(A) = 0$, constant functions do not provide minimal graphs, so there is no maximum principle, in the sense that the supremum (or infimum) of a solution to the minimal graph equation may be strict and attained in the interior of the domain.

In this paper, we consider the convex domain $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ with piecewise smooth boundary and exhibit the partial differential equation (PDE) that a function $u : \Omega \to \mathbb{R}$ must satisfy for its π-graph

$$\operatorname{graph}(u) = \{(x, y, u(x, y)) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid (x, y, 0) \in \Omega\}$$

to have prescribed mean curvature function. Such a PDE has different behaviours that depend on the trace and on the determinant of A. For example, when $\operatorname{trace}(A) = 0$, if u is such that $\operatorname{graph}(u)$ has nonnegative mean curvature $H \geq 0$ with respect to the upwards orientation, then it satisfies the maximum principle

$$\sup_{\partial\Omega} u = \sup_{\Omega} u. \tag{1.1}$$

This property was first observed by Meeks III *et al.* in [9] (we state this result as Lemma 3.1, below), and we remark that (1.1) does not hold when $\operatorname{trace}(A) > 0$, even in the stronger case $H \equiv 0$: a minimal graph that is constant along its boundary necessarily assumes an interior maximum and it is not constant, as horizontal planes (representing constant functions) are no longer minimal. It becomes a natural question to ask if there is a *maximal oscillation* that these minimal graphs that are constant along the boundary can attain, and this question is answered in this paper via height estimates of PDEs.

Let us describe some of the main results of this paper. In Section 3, given $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ and a parameter $\alpha \in \mathbb{R}$, we obtain, in Theorem 3.2, a constant $C(\alpha) = C(\operatorname{diam}(\Omega), \alpha)$ such that if $u : \Omega \to \mathbb{R}$ is a function where $\operatorname{graph}(u)$ has nonnegative mean curvature, then

$$\sup_{\partial\Omega} u \leq \alpha \Rightarrow \sup_{\Omega} u \leq \sup_{\partial\Omega} u + C(\alpha). \tag{1.2}$$

Still in Section 3, we prove that the dependence of α in (1.2) is essential (Theorem 3.3) for the validity of the result, in the sense that it is not possible to obtain some constant $C = C(\Omega)$ such that every $u : \Omega \to \mathbb{R}$, where $\operatorname{graph}(u)$ has nonnegative mean curvature satisfies the uniform height estimate

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C.$$

We also use, in Theorem 3.5, the freedom of the parameter α to obtain that, when $\operatorname{trace}(A) > 0$, the oscillation of a family of solutions to the problem

$$\begin{cases} \operatorname{graph}(u) \text{ is a minimal surface of } \mathbb{R}^2 \rtimes_A \mathbb{R} \\ u|_{\partial\Omega} = \alpha \in \mathbb{R} \end{cases}$$

converges to zero when α approaches $+\infty$. Moreover, we prove it goes to $+\infty$, if $\alpha \to -\infty$.

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We finish the paper in Section 4, where we bring techniques from Killing graphs, in addition to the estimates on the coefficients of the mean curvature operator obtained on Section 3, to generalise an argument of Menezes [11] to any semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, and obtain the existence of minimal π-graphs which are similar to the fundamental piece of the doubly periodic Scherk surface of $\mathbb{R}^3$, in Theorem 4.1.

2. Semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$

This section gives a brief review of semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$. We follow the notation and construction of Meeks III and Pérez [10].

Let H, V to be two groups and let $\varphi : V \to \mathrm{Aut}(H)$ be a group homomorphism between V and the group of automorphisms of H. Then the *semidirect product between H and V with respect to φ*, denoted by $G = H \rtimes_\varphi V$, is the Cartesian product $H \times V$ endowed with the group operation $* : G \times G \to G$ given by

$$(h_1, v_1) * (h_2, v_2) = (h_1 \cdot \varphi_{v_1}(h_2), v_1 v_2).$$

With this group operation, both H and V can be viewed as subgroups of G and $H \lhd G$ is identified as a normal subgroup of G. This construction generalises the notion of direct products of groups, where the operation on the Cartesian product $H \times V$ would be the product operation $(h_1, v_1) * (h_2, v_2) = (h_1 h_2, v_1 v_2)$.

Even in the particular case of $H = \mathbb{R}^2$ and $V = \mathbb{R}$ being two abelian groups, it is possible to obtain a great variety of groups via the semidirect product of $\mathbb{R}^2$ and $\mathbb{R}$, depending uniquely on the choice of the (now one-parameter) family of automorphisms of $\mathbb{R}^2$. Precisely, with the exceptions of $SU(2)$ (which is not diffeomorphic to $\mathbb{R}^3$) and $\widetilde{PSL}(2, \mathbb{R})$ (which has no normal subgroup of dimension two), it is possible to construct all three-dimensional simply connected Lie groups using the following setting. Fix a matrix $A \in M_2(\mathbb{R})$,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and consider φ the one-parameter subgroup of automorphisms of $\mathbb{R}^2$ generated by the exponential map

$$\begin{array}{rcl} \varphi : \mathbb{R} & \to & \mathrm{Aut}(\mathbb{R}^2), \\ z & \mapsto & e^{Az} : \mathbb{R}^2 \to \mathbb{R}^2. \end{array}$$

Then $\mathbb{R}^2 \rtimes_A \mathbb{R} = \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ is the semidirect product between $\mathbb{R}^2$ and $\mathbb{R}$ with respect to φ: that is, the set $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ endowed with the group operation $*$ defined via

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + e^{Az_1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_1 + z_2 \right).$$

Using the notation of [10], denote the exponential map e^{Az} by

$$e^{Az} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix},$$

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and observe that the vector fields defined by

$$E_1(x, y, z) = a_{11}(z)\partial_x + a_{21}(z)\partial_y, \quad E_2(x, y, z) = a_{12}(z)\partial_x + a_{22}(z)\partial_y, \quad E_3 = \partial_z$$

are left-invariant and extend the canonical basis $\{\partial_x(0), \partial_y(0), \partial_z(0)\}$ at the origin of $\mathbb{R}^3$. Moreover, if we let

$$F_1 = \partial_x, \quad F_2 = \partial_y, \quad F_3(x, y, z) = (ax + by)\partial_x + (cx + dy)\partial_y + \partial_z,$$

it follows that each F_i is a right-invariant vector field of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, so they are Killing fields with respect to any left-invariant metric of $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

The metric to be considered on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is the *canonical left-invariant metric*, that is, the one given by stating that $\{E_1, E_2, E_3\}$ are unitary and orthogonal to each other everywhere. In particular, since

$$\partial_x(x, y, z) = a_{11}(-z)E_1 + a_{21}(-z)E_2,$$
$$\partial_y(x, y, z) = a_{12}(-z)E_1 + a_{22}(-z)E_2,$$

we can express the metric of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ in coordinates as

$$\begin{aligned} ds^2 = {} & [a_{11}(-z)^2 + a_{21}(-z)^2]\, dx^2 + [a_{12}(-z)^2 + a_{22}(-z)^2]\, dy^2 + dz^2 \\ & + [a_{11}(-z)a_{12}(-z) + a_{21}(-z)a_{22}(-z)](dx \otimes dy + dy \otimes dx). \end{aligned}$$

Note that, as $e^{-Az} = (e^{Az})^{-1}$ and $\det(e^{Az}) = e^{z\operatorname{trace}(A)}$,

$$\begin{pmatrix} a_{11}(-z) & a_{12}(-z) \\ a_{21}(-z) & a_{22}(-z) \end{pmatrix} = e^{-z\operatorname{trace}(A)} \begin{pmatrix} a_{22}(z) & -a_{12}(z) \\ -a_{21}(z) & a_{11}(z) \end{pmatrix},$$

and we can introduce the notation

$$\begin{aligned} Q_{11}(z) &= \langle \partial_x, \partial_x \rangle = e^{-2z\operatorname{trace}(A)}[a_{21}(z)^2 + a_{22}(z)^2], \\ Q_{22}(z) &= \langle \partial_y, \partial_y \rangle = e^{-2z\operatorname{trace}(A)}[a_{11}(z)^2 + a_{12}(z)^2], \\ Q_{12}(z) &= \langle \partial_x, \partial_y \rangle = -e^{-2z\operatorname{trace}(A)}[a_{11}(z)a_{21}(z) + a_{12}(z)a_{22}(z)] \end{aligned} \tag{2.1}$$

to obtain that the metric ds^2 is expressed by

$$ds^2 = Q_{11}(z)\, dx^2 + Q_{22}(z)\, dy^2 + dz^2 + Q_{12}(z)(dx \otimes dy + dy \otimes dx).$$

If $A, B \in M_2(\mathbb{R})$ are two congruent matrices, i.e., if there is some orthogonal matrix $P \in O(2)$ such that $B = PAP^{-1}$, then the groups $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and $\mathbb{R}^2 \rtimes_B \mathbb{R}$, endowed with their respective canonical left-invariant metrics, are *isomorphic* and *isometric*, and the map that makes such identification is a simple rotation on horizontal planes induced by P: that is,

$$\begin{aligned} \psi : \mathbb{R}^2 \rtimes_A \mathbb{R} &\to \mathbb{R}^2 \rtimes_B \mathbb{R} \\ (x, y, z) &\mapsto (P(x, y), z). \end{aligned} \tag{2.2}$$

The Lie brackets of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ are given by

$$[E_1, E_2] = 0, \quad [E_3, E_1] = aE_1 + cE_2, \quad [E_3, E_2] = bE_1 + dE_2,$$

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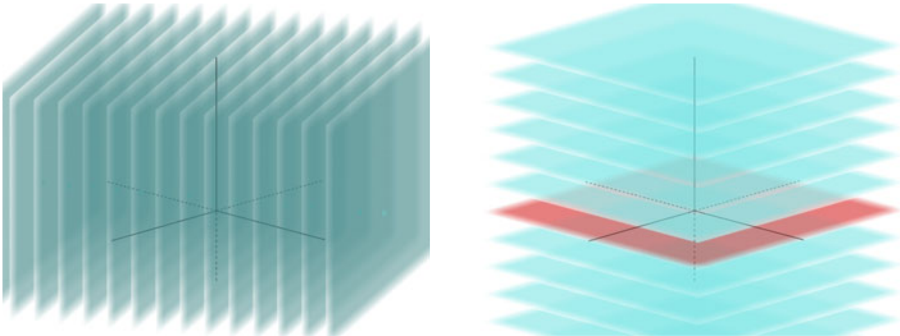

(a) (b)

FIGURE 1. On semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$, every *vertical* plane is a minimal surface. Horizontal planes are flat, have constant mean curvature $H = \mathrm{trace}(A)/2$ and the subgroup $\mathbb{H} = \mathbb{R}^2 \rtimes_A \{0\}$ (highlighted in the above right picture) is normal in $\mathbb{R}^2 \rtimes_A \mathbb{R}$. (a) A foliation of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by vertical (minimal) planes. (b) The foliation of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by horizontal (constant mean curvature) planes.

so the Levi-Civita equation implies that the Riemannian connection of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ satisfies

$$\begin{array}{l|l|l} \nabla_{E_1}E_1 = aE_3 & \nabla_{E_1}E_2 = \frac{b+c}{2}E_3 & \nabla_{E_1}E_3 = -aE_1 - \frac{b+c}{2}E_2 \\ \nabla_{E_2}E_1 = \frac{b+c}{2}E_3 & \nabla_{E_2}E_2 = dE_3 & \nabla_{E_2}E_3 = -\frac{b+c}{2}E_1 - dE_2 \\ \nabla_{E_3}E_1 = \frac{c-b}{2}E_2 & \nabla_{E_3}E_2 = \frac{b-c}{2}E_1 & \nabla_{E_3}E_3 = 0. \end{array} \tag{2.3}$$

We notice two important properties of planes in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ (see Figure 1). First, we observe that the metric ds^2 is invariant by rotations of angle π around the vertical lines $\{(x_0, y_0, z) \mid z \in \mathbb{R}\}$, and hence vertical planes are minimal surfaces of $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Moreover, horizontal planes $\{z = c\}$ have E_3 as a unitary normal vector field, so they have constant mean curvature (with respect to the upward orientation) given by $H = \mathrm{trace}(A)/2$. In particular, horizontal planes of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ are minimal if and only if $\mathrm{trace}(A) = 0$.

However, the difference between the cases $\mathrm{trace}(A) = 0$ and $\mathrm{trace}(A) \neq 0$ goes further than horizontal planes being minimal: concerning the classification of simply connected Lie groups of dimension three, we notice that Meeks III and Pérez [10] proved that any *nonunimodular* (a group G is said to be unimodular if $\det(\mathrm{Ad}_g) = 1$ for all $g \in G$) Lie group of dimension three is isomorphic and isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, endowed with its left-invariant metric, where $A \in M_2(\mathbb{R})$ is such that $\mathrm{trace}(A) \neq 0$ [10, Lemma 2.11]. Moreover, they also proved that, with the exceptions of $SU(2)$ and $\widetilde{PSL}(2, \mathbb{R})$, all other unimodular metric Lie groups are isomorphic and isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, with $\mathrm{trace}(A) = 0$ [10, Section 2.6 and Theorem 2.15]. Herein, we refer to the cases $\mathrm{trace}(A) = 0$ or $\mathrm{trace}(A) \neq 0$, respectively, as the *unimodular* and *nonunimodular* case.

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3. Mean curvature equation and height estimates

In this section, we consider a smooth open domain $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ and a function $u : \Omega \to \mathbb{R}$. The π-graph of u is

$$\Sigma = \mathrm{graph}(u) = \{(x, y, u(x,y)) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid (x, y, 0) \in \Omega\}.$$

When oriented with respect to the upwards direction, the mean curvature of Σ is

$$H = \frac{e^{2u\mathrm{trace}(A)}}{2W^3}[u_{xx}(Q_{22}(u) + u_y^2) + u_{yy}(Q_{11}(u) + u_x^2) - 2u_{xy}(Q_{12}(u) + u_x u_y) \\ + G_1(u)u_x^2 + G_2(u)u_y^2 + G_3(u)u_x u_y + (a + d)e^{-2u\mathrm{trace}(A)}], \tag{3.1}$$

where $Q_{ij} : \mathbb{R} \to \mathbb{R}$ are the coefficients of the metric of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, defined in (2.1), $G_i : \mathbb{R} \to \mathbb{R}$ are the functions given by

$$\begin{aligned} G_1(z) &= e^{-2z\mathrm{trace}(A)}((2a + d)a_{11}(z)^2 + (a + 2d)a_{12}(z)^2 + (b + c)a_{11}(z)a_{12}(z)), \\ G_2(z) &= e^{-2z\mathrm{trace}(A)}((2a + d)a_{21}(z)^2 + (a + 2d)a_{22}(z)^2 + (b + c)a_{21}(z)a_{22}(z)), \\ G_3(z) &= e^{-2z\mathrm{trace}(A)}((4a + 2d)a_{11}(z)a_{21}(z) + (2a + 4d)a_{12}(z)a_{22}(z) \\ &\qquad + (b + c)(a_{11}(z)a_{22}(z) + a_{12}(z)a_{21}(z))), \end{aligned}$$

and W is

$$\begin{aligned} W(z, p) &= \sqrt{1 + (a_{11}(z)p_1 + a_{21}(z)p_2)^2 + (a_{12}(z)p_1 + a_{22}(z)p_2)^2} \\ &= \sqrt{1 + e^{2z\mathrm{trace}(A)}(Q_{22}(z)p_1^2 - 2Q_{12}(z)p_1 p_2 + Q_{11}(z)p_2^2)}. \end{aligned}$$

Following the above notation, we define the *mean curvature operator* by

$$Q(u) = u_{xx}(Q_{22}(u) + u_y^2) + u_{yy}(Q_{11}(u) + u_x^2) + 2u_{xy}(Q_{12}(u) - u_x u_y) \\ + G_1(u)u_x^2 + G_2(u)u_y^2 + G_3(u)u_x u_y + (a + d)e^{-2u\mathrm{trace}(A)}, \tag{3.2}$$

so graph(u) is a minimal surface of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ if and only if u satisfies $Q(u) = 0$ in Ω.

Note that Q is a quasilinear elliptic operator, as the matrix

$$Q(z, p) = \begin{pmatrix} Q_{22}(z) + p_2^2 & Q_{12}(z) - p_1 p_2 \\ Q_{12}(z) - p_1 p_2 & Q_{11}(z) + p_1^2 \end{pmatrix}$$

is positive definite for every $z \in \mathbb{R}$ and $p = (p_1, p_2) \in \mathbb{R}^2$, which is easy to see using the relation

$$Q_{11}(z)Q_{22}(z) - Q_{12}(z)^2 = e^{-2z\mathrm{trace}(A)}.$$

In the papers of Meeks III *et al.* [7–9], some work has been done in order to understand constant mean curvature π-graphs: the fact that $\mathbb{R}^2 \rtimes_A \mathbb{R}$ admits a foliation by parallel horizontal planes of constant mean curvature $H = \mathrm{trace}(A)/2$ determines much of the structure of those graphs. For instance, using this property and the mean curvature comparison principle, they are able to prove the following lemma.

Lemma 3.1 [9, Assertion 15.5]. *Let $D \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ be a convex compact disc and let $C = \partial D$ be its boundary. Consider $\pi(x, y, z) = (x, y, 0)$, the vertical projection. If $\Gamma \subseteq \pi^{-1}(C)$ is a closed simple curve such that the projection $\pi : \Gamma \to C$ monotonically parametrises (this means that $\pi(\Gamma) \subset \partial\Omega$ and $\pi^{-1}(\{p\}) \cap \Gamma$ is either a single point or a compact interval for every $p \in \partial\Omega$) C and $h : \Gamma \to \mathbb{R}$ is the height function, let $c_0 = \inf_\Gamma h$ and $c_1 = \sup_\Gamma h$. If Σ is a compact minimal surface with $\partial\Sigma = \Gamma$, it follows that:*

(1) *if* $\operatorname{trace}(A) \geq 0$, *then* $\Sigma \subseteq \pi^{-1}(D) \cap \{z \geq c_0\}$; *and*
(2) *if* $\operatorname{trace}(A) \leq 0$, *then* $\Sigma \subseteq \pi^{-1}(D) \cap \{z \leq c_1\}$.

In the particular case of graphs, Lemma 3.1 implies that a minimal graph over some smooth domain $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$, which is compact and convex, satisfies the maximum principle if $\operatorname{trace}(A) \leq 0$, satisfies the minimum principle if $\operatorname{trace}(A) \geq 0$ and satisfies both in the unimodular case only. However, when $\operatorname{trace}(A) > 0$ no uniform upper bound is obtained, nor is a lower bound when $\operatorname{trace}(A) < 0$. This motivates the search for height estimates for minimal graphs, which is the next result. Perhaps, the proof of Theorem 3.2 is as interesting as the result itself, as it gives some information about the behaviour of the operator Q in the many possible settings for the matrix A. Such properties will be used in the proof of Theorem 3.5, and also in Section 4 to obtain the existence of minimal Killing graphs that converge to the Scherk-like fundamental piece of Theorem 4.1.

Theorem 3.2. *Let $A \in M_2(\mathbb{R})$ and let $\mathbb{R}^2 \rtimes_A \mathbb{R}$ be a semidirect product endowed with its canonical left-invariant metric. Let $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ be a bounded, convex domain and let $\alpha \in \mathbb{R}$ be any given constant. Then there exists a constant $C(\alpha) = C(\operatorname{diam}(\Omega), \alpha)$ such that, for every u satisfying $Q(u) \geq 0$ and $\sup_{\partial\Omega} u \leq \alpha$, it holds that*

$$\sup_\Omega u \leq \alpha + C(\alpha).$$

In particular, there is a constant C depending on $\operatorname{diam}(\Omega)$ and on $\sup_{\partial\Omega} u$ such that every $u : \Omega \to \mathbb{R}$ whose graph has nonnegative mean curvature function with respect to the upwards orientation satisfies

$$\sup_\Omega u \leq \sup_{\partial\Omega} u + C\Big(\sup_{\partial\Omega} u\Big).$$

The proof of Theorem 3.2 uses techniques from quasilinear elliptic PDEs, mainly the comparison principle. For instance, Theorem 10.1 of [5] gives us that if R is a quasilinear elliptic operator of the form

$$R(w) = \sum_{i,j=1}^{2} a_{ij}(x, \operatorname{grad}(w)) w_{x_i x_j} + b(x, w, \operatorname{grad}(w)), \tag{3.3}$$

for C^2 functions $w : \Omega \to \mathbb{R}$, where a_{ij} and b are smooth functions and b is such that for each $x \in \Omega$ and $p \in \mathbb{R}^2$ the function $z \mapsto b(x, z, p)$ is nonincreasing, then, given u, $v : \Omega \to \mathbb{R}$ such that $R(u) \geq R(v)$ in Ω and $u \leq v$ in $\partial\Omega$, then $u \leq v$ in Ω.

However, the operator Q given by (3.2) does not satisfy the hypothesis of such a comparison principle (or of its generalisations), as the coefficients of the second-order terms of $Q(u)$ *depend* on u. This happens because translations $(x, y, z) \mapsto (x, y, z + t)$ are not isometries of $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Hence, we are not able to prove uniqueness of solutions to the minimal graph equation and we also need to use a indirect approach to find the height estimates of Theorem 3.2.

In order to prove Theorem 3.2, we define a quasilinear operator R related to Q, for which the comparison principle holds. Then, we find an *ad hoc* positive function $v : \Omega \to \mathbb{R}$, whose construction will depend only on Ω and on α such that $R(v + \alpha) \le R(u)$. Then, as $u \le \alpha \le v + \alpha$ along $\partial\Omega$, it will follow that $u \le v + \alpha$ in Ω, and we can let $C(\alpha, \Omega)$ be given by $C = \sup_\Omega v$.

Proof of Theorem 3.2. First, we notice that, when $\operatorname{trace}(A) \le 0$, the result is trivial with $C = 0$ and without the need for an α, by Lemma 3.1. Thus we will suppose that $\operatorname{trace}(A) > 0$ and focus on the nonunimodular case. Without loss of generality, after a homothety of the metric, we may assume that $\operatorname{trace}(A) = 2$ and that A is written as

$$A = \begin{pmatrix} 1 + a & b \\ c & 1 - a \end{pmatrix}, \tag{3.4}$$

for some a, b, $c \in \mathbb{R}$. We divide the proof into two cases, starting when A is not a diagonal matrix.

Case 1. Suppose that A is not a diagonal matrix.

We begin by proving the following key claim, which will also be used in Section 4.

Claim 1. Let the functions Q_{ij} be the ones defined by (2.1) with respect to the matrix A of (3.4), where either $b \neq 0$ or $c \neq 0$. Then, there is some $\lambda > 0$ such that at least one of the following holds, for every $z \in \mathbb{R}$:

(i) $Q_{22}(z)e^{2z} > \lambda$;
(ii) $Q_{11}(z)e^{2z} > \lambda$.

Moreover, if $a^2 + bc \le 0$, both (i) and (ii) hold, and if $a^2 + bc > 0$, then $b \neq 0$ is equivalent to (i) and $c \neq 0$ is equivalent to (ii).

Proof of Claim 1. We prove Claim 1 in each of three possibilities to the exponential of A. First, write $A = I + A_0$, where I is the identity matrix and A_0 is the traceless part of A given by

$$A_0 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

Since I and A_0 commute, $e^{Az} = e^{Iz + A_0 z} = e^{Iz} e^{A_0 z}$, thus

$$e^{Az} = e^z \begin{pmatrix} a^0_{11}(z) & a^0_{12}(z) \\ a^0_{21}(z) & a^0_{22}(z) \end{pmatrix},$$

where we denote the coefficients of the exponential $e^{A_0 z}$ by $a^0_{ij}(z)$. Then $a_{ij}(z) = e^z a^0_{ij}(z)$, and it follows that

$$Q_{11}(z)e^{2z} = e^{-4z}[a_{21}(z)^2 + a_{22}(z)^2]e^{2z} = a^0_{21}(z)^2 + a^0_{22}(z)^2. \tag{3.5}$$

Analogously,

$$Q_{22}(z)e^{2z} = a^0_{11}(z)^2 + a^0_{12}(z)^2. \tag{3.6}$$

Note that the characteristic equation of A_0 is $0 = \det(A_0 - tI) = t^2 - (a^2 + bc)$ so, if we let $d = \sqrt{|a^2 + bc|}$, the exponential of A_0 is

$$e^{A_0 z} = \begin{pmatrix} \cos(dz) + \dfrac{a}{d}\sin(dz) & \dfrac{b}{d}\sin(dz) \\ \dfrac{c}{d}\sin(dz) & \cos(dz) - \dfrac{a}{d}\sin(dz) \end{pmatrix} \quad \text{when } a^2 + bc < 0, \tag{3.7}$$

$$e^{A_0 z} = \begin{pmatrix} 1 + az & bz \\ cz & 1 - az \end{pmatrix} \quad \text{when } a^2 + bc = 0, \tag{3.8}$$

$$e^{A_0 z} = \begin{pmatrix} \cosh(dz) + \dfrac{a}{d}\sinh(dz) & \dfrac{b}{d}\sinh(dz) \\ \dfrac{c}{d}\sinh(dz) & \cosh(dz) - \dfrac{a}{d}\sinh(dz) \end{pmatrix} \quad \text{when } a^2 + bc > 0. \tag{3.9}$$

We remark that the constant $a^2 + bc$ is linked with the Milnor D-invariant of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, which is defined by $D = \det(A) = 1 - (a^2 + bc)$. So each case $a^2 + bc > 0$, $a^2 + bc = 0$ and $a^2 + bc < 0$ is in correspondence with $D < 1$, $D = 1$ and $D > 1$, respectively.

Let $f(z) = a^0_{11}(z)^2 + a^0_{12}(z)^2$ and $g(z) = a^0_{21}(z)^2 + a^0_{22}(z)^2$. We will prove that there is some $\lambda > 0$ such that either $f(z) > \lambda$ or $g(z) > \lambda$, and this proves the claim, by (3.5) and (3.6).

Note that both f and g are always positive, since the existence of some $z_0 \in \mathbb{R}$ such that $f(z_0) = 0$ or $g(z_0) = 0$ would imply that $\det(e^{A_0 z_0}) = 0$, which is an absurdity. Hence, we just need to check the asymptotic behaviour of f and g.

If $a^2 + bc < 0$, the existence of λ, as claimed, follows directly from the fact that both f and g are periodic and positive, by (3.7). If $a^2 + bc = 0$, then f and g are respectively

$$f(z) = (1 + az)^2 + (bz)^2 = (a^2 + b^2)z^2 + 2az + 1,$$
$$g(z) = (1 - az)^2 + (cz)^2 = (a^2 + c^2)z^2 - 2az + 1,$$

which are both strictly positive at infinity for any choice of a, b, c, so we also have the existence of λ in this case. Finally, if $a^2 + bc > 0$, then f and g are given by

$$f(z) = \left(\cosh(dz) + \frac{a}{d}\sinh(dz)\right)^2 + \left(\frac{b}{d}\sinh(dz)\right)^2,$$
$$g(z) = \left(\cosh(dz) - \frac{a}{d}\sinh(dz)\right)^2 + \left(\frac{c}{d}\sinh(dz)\right)^2.$$

If (i) was not true, either $\lim_{z\to-\infty} f(z) = 0$ or $\lim_{z\to+\infty} f(z) = 0$, and hence $b = 0$. Also, if $\lim_{z\to-\infty} g(z) = 0$ or $\lim_{z\to+\infty} g(z) = 0$, we would have $c = 0$. This shows that

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if $b \neq 0$, then (i) holds and, if $c \neq 0$, then (ii) holds. As A is not a diagonal matrix, at least one of (i) and (ii) is true, which finishes the proof of the claim. □

To proceed with the proof of the first case of Theorem 3.2, we prove the existence of $\Lambda > 0$ such that $G_1(z) \leq \Lambda Q_{22}(z)$ and $G_2(z) \leq \Lambda Q_{11}(z)$. By definition,

$$\begin{aligned}\frac{G_1(z)}{Q_{22}(z)} &= \frac{e^{-4z}[(3+a)a_{11}(z)^2 + (3-a)a_{12}(z)^2 + (b+c)a_{11}(z)a_{12}(z)]}{e^{-4z}[a_{11}(z)^2 + a_{12}(z)^2]}\\ &= 3 + a\frac{a_{11}(z)^2 - a_{12}(z)^2}{a_{11}(z)^2 + a_{12}(z)^2} + (b+c)\frac{a_{11}(z)a_{12}(z)}{a_{11}(z)^2 + a_{12}(z)^2}\\ &\leq 3 + |a| + \frac{|b+c|}{2} = \Lambda, \end{aligned} \tag{3.10}$$

and, *mutatis mutandis*, the same estimate holds for the quotient $G_2(z)/Q_{11}(z)$.

Next, using the existence of λ and Λ, as before, we prove the first case of the theorem. Fix any constant $\alpha \in \mathbb{R}$ and let u be any function that satisfies $Q(u) \geq 0$ and $\sup_{\partial\Omega} u \leq \alpha$.

First, assume that (i) holds and let R be the quasilinear elliptic operator defined as

$$\begin{aligned} R(w) = w_{xx}\Big(\frac{Q_{22}(u) + w_y^2}{Q_{22}(u)}\Big) + w_{yy}\Big(\frac{Q_{11}(u) + w_x^2}{Q_{22}(u)}\Big) + 2w_{xy}\Big(\frac{Q_{12}(u) - w_x w_y}{Q_{22}(u)}\Big)\\ + \frac{G_1(u)}{Q_{22}(u)}w_x^2 + \frac{G_2(u)}{Q_{22}(u)}w_y^2 + \frac{G_3(u)}{Q_{22}(u)}w_x w_y + 2\frac{e^{-2u}}{Q_{22}(u)}e^{-2w}. \end{aligned}$$

Note that R is defined in order to have two features. First, when $w = u$, we have $R(u) = Q(u)/Q_{22}(u) \geq 0$. Second, using the notation of (3.3), we have that the coefficients a_{ij} of R do not depend on w, only on the space variable and on the derivatives of w. Also, the function $z \mapsto b(x, z, p)$ is nonincreasing for every x and p fixed, and thus R satisfies the hypothesis of the comparison principle (although, as noticed before, Q does not).

In order to finish the proof of Case 1 (when (i) holds), we will build a nonnegative function $v : \Omega \to \mathbb{R}$ that will depend uniquely on Ω and on α such that $R(v + \alpha) \leq 0 \leq R(u)$. As $u \leq \alpha \leq \alpha + v$ on $\partial\Omega$, it will follow, from the comparison principle, that $u \leq v + \alpha$ in Ω, and this will finish the proof.

As Ω is a bounded domain, after a horizontal translation (which is an isometry of the ambient space) we may assume, without loss of generality, that it is contained in a strip

$$\Omega \subseteq \{(x, y, 0) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid 1 < x < M\},$$

for some $M > 1$. Let $v(x, y) = \ln(lx)/L$, where l, $L > 0$ are constants yet to be defined. By the definition of R and v and by the existence of λ and Λ as before,

$$\begin{aligned} R(v + \alpha) &= v_{xx} + \frac{G_1(u)}{Q_{22}(u)}v_x^2 + 2\frac{e^{-2u}}{Q_{22}(u)}e^{-2(v+\alpha)}\\ &< v_{xx} + \Lambda v_x^2 + \frac{2}{\lambda}e^{-2(v+\alpha)}. \end{aligned}$$

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Then, using that $e^v = (lx)^{1/L}$, $v_x = 1/Lx$ and $v_{xx} = -1/Lx^2$, we obtain

$$\begin{aligned} R(v+\alpha) &< -\frac{1}{Lx^2} + \Lambda\frac{1}{L^2x^2} + \frac{2}{\lambda e^{2\alpha}}(lx)^{-2/L} \\ &= \frac{1}{Lx^2}\Big[-1 + \frac{\Lambda}{L} + \frac{2L}{\lambda e^{2\alpha} l^{2/L}} x^{(2L-2)/L}\Big]. \end{aligned} \tag{3.11}$$

Take $L = 1 + \Lambda$. As $1 < x < M$, it follows that

$$R(v+\alpha) < \frac{1}{(1+\Lambda)x^2}\Big[-\frac{1}{1+\Lambda} + 2\frac{1+\Lambda}{\lambda e^{2\alpha} l^{2/(1+\Lambda)}} M^{2\Lambda/(1+\Lambda)}\Big], \tag{3.12}$$

and we can choose l big enough (in particular, we may assume $l \geq 1$, so that $v > 0$ in Ω) such that

$$-\frac{1}{1+\Lambda} + 2\frac{1+\Lambda}{\lambda e^{2\alpha} l^{2/(1+\Lambda)}} M^{2\Lambda/(1+\Lambda)} < 0, \tag{3.13}$$

so $R(v+\alpha) < 0$. We remark that the choice of l and L depends uniquely on λ, Λ, α and M, so it does not depend on u.

As R satisfies the hypothesis of the comparison principle and $v + \alpha \geq u$ on $\partial\Omega$, it follows that $\sup_\Omega u \leq \sup_\Omega v + \alpha$. Finally, we set $C = \sup_\Omega v$, and the theorem follows when A is not diagonal and (i) holds.

Still in Case 1, with A not being a diagonal matrix, if (i) was not true, then $b = 0$ and $c \neq 0$ so then (ii) would hold. In this case, let

$$\begin{aligned} R(w) = w_{xx}\Big(\frac{Q_{22}(u) + w_y^2}{Q_{11}(u)}\Big) &+ w_{yy}\Big(\frac{Q_{11}(u) + w_x^2}{Q_{11}(u)}\Big) + 2w_{xy}\Big(\frac{Q_{12}(u) - w_x w_y}{Q_{11}(u)}\Big) \\ &+ \frac{G_1(u)}{Q_{11}(u)}w_x^2 + \frac{G_2(u)}{Q_{11}(u)}w_y^2 + \frac{G_3(u)}{Q_{11}(u)}w_x w_y + 2\frac{e^{-2u}}{Q_{11}(u)}e^{-2w}. \end{aligned}$$

From here, just proceed as before, but using $v(x, y) = \ln(ly)/L$ and making appropriate choices for l and L, to finish the proof of Case 1.

Case 2. Assume that A is a diagonal matrix

$$A = \begin{pmatrix} 1+a & 0 \\ 0 & 1-a \end{pmatrix}.$$

In this case, $a_{11}(z) = e^{(1+a)z}$, $a_{22}(z) = e^{(1-a)z}$ and $a_{12}(z) = a_{21}(z) = 0$. It follows that the operator Q is given by

$$\begin{aligned} Q(u) = u_{xx}(e^{-2(1-a)u} + u_y^2) &+ u_{yy}(e^{-2(1+a)u} + u_x^2) - 2u_{xy}(u_x u_y) \\ &+ (3+a)e^{-2(1-a)u}u_x^2 + (3-a)e^{-2(1+a)u}u_y^2 + 2e^{-4u}. \end{aligned}$$

If $a \geq 0$, we define R as the operator

$$\begin{aligned} R(w) = w_{xx}(1 + e^{2(1-a)u}w_y^2) &+ w_{yy}(e^{-4au} + e^{2(1-a)u}w_x^2) - 2w_{xy}(e^{2(1-a)u}w_x w_y) \\ &+ (3+a)w_x^2 + (3-a)e^{-4au}w_y^2 + 2e^{-2(1+a)w} \end{aligned}$$

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and, if $a < 0$, R will be defined as

$$R(w) = w_{xx}(e^{4au} + e^{2(1+a)u}w_y^2) + w_{yy}(1 + e^{2(1+a)u}w_x^2) - 2w_{xy}(e^{2(1+a)u}w_xw_y) \\ + (3+a)e^{4au}w_x^2 + (3-a)w_y^2 + 2e^{-2(1-a)w}.$$

Now, we just set v to again be $v(x, y) = \ln(lx)/L$ when $a \geq 0$, and $v(x, y) = \ln(ly)/L$ when $a < 0$. The proof will follow, as in the previous case, using $\Lambda = 3 + |a|$ and $\lambda = 1$. □

Next, we prove that the dependence of α on the constant C of Theorem 3.2 cannot be removed. Precisely, we prove the following theorem.

Theorem 3.3. *Let A be a matrix as in (3.4) and let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a nonunimodular semidirect product endowed with its canonical left-invariant metric. Let $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ be a bounded, convex domain. Then, for every constant $C > 0$, there exists some function $u : \Omega \to \mathbb{R}$ satisfying $Q(u) = 0$ and also*

$$\sup_{\Omega} u > \sup_{\partial\Omega} u + C. \tag{3.14}$$

The proof of Theorem 3.3 above is by contradiction and consists of using the vertical translation that arises from the group structure to translate a family of solutions tending to $-\infty$, all to height zero. We prove that if Theorem 3.3 was false, such a family would be uniformly bounded, and this would generate a contradiction with the following theorem, due to Meeks III *et al.* [9].

Theorem 3.4 [9, Theorem 15.4]. *Let X be a nonunimodular metric Lie group which is isomorphic and isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, $A \in M_2(\mathbb{R})$. Suppose that $\Gamma(n) \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ is a sequence of C^2 simple closed convex curves with $e = (0, 0, 0) \in \Gamma(n)$ such that the geodesic curvatures of $\Gamma(n)$ converge uniformly to zero and the curves $\Gamma(n)$ converge in compact subsets to a line L with $e \in L$ as $n \to \infty$. Then, for any sequence $M(n)$ of compact branched minimal discs with $\partial M(n) = \Gamma(n)$, the surfaces $M(n)$ converge C^2 in compact subsets, as $n \to \infty$, to the vertical half-plane $\pi^{-1}(L) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$.*

Proof of Theorem 3.3. We begin by proving the following claim.

Claim 2. Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle centred on the origin of $\mathbb{R}^2$. Let $A \in M_2(\mathbb{R})$ be a matrix with $\operatorname{trace}(A) = 2$, as in (3.4), and let e^{Az} be its exponential map. Then, there is a point $p \in \mathbb{S}^1$ and an increasing sequence $(z_n)_{n\in\mathbb{N}} \in (0, +\infty)$ such that the curves $\Gamma_n = e^{Az_n}(\mathbb{S}^1 - p)$, defined by the homothety by e^{Az_n} of the translated circle $\mathbb{S}^1 - p$, satisfy the hypothesis of Theorem 3.4 at the origin: that is, the geodesic curvature of Γ_n at zero converges to zero and Γ_n converges to a line L in compacts, with $0 \in L$.

Proof of Claim 2. Denote the traceless part of A by A_0 and observe that $e^{Az} = e^z e^{A_0 z}$. Then $e^{Az}\mathbb{S}^1 = e^z(e^{A_0 z}\mathbb{S}^1)$ is a homothety by e^z of the curve $e^{A_0 z}\mathbb{S}^1$. Let $d = \sqrt{|a^2 + bc|}$ and divide the proof of the three aforementioned cases given in (3.7)–(3.9).

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If $a^2 + bc < 0$, we let $p \in \mathbb{S}^1$ be any point and define $z_n = (2n\pi)/d$. Then $e^{A_0 z_n} = \mathrm{Id}$, so $e^{Az_n}\mathbb{S}^1$ is a circle of radius e^{2z_n} centred at the origin, and $\Gamma_n = e^{Az_n}(\mathbb{S}^1 - p)$ is a circle through the origin with radius e^{2z_n}. As $z_n \to \infty$, Γ_n will converge to a line L through zero, and the claim is proved, in this case.

If $a^2 + bc = 0$, then $e^{A_0 z}$ is given by (3.8) and $e^{A_0 z}\mathbb{S}^1$ is an ellipse. The homotheties of an ellipse by e^n admit a point where its geodesic curvature converges to zero and, after a translation, it converges to a line in compacts, which proves the claim in this second case.

Finally, if $a^2 + bc > 0$, $e^{A_0 z}$ is given by (3.9). If $bc \neq 0$, then $d \neq |a|$ and, if z is big enough, $\cosh(dz) \simeq e^{dz}/2$ and $\sinh(dz) \simeq e^{dz}/2$, so

$$e^{A_0 z} \simeq \frac{e^{dz}}{2d}\begin{pmatrix} d+a & b \\ c & d-a \end{pmatrix},$$

and $e^{Az}\mathbb{S}^1$ is asymptotic to a homothety of $e^{(d+2)z}$ of an ellipse, which has the desired properties. The last case to be treated is when $d^2 = a^2 + bc = a^2 > 0$. Then

$$e^{A_0 z} = \begin{pmatrix} e^{dz} & \dfrac{b}{d}\sinh(dz) \\ \dfrac{c}{d}\sinh(dz) & e^{-dz} \end{pmatrix} \simeq \frac{e^{dz}}{d}\begin{pmatrix} d & b \\ c & de^{-2dz} \end{pmatrix}$$

and, for z large enough, it follows that $e^{A_0 z}\mathbb{S}^1$ is asymptotic to a line segment, with multiplicity two. Now the convergence of $e^{Az}\mathbb{S}^1$ depends on the two possible cases, $0 < d \leq 1$ or $d > 1$. If $d \leq 1$, then the homothety of e^z on $e^{A_0 z}$ will open the segment and make it asymptotic to an ellipse, which again admits a point p, as claimed. If $d > 1$, then the action of e^z still makes $e^{Az}\mathbb{S}^1$ converge to a line in compacts, so the claim is proved. □

Now we prove Theorem 3.3, arguing by contradiction. Suppose that there is $C > 0$ such that, for every solution of $Q(u) = 0$ in Ω,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C. \tag{3.15}$$

In particular, the same estimate holds for any bounded, smooth domain contained in Ω. Let $r > 0$ be such that a Euclidean ball B_r with radius r is contained in Ω, let $\mathbb{S}^1(r) = \partial B_r$ be the circle that bounds B_r and let $p \in \mathbb{S}^1(r)$ and $(z_n)_{n\in\mathbb{N}}$ be the ones given by Claim 2. Consider, for each $n \in \mathbb{N}$, the problem

$$\begin{cases} Q(u) = 0 & \text{in } B_r, \\ u = -z_n & \text{on } \partial B_r. \end{cases} \tag{3.16}$$

The existence result, due to Meeks III *et al.* [9, Theorem 15.1], implies that (3.16) admits a solution $u_n : B_r \to \mathbb{R}$, and, from (3.15), it follows that, for every $n \in \mathbb{N}$, u_n satisfies

$$\sup_{B_r} u_n \leq \sup_{\partial\Omega} u_n + C = -z_n + C.$$

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We translate the functions u_n vertically to height zero, using the left translation of the group $L_{(0,0,z_n)}$ to obtain a contradiction. If $\Sigma_n = \text{graph}(u_n)$, then

$$\begin{aligned} L_{(0,0,z_n)}\Sigma_n &= \{L_{(0,0,z_n)}(x, y, u_n(x, y)) \mid (x,\ y) \in B_r\} \\ &= \left\{\left(e^{Az_n}\begin{pmatrix} x \\ y \end{pmatrix}, u_n(x, y) + z_n\right),\ (x,\ y) \in B_r\right\} \\ &= \left\{\left(\widetilde{x}, \widetilde{y}, u_n\left(e^{-Az_n}\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix}\right) + z_n\right),\ (\widetilde{x}, \widetilde{y}) \in e^{Az_n}B_r\right\}. \end{aligned}$$

Hence, if we let $v_n : e^{Az_n}B_r \to \mathbb{R}$ be the function defined by

$$v_n(x, y) = u_n\left(e^{-Az_n}\begin{pmatrix} x \\ y \end{pmatrix}\right) + z_n,$$

it follows that the graph of v_n is a left-translate of the graph of u_n and, in particular, its graph $\widetilde{\Sigma}_n = L_{(0,0,z_n)}\Sigma_n$ is a minimal surface of $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Moreover, these graphs $\widetilde{\Sigma}_n$ satisfy the hypothesis of Theorem 3.4, and thus they should converge, in compact sets, to a vertical half-plane. However, it holds that

$$\sup_{e^{Az_n}B_r} v_n = \sup_{B_r} u_n + z_n \le C,$$

so the sequence v_n is uniformly bounded, which generates a contradiction that proves Theorem 3.3. □

Note that the last proof shows more than the existence of *a* function u as on (3.14) for a fixed constant C. We actually proved that *any* sequence of functions with values along the boundary converging to $-\infty$ should have unbounded oscillation. In particular, using the notation of Theorem 3.2, it follows that, when $\alpha \to -\infty$, it is necessary that $C(\alpha) \to +\infty$. It is also possible to prove that $C(\alpha)$ may be chosen more carefully to satisfy $C(\alpha) \to 0$ when $\alpha \to +\infty$ (when $\text{trace}(A) > 0$). We make this analysis in the next result and in Corollary 3.6.

Theorem 3.5. *Let $A \in M_2(\mathbb{R})$ and let $\mathbb{R}^2 \rtimes_A \mathbb{R}$ be a semidirect product endowed with its canonical left-invariant metric. Let $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ be some open bounded smooth domain, let $k \in \mathbb{Z}$ be given and let u_k be a solution to the problem*

$$\begin{cases} Q(u) = 0 & \text{in } \Omega, \\ u = k & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

Then, if $\text{osc}_\Omega(u) = \sup_\Omega(u) - \inf_\Omega(u)$ denotes the oscillation of a function u in Ω, the following hold.

(1) *If $\text{trace}(A) = 0$, then $u_k \equiv k$ is the constant function.*
(2) *If $\text{trace}(A) > 0$, then $u_k > k$ in Ω. Moreover,*

$$\lim_{k\to-\infty} \text{osc}_\Omega(u_k) = +\infty \quad \text{and} \quad \lim_{k\to+\infty} \text{osc}_\Omega(u_k) = 0.$$

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(3) *If* $\operatorname{trace}(A) < 0$, *then* $u_k < k$ *in* Ω. *Moreover,*

$$\lim_{k\to-\infty} \operatorname{osc}_\Omega(u_k) = 0 \quad \text{and} \quad \lim_{k\to+\infty} \operatorname{osc}_\Omega(u_k) = +\infty.$$

Proof. If $\operatorname{trace}(A) = 0$, it is clear that $u_k \equiv k$ is the unique solution to (3.17), by Lemma 3.1, which proves (1). Also, as the change $A \to -A$ gives rise to an isometry, $(x, y, z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mapsto (-x, -y, -z) \in \mathbb{R}^2 \rtimes_{-A} \mathbb{R}$, (3) follows from (2), so we can simply prove the case of $\operatorname{trace}(A) > 0$ and, as previously, it is without loss of generality that we assume that $\operatorname{trace}(A) = 2$, so A is written as in (3.4).

From Lemma 3.1, it follows that $u_k \geq k$ in Ω and, if at an interior point $x \in \Omega$ the function u_k attains its minimum $u_k(x) = k$, then the mean curvature comparison principle, applied to $\Sigma_k = \operatorname{graph}(u_k)$ and to the plane $\{z = k\}$, will imply that the mean curvature of Σ_k is at least as big as that of the plane, which is $1 > 0$. This is a contradiction that proves that $u_k > k$ in Ω.

The second part of (2) follows similarly to the proof of Theorem 3.3. If the oscillation of u_k was not going to $+\infty$ when $k \to -\infty$, then we could translate all the minimal surfaces $\Sigma_k = \operatorname{graph}(u_k)$ to height zero and obtain a contradiction with Theorem 3.4.

It remains to prove that the oscillation of u_k goes to zero when k approaches $+\infty$. In order to do so, it suffices to prove that the constant $C(\alpha)$ can be chosen to go to zero when $\alpha \to \infty$.

Recall the proof of Theorem 3.2: $C = C(\alpha)$ was chosen depending on many parameters l, L, λ, Λ, M and α. The constants λ and Λ depend only on the ambient space, as they come from estimates of the coefficients of the operator Q. The constant M depends uniquely on the diameter of Ω, so it is also fixed. In the proof of Theorem 3.2, the free parameters we could work with were l and L, which depend on the previous ones and on the *a priori* constant α. Using an appropriate choice of l and L, we obtained the expression for C given by

$$C = \frac{\ln(lM)}{L}.$$

The key steps for choosing l and L were between (3.11)–(3.13). However, these steps were carried out by considering the worst case, where the number α was a *negatively large* number, so we began by choosing L and then arrived at the definition of a big enough l, in order to compensate $e^{2\alpha}$, which was thought to be close to zero. Now, we are taking $\alpha_k = k$ to be *positively large*, so we follow a different approach.

Using the notation of the proof of Theorem 3.2, let $L = \Lambda + j$, where $j \in \mathbb{N}$ is yet to be chosen, and take $l = 1$, to obtain, similarly to (3.12), the inequality

$$R(v + k) < \frac{1}{(\Lambda + j)x^2}\left[-\frac{j}{\Lambda + j} + 2\frac{\Lambda + j}{\lambda e^{2k}} M^{2-2/(\Lambda+j)}\right]. \tag{3.18}$$

Then we proceed as before, and try to find some $j \in \mathbb{N}$ such that the right-hand side of (3.18) becomes negative. Such j exists if and only if it satisfies

$$\frac{(\Lambda + j)^2}{jM^{2/(\Lambda+j)}} < \frac{\lambda}{2M^2} e^{2k}. \tag{3.19}$$

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There is $k_0 \in \mathbb{N}$ big enough such that, for every $k \geq k_0$, it is possible to find $j \in \mathbb{N}$ satisfying (3.19). For $k \geq k_0$, denote by $j(k)$ the *largest* $j \in \mathbb{N}$ such that (3.19) holds (as the left-hand side is unbounded with j this is well defined). By taking $L = \Lambda + j(k)$, we use (3.18) to obtain, as in Theorem 3.2, that there exists a constant $C(k) = C(\Omega, k)$ given by

$$C(k) = \frac{\ln(M)}{\Lambda + j(k)}$$

such that every $u : \Omega \to \mathbb{R}$, where

$$\begin{cases} Q(u) \geq 0 & \text{in } \Omega, \\ u \leq k & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$\sup_{\Omega}(u) \leq k + \frac{\ln(M)}{\Lambda + j(k)}.$$

Note this is the same result as in Theorem 3.2 but for a different constant C, and only for $\alpha = k \geq k_0$. In particular, the functions u_k satisfy, for k large enough, that

$$\sup_{\Omega} u_k \leq k + \frac{\ln(M)}{\Lambda + j(k)},$$

and hence

$$\operatorname{osc}_{\Omega}(u_k) = \sup_{\Omega} u_k - k \leq \frac{\ln(M)}{\Lambda + j(k)}.$$

Finally, as the right-hand side of (3.19) is unbounded with respect to k, it follows that $\lim_{k\to\infty} j(k) = +\infty$, so

$$\lim_{k\to+\infty} \frac{\ln(M)}{\Lambda + j(k)} = 0,$$

and the oscillation of u_k also tends to zero when $k \to +\infty$. This finishes the proof of (2) and of the theorem. □

This proof has the next result as a consequence.

Corollary 3.6. *Let* $\mathbb{R}^2 \rtimes_A \mathbb{R}$ *be a nonunimodular semidirect product with* $\operatorname{trace}(A) > 0$ *and let* $C(\alpha)$ *be the constant given by Theorems 3.2 and by the proof of Theorem 3.5. Then*

$$\lim_{\alpha\to-\infty} C(\alpha) = +\infty, \quad \lim_{\alpha\to+\infty} C(\alpha) = 0.$$

In particular, if $u_L : \Omega \to \mathbb{R}$ *is a function satisfying*

$$\begin{cases} Q(u) \geq 0 & \text{in } \Omega, \\ \sup_{\partial\Omega} u = L \in \mathbb{R}, & \end{cases}$$

then

$$\lim_{L\to-\infty}\Big(\sup_{\Omega} u_L - L\Big) = +\infty, \quad \lim_{L\to+\infty}\Big(\sup_{\Omega} u_L - L\Big) = 0.$$

4. Scherk-like fundamental pieces

In this section, we use the tools developed in the study of the mean curvature operator, together with Killing graph techniques, to obtain an existence result of *Scherk-like fundamental pieces*, which are minimal π-graphs on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ that assume the value zero along a piecewise smooth curve $\gamma \subset \mathbb{R}^2 \rtimes_A \{0\}$ and have $\gamma \cup (\{p_1\} \times [0, \infty)) \cup (\{p_2\} \times [0, \infty))$ as boundary, where p_1 and p_2 are the endpoints of γ.

In the ambient space of an *unimodular* group $\mathbb{R}^2 \rtimes_A \mathbb{R}$, Menezes [11] proved the existence of *complete* (without boundary) minimal surfaces, similar to the singly and to the doubly periodic Scherk minimal surfaces of $\mathbb{R}^3$. We would like to take a moment to give the main steps of the proof of Menezes for the existence of a doubly periodic example.

Sketch of the proof of Theorem 2 of [11]. Let $\Delta \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ be a triangle with vertices

$$o = (0, 0, 0), \quad p_1 = (a, 0, 0), \quad p_2 = (0, a, 0),$$

for some $a > 0$. Let P_c be the polygon given by the union of segments

$$P_c = \overline{op_1} \cup \overline{p_1 p_1(c)} \cup \overline{p_1(c)p_2(c)} \cup \overline{p_2(c)p_2} \cup \overline{p_2 o}, \tag{4.1}$$

where $p_1(c) = (a, 0, c)$ and $p_2(c) = (0, a, c)$. Reference [9, Theorem 15.1] implies the existence of a minimal π-graph Σ_c with $\partial\Sigma_c = P_c$.

Then one key property was observed: any Σ with such boundary is a *Killing graph* over the vertical domain $\Omega_c = \{(t, a - t, s) \mid 0 \le t \le a,\ 0 \le s \le c\}$ with respect to the horizontal Killing field $\partial_x + \partial_y$, and thus Σ_c is the *unique* minimal surface with Γ_c as boundary.

This implies that Σ_c is stable and that the variation $c \mapsto \Sigma_c$ is continuous. By making $c \to \infty$, and using curvature estimates due to Rosenberg *et al.* [13] for stable surfaces in homogeneous manifolds, it is possible to show the convergence of Σ_c to some surface Σ_∞, which is nowhere vertical and has boundary

$$\partial\Sigma_\infty = P_\infty = \overline{op_1} \cup (\{p_1\} \times [0, \infty)) \cup \overline{op_2} \cup (\{p_2\} \times [0, \infty)).$$

Finally, use the ambient isometries to rotate Σ_∞ along the two segments $\overline{op_1}$ and $\overline{op_2}$ to obtain a complete minimal π-graph on $\mathbb{R}^2 \rtimes_A \mathbb{R}$, which can be extended periodically by horizontal translations. □

On this subject, our contribution is an extension of the above result to any semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Although, in the general case, our method does not produce examples without boundary, in the setting of unimodular groups, our proof, which differs from the one of Menezes, re-obtains the same result explained above. We state our result as follows.

Theorem 4.1. *Let $\mathbb{R}^2 \rtimes_A \mathbb{R}$ be a semidirect product, where $A \in M_2(\mathbb{R})$ is any matrix with* $\operatorname{trace}(A) \ge 0$. *There exists $L_0 = L_0(\operatorname{trace}(A), \det(A)) > 0$ (and $L_0 = \infty$ when* $\operatorname{trace}(A) = 0$*) such that if $p_1, p_2 \in \mathbb{R}^2 \rtimes_A \{0\}$ satisfy $d(p_1, p_2) \le L_0$, then, for any*

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piecewise smooth curve $\gamma \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ *with endpoints* p_1, p_2 *which is a convex graph over the segment* $\alpha = \overline{p_1 p_2}$ *and meets* α *on angles less than* $\pi/2$*, there exists a minimal surface* Σ *which is a* π*-graph with boundary*

$$\partial\Sigma = \gamma \cup (\{p_1\} \times [0, +\infty)) \cup (\{p_2\} \times [0, +\infty)).$$

Moreover, Σ *is nowhere vertical, it is the unique minimal surface on* $\mathbb{R}^2 \rtimes_A \mathbb{R}$ *with such boundary and it is a Killing graph over the vertical domain* $\Omega_\infty = \alpha \times [0, +\infty)$.

Remark. Our construction works in some well-studied spaces, for example in the product space $\mathbb{H}^2 \times \mathbb{R}$, which is isometric and isomorphic to the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, when we choose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In $\mathbb{H}^2 \times \mathbb{R}$, Scherk-like graphs have been studied already, and even more general results have been obtained (for example, in the work of Nelli and Rosenberg [12] and in the work of Hauswirth *et al.* [6]). However, the isometry between $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ maps $\mathbb{R}^2 \rtimes_A \{0\}$ not to $\mathbb{H}^2 \times \{0\}$, as would appear at first sight, but to a horocylinder (that is, the product of a horocycle of $\mathbb{H}^2$ with $\mathbb{R}$), so the orientation of our graphs is not that usually studied in this space.

The proof of Theorem 4.1 is given in Section 4.2. If $\operatorname{trace}(A) > 0$, when considering polygons P_c as in (4.1), there is a minimal π-graph Σ_c with boundary P_c. However, as the maximum principle does not hold, there is no reason for Σ_c be a Killing graph over Ω_c and we do not have curvature estimates. Furthermore, we do not have the tools to ensure the continuity of the family Σ_c, which makes it impossible to use geometric barriers. It becomes clear that, when $\operatorname{trace}(A) \neq 0$, another sequence of surfaces Σ_c should be constructed, or other tools (such as stability of minimal π-graphs – a question that remains open) should be developed.

Our approach will be as follows. Instead of considering minimal π-graphs over a domain on $\mathbb{R}^2 \rtimes_A \{0\}$, we will look to the problem *horizontally*, and consider an exhaustion of the *half-strip* $\Omega_\infty = \alpha \times [0, +\infty)$ by subdomains Ω_c, on which is possible to find a family of minimal Killing graphs with prescribed boundary. Then we use techniques from Killing graphs and elliptic PDEs to obtain the convergence of such a family to another minimal Killing graph Σ_∞. Then we go back to the problem *vertically* (as the intermediate Killing graphs will also be π-graphs, by a result of Meeks III, Mira, Pérez and Ros), and we apply the geometric barriers used by Menezes to see that the surface Σ_∞ is, as claimed, a π-graph, nowhere vertical.

4.1. A good exhaustion of Ω_∞. The next proposition is crucial to the construction described above, as it gives the exhaustion of Ω_∞ by domains Ω_c, where it is possible to find minimal Killing graphs with the prescribed boundary (see Figure 2).

Proposition 4.2. *Let* $\mathbb{R}^2 \rtimes_A \mathbb{R}$ *be a semidirect product where* $\operatorname{trace}(A) \geq 0$*. Then there exists a constant* $L_0 = L_0(A)$ *that depends uniquely on* A *such that, for every two points*

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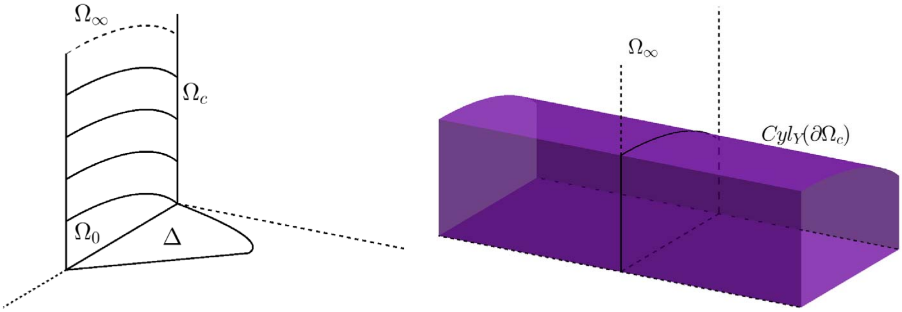

FIGURE 2. (a) The horizontal domain Δ and the exhaustion of Ω_∞ by subdomains Ω_c whose Killing cylinder (b) has mean curvature vector pointing inwards.

$p_1,\ p_2 \in \mathbb{R}^2 \rtimes_A \{0\}$, if $\alpha = \overline{p_1 p_2}$ is the segment joining p_1 and p_2, Ω_∞ is the vertical domain

$$\Omega_\infty = \alpha \times [0,\ +\infty),$$

and if $L = \text{length}(\alpha) < L_0$, then Ω_∞ admits a continuous exhaustion $\{\Omega_c\}_{c>0}$ by domains Ω_c with boundary given by α, a graph over α (called α_c) and the two vertical segments joining the endpoints of α and α_c.

Moreover, such an exhaustion is such that the Killing cylinder over $\partial\Omega_c$ with respect to the horizontal Killing field $Y_\theta = \sin(\theta)\partial_x + \cos(\theta)\partial_y$ has mean curvature vector pointing inwards, where θ is such that Y_θ is normal to Ω_∞ at $z = 0$.

PROOF. Let $p_1, p_2 \in \mathbb{R}^2 \rtimes_A \{0\}$ be any two points and, after a rotation on A as in (2.2) and a horizontal translation of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, we may assume, without loss of generality, that $p_1 = (0, 0, 0)$ and $p_2 = (L, 0, 0)$ for some $L > 0$. We are going to show that if L is sufficiently small, then we can find the exhaustion, as claimed.

In this setting, α is the segment $\alpha = \{(x,\ 0,\ 0) \mid 0 \le x \le L\}$ and Ω_∞ is the half-strip

$$\Omega_\infty = \{(x,\ 0,\ z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid 0 \le x \le L,\ z \ge 0\},$$

transversal to the Killing field $Y = \partial_y$. Such assumptions will be kept until the end of the paper.

If $\text{trace}(A) = 0$, then the result is trivial (and without the need for an upper bound L_0), by taking α_c to be the translate of α to height c given by $\alpha_c = \{(x, 0, c) \mid 0 \le x \le L\}$, as horizontal planes are minimal. Then, until the end of the proof, we will treat the nonunimodular case and again we assume, without loss of generality, that $\text{trace}(A) = 2$, so A is a matrix as in (3.4). We will exhibit the curves α_c explicitly, and then we will prove that they have the desired properties.

First, we treat the case where A is not diagonal and either $a^2 + bc \le 0$ or $b \ne 0$. Let λ, Λ be the constants related to the matrix A via (i) of Claim 1 and (3.10). Let

$$L_0 = \sqrt{\frac{\lambda}{2\Lambda}}\,\frac{\pi}{2}$$

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and, for $L \le L_0$, we let $f : [0, L] \to \mathbb{R}$ be

$$f(x) = \frac{1}{\Lambda} \ln\left(\frac{\cos\left(\sqrt{\frac{2\Lambda}{\lambda}}x\right)}{\cos\left(\sqrt{\frac{2\Lambda}{\lambda}}L\right)}\right). \tag{4.2}$$

Note that f is well defined, as $0 \le x \le L < L_0$ implies that

$$\cos\left(\sqrt{\frac{2\Lambda}{\lambda}}x\right) \ge \cos\left(\sqrt{\frac{2\Lambda}{\lambda}}L\right) > 0,$$

so the quotient in (4.2) is larger than (or equal to) one. In particular $f \ge 0$, with $f(x) = 0 \iff x = L$, and, for $c > 0$, we define $f_c = f + c$ and let $\alpha_c = \text{graph}(f_c) \subseteq \Omega_\infty$. Using such f_c, we define

$$\Omega_c = \{(x, 0, z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid 0 \le x \le L, 0 \le z \le f_c(x)\},$$

and it follows that $\{\Omega_c\}_{c>0}$ is a continuous exhaustion of Ω_∞. Next, we show that the Killing cylinder of the boundary of Ω_c with respect to ∂_y is mean convex, i.e., has mean curvature vector pointing inwards.

The ∂_y-Killing cylinder of $\partial\Omega_c$ has four smooth components (see Figure 2(b)): one is a subdomain of a horizontal plane, so it has mean curvature one pointing upwards, while two are contained on vertical planes and thus are minimal. The last component is the one corresponding to α_c, and it is a π-graph of the function $u_c(x, y) = f_c(x)$. Hence (3.1) implies that its mean curvature, when oriented upwards, is

$$H = \frac{e^{4f_c}}{2W^3}[Q_{22}(f_c)f_c'' + G_1(f_c)(f_c')^2 + 2e^{-4f_c}].$$

From the proof of Theorem 3.2, we obtain that $G_1/Q_{22} \le \Lambda$. Moreover, Claim 1 implies that $Q_{22}(z) > \lambda e^{-2z}$, and hence

$$H \le \frac{e^{4f_c}}{2W^3} Q_{22}(f_c)\left[f'' + \Lambda(f')^2 + 2\frac{e^{-2f_c}}{\lambda}\right],$$

whenever A is not diagonal and satisfies either $b \ne 0$ or $a^2 + bc \le 0$. In particular, as $f_c \ge 0$,

$$H \le \frac{e^{4f_c}}{2W^3} Q_{22}(f_c)\left[f'' + \Lambda(f')^2 + \frac{2}{\lambda}\right]. \tag{4.3}$$

Note that f was chosen in such a way that it satisfies the ordinary differential equation

$$f'' + \Lambda(f')^2 + \frac{2}{\lambda} = 0, \tag{4.4}$$

so, from (4.4) and (4.3), we obtain that $H \le 0$, with respect to the upward orientation, and hence the mean curvature vector of the Killing cylinder around α_c points downwards, as desired.

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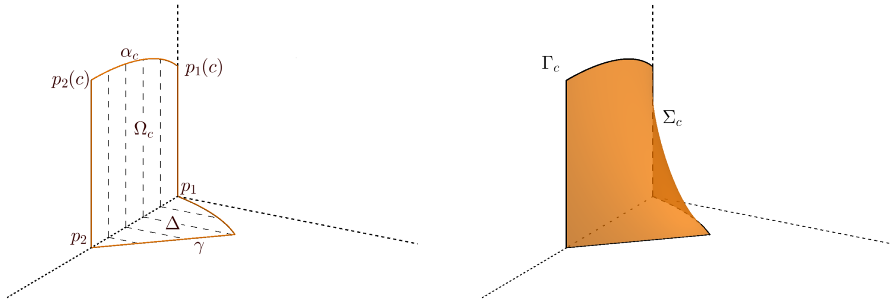

(a) (b)

FIGURE 3. The surface Σ_c (b) is both a π-graph over Δ and a ∂_y-Killing graph over Ω_c, with $\partial\Sigma = \Gamma_c$ (a).

This finishes the proof when A is not diagonal and either $a^2 + bc \leq 0$ or $b \neq 0$. Now, we treat the simpler case of A being given by

$$A = \begin{pmatrix} 1+a & 0 \\ c & 1-a \end{pmatrix}.$$

It follows, from (3.9), that $Q_{22}(z) = e^{2(a-1)z}$ and $G_1(z) = (3+a)e^{2(a-1)z}$. Thus, the mean curvature of the π-graph of $u(x, y) = f(x)$ is

$$H = \frac{e^{2(a+1)f}}{2W^3}[f'' + (3+a)(f')^2 + 2e^{-2(1+a)f}],$$

and we can finish the proof similarly to the previous case. □

4.2. Existence of Scherk-like graphs – proof of Theorem 4.1. This section proves Theorem 4.1 via a standard argument of convergence. However, sometimes we look at the graphs vertically (as π-graphs), with geometrically defined barriers, and sometimes horizontally (as ∂_y-Killing graphs), so that we can use techniques of Killing graphs and elliptic PDEs.

PROOF OF THEOREM 4.1. Let $A \in M_2(\mathbb{R})$ be any matrix with trace$(A) \geq 0$ and let $L_0 > 0$ be the one given by Proposition 4.2. Let $p_1,\ p_2 \in \mathbb{R}^2 \rtimes_A \{0\}$ be such that $d(p_1,\ p_2) = L < L_0$ and, without loss of generality, assume that $p_1 = (0, 0, 0)$ and $p_2 = (L, 0, 0)$.

Let $\alpha = \{(x, 0, 0) \mid 0 \leq x \leq L\}$ be the segment joining p_1 and p_2 and let $g : [0,\ L] \to \mathbb{R}$ be a convex, piecewise smooth function, with $g(0) = g(L) = 0$ and meeting α on angles smaller than $\pi/2$ at zero and L, that defines a piecewise smooth curve $\gamma \subseteq \mathbb{R}^2 \rtimes_A \{0\}$, given by

$$\gamma = \{(x,\ g(x),\ 0) \in \mathbb{R}^2 \rtimes_A \{0\} \mid 0 \leq x \leq L\},$$

with endpoints $p_1,\ p_2$ such that $\alpha \cup \gamma$ bounds a convex domain $\Delta \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ (as in Figure 3(a)).

Let $\Omega_\infty = \alpha \times [0,\ +\infty)$ and, following the notation of Proposition 4.2, let, for each $c \geq 0$,

$$\Omega_c = \{(x,\ 0,\ z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid 0 \leq x \leq L,\ \ 0 \leq z \leq f_c(x)\}.$$

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Let $\alpha_c = \{(x,\ 0,\ f_c(x)) \mid 0 \le x \le L\}$ be the graph of f_c, in such a way that its ∂_y-Killing cylinder

$$\mathrm{Cyl}_{\partial_y}(\alpha_c) = \{(x,\ y,\ f_c(x)) \mid 0 \le x \le L,\ y \in \mathbb{R}\}$$

has mean curvature vector pointing downwards. We denote by

$$p_1(c) = (0, 0, f_c(0)), \quad p_2(c) = (L,\ 0,\ f_c(L))$$

the endpoints of α_c and, for $c \ge 0$, we let Γ_c be the simple closed curve in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ given by (see Figure 3(a))

$$\Gamma_c = \gamma \cup \overline{p_1 p_1(c)} \cup \alpha_c \cup \overline{p_2 p_2(c)}.$$

Claim 3. The curve Γ_c, as above, bounds a *unique* minimal π-graph Σ_c over Δ, which is also a ∂_y-Killing graph over Ω_c.

Proof of Claim 3. First, we notice that Γ_c monotonically parametrises $\partial\Delta$, and then we can use [9, Theorem 15.1] to obtain a minimal π-graph Σ_c with boundary $\partial\Sigma_c = \Gamma_c$.

Next, we show that Σ_c is a ∂_y-Killing graph, in the sense that there is a function $g_c : \overline{\Omega}_c \to \mathbb{R}$, smooth up to the boundary, such that $R(g_c) = 0$, where R will stand for the elliptic operator of the mean curvature of minimal ∂_y-Killing graphs and

$$\Sigma_c = Gr_{\partial_y}(g_c) = \{(x,\ g_c(x,z),\ z) \mid (x, 0, z) \in \Omega_c\}. \tag{4.5}$$

Note that, as Σ_c is a π-graph, there is a function $u_c : \Delta \to \mathbb{R}$ such that

$$\Sigma_c = \mathrm{graph}(u_c) = \{(x, y, u_c(x, y)) \mid (x, y, 0) \in \Delta\}. \tag{4.6}$$

We claim that Σ_c is contained in the ∂_y-Killing cylinder over Ω_c, so $0 \le u_c(x, y) \le f_c(x)$. Indeed, it follows directly from the mean curvature comparison principle that $u > 0$ in the interior of Δ, so we show that $u_c(x, y) \le f_c(x)$ for every $(x, y, 0) \in \Delta$. Arguing by contradiction, if there was an interior point $(x_0,\ y_0,\ 0) \in \Delta$ such that $u_c(x_0,\ y_0) > f_c(x_0)$, then we could consider the family $\mathrm{Cyl}_{\partial_y}(\alpha_t)$, for $t > c$, and obtain a last contact point, interior for both Σ_c and $\mathrm{Cyl}_{\partial_y}(\alpha_t)$, so the mean curvature of $\mathrm{Cyl}_{\partial_y}(\alpha_t)$ would point upwards. This would contradict Proposition 4.2 so it proves that $u_c(x, y) \le f_c(x)$ for every $(x, y, 0) \in \Delta$.

Let $q = (x, 0, z) \in \Omega_c$ be an interior point and consider $O(q) = \{(x, y, z) \mid y \in \mathbb{R}\}$, the orbit of q with respect to the flux φ_t of the Killing field ∂_y. Note that $O(q) \cap \Sigma_c$ is never empty for $q \in \Omega_c$, otherwise Σ_c would not be simply connected. Hence it would not be a π-graph over Δ.

Moreover, the intersection $O(q) \cap \Sigma_c$ cannot contain more than one point: if there were two points $q_i = \varphi_{t_i}(q) \in \Sigma_c$, with $t_1 < t_2$, then, for $t_0 = t_2 - t_1 > 0$, $\varphi_{t_0}(\Sigma_c) \cap \Sigma_c \neq \emptyset$. Now, as $\varphi_t(\partial\Sigma_c) \cap \Sigma_c = \emptyset$ for all $t \neq 0$, by construction, we could consider the last contact point between $\varphi_t(\Sigma_c) \cap \Sigma_c$, and it would be interior for both Σ_c and $\varphi_t(\Sigma_c)$. This would contradict the maximum principle.

This defines a function $g_c : \Omega_c \to \mathbb{R}$, which satisfies the relation $(x, g_c(x, z), z) = \Sigma_c \cap O(x, 0, z)$ and thus Σ_c can be written as in (4.5). However, we still do not have the

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regularity of g_c. In order to prove that g_c is smooth, we begin by proving that the norm of $\operatorname{grad}(g_c)$ is bounded in Ω_c.

Let $q \in \Omega_c$ be any interior point and consider a small open ball $B = B_{\Omega_c}(q, r) \subseteq \Omega_c$ such that $\mathrm{Cyl}_{\partial_y}(\partial B)$ has mean curvature vector pointing inwards. Consider the problem over $\overline{B}$ given by

$$\begin{cases} R(w) = 0 & \text{in } B, \\ w|_{\partial B} = g_c|_{\partial B}, \end{cases} \tag{4.7}$$

where R is the mean curvature operator for ∂_y-Killing graphs. In other words, we are looking for a minimal ∂_y-Killing graph over a small ball on Ω_c that coincides with Σ_c on its boundary.

If $\Phi := g_c|_{\partial B}$ was of class $C^{2,\alpha}$, we could simply use the existence result due to Dajczer and de Lira [3, Theorem 1][1] to obtain a solution to (4.7). However, at this point we can only guarantee that Φ is of class C^0, so we need to use an approximation argument. Let $(\Phi_n^{\pm})_{n\in\mathbb{N}} \subseteq C^{2,\alpha}(\partial B)$ be two sequences of $C^{2,\alpha}$ functions, converging to Φ and such that

$$\Phi_n^- \le \Phi_{n+1}^- \le \Phi \le \Phi_{n+1}^+ \le \Phi_n^+, \tag{4.8}$$

for every $n \in \mathbb{N}$. By [3, Theorem 1], there are functions $w_n^{\pm} \in C^{2,\alpha}(\overline{B})$ with minimal ∂_y-Killing graphs and such that $w_n^{\pm}|_{\partial B} = \Phi_n^{\pm}$. From (4.8) we obtain that the sequences $(w_n^{\pm})_{n\in\mathbb{N}}$ are also monotone, $(w_n^-)_{n\in\mathbb{N}}$ is nondecreasing, $(w_n^+)_{n\in\mathbb{N}}$ is nonincreasing and both are uniformly bounded. To obtain the convergence of the sequences $w_n^{\pm}$ to a solution of (4.7), we use some recent gradient estimates for Killing graphs obtained by Casteras and Ripoll in [1].

THEOREM 4.3 [1, Theorem 4]. *Let M be a Riemannian manifold and let Y be a Killing field. Let Ω be a Killing domain in M and let $o \in \Omega$ and $r > 0$ such that the open geodesic ball $B_\Omega(o, r)$ is contained in Ω. Let $u \in C^3(B_\Omega(o, r))$ be a negative function whose Y-Killing graph has constant mean curvature H. Then there is a constant L depending only on $u(o)$, r, $|Y|$ and H such that $\|\operatorname{grad}(u)(o)\| \le L$.*

All functions $w_n^{\pm}$ have uniform bounds on their C^0 norm, and thus Theorem 4.3 above implies that there are uniform gradient estimates on compact subsets of B. This implies that both sequences will converge on the C^2 norm to a function $w \in C^2(B) \cap C^0(\overline{B})$, which is a solution of (4.7). Now, just use the flux of ∂_y and the same translation argument as before to obtain that w coincides with g_c in B. Hence the gradient of g_c is bounded on interior points of Ω_c, as claimed.

Next, we use the relation $(x, g_c(x, z), z) = (x, y, u_c(x, y))$ to prove that g_c is actually smooth *up to the boundary*, with the unique exceptions of p_1, p_2, $p_1(c)$, $p_2(c)$ (where $\partial\Omega_c$ is not smooth), and the finite number of points where g is not differentiable. Note that u_c is smooth up to the boundary (except on the points where $\partial\Delta$ is not differentiable) and that the gradient of u_c is never horizontal on $\partial\Delta$, by the boundary

[1] We notice that the hypothesis on the Ricci curvature on [3] is used uniquely to obtain an *a priori* estimate for the height of the graph, which is satisfied in our setting by the maximum principle.

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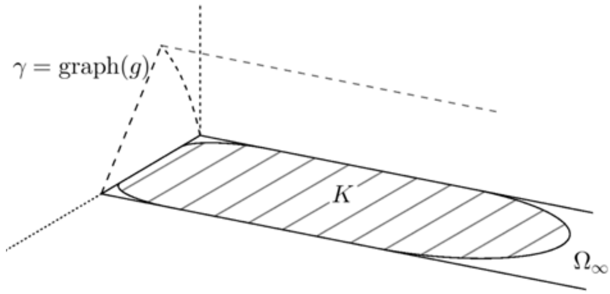

FIGURE 4. $\Omega_\infty \subseteq P$ viewed horizontally: on each compact set $K \subseteq \overline{\Omega}_\infty$ there are uniform gradient estimates.

maximum principle. Moreover, it follows from the last argument that $\text{grad}(u_c)$ never vanishes on interior points of Δ, so g_c is also smooth up to the boundary, with the exceptions given above.

This proves that any minimal π-graph S with $\partial S = \Gamma_c$ is a Killing graph. It follows easily that Σ_c is unique, proving Claim 3. □

We proceed with the proof of Theorem 4.1, by noticing that that the uniqueness of Σ_c, given by Claim 3, implies that the correspondence $c \mapsto g_c$ is continuous. Moreover, by its construction, we have that each g_c satisfies, on the boundary of Ω_c,

$$g_c(0, z) = g_c(L, z) = 0, \quad g_c|_{\alpha_c} = 0, \quad g_c(x, 0) = g(x).$$

Again, as Σ_c is a π-graph over Δ, it is contained on the π-cylinder over Δ, and this can be translated to the horizontal setting as the inequality

$$0 \le g_c(x, z) \le g(x), \tag{4.9}$$

for every $(x, 0, z) \in \Omega_c$. Moreover, the usual argument using the flux of ∂_y shows that the sequence g_c is monotonically increasing with c. In particular, as it is bounded, the sequence will converge pointwise for some function $g_\infty : \Omega_\infty \to \mathbb{R}$, such that $0 \le g_\infty \le g$. The next claim shows that the convergence is actually on the C^2 norm, so $Gr_{\partial_y}(g_\infty) = \Sigma_\infty$ is a minimal surface of $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

Claim 4. When $c \to \infty$, the functions g_c converge on the C^2 norm to $g_\infty : \Omega_\infty \to \mathbb{R}$.

Proof of Claim 4. To prove this claim, we use the same argument as that of Claim 3, via gradient and height estimates for Killing graphs. Let $K \subseteq \overline{\Omega}_\infty$ be a compact set in $\overline{\Omega}_\infty$ with $C^{2,\alpha}$ boundary, as in Figure 4. As it holds that $g_c(x, z) \le g(x)$, it follows that the height of g_c is uniformly bounded on K, so we can use Theorem 4.3 to obtain an uniform bound for the norm of the gradient of g_c on interior points of K.

Note that (4.9), together with the assumption that the angle γ makes with α at p_1 and p_2 is less than $\pi/2$, implies that every g_c satisfies a uniform gradient estimate also along the boundary of K. As $g_c|_K \in C^{2,\alpha}(K)$ is smooth up to the boundary, this implies a uniform estimate for the gradient of g_c on K.

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Now, by taking an exhaustion of Ω_∞ by compact sets and, by using a standard argument, we obtain that a subsequence of (g_c) converges to g_∞ on the C^2 norm. In particular, as the sequence is monotone and converges pointwise, it follows that the convergence is smooth on the whole Ω_∞. □

From this claim we obtain that Σ_∞ is a minimal surface of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and that its boundary is

$$\partial\Sigma_\infty = \Gamma_\infty = \gamma \cup (\{p_1\} \times [0, \infty)) \cup (\{p_2\} \times [0, \infty)).$$

In order to finish the proof of Theorem 4.1, it remains to show that Σ_∞ is nowhere vertical and that it is unique. The uniqueness comes directly from the fact that it was built as a Killing graph, and that every other surface with such boundary is contained on the ∂_y-Killing cylinder over Ω_∞.

To show that Σ_∞ is nowhere vertical, we go back to analyse the problem using π-graphs. First, if there was an interior point $p \in \Sigma_\infty$ such that $T_p\Sigma_\infty$ was vertical, Σ_∞ and $T_p\Sigma_\infty$ would be two minimal surfaces of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ tangential to each other at p. Then, there would be at least two curves, meeting transversely at p on the intersection $T_p\Sigma_\infty \cap \Sigma_\infty$, so Σ_∞ cannot be a π-graph on a neighbourhood of p. Hence, it is a π-cylinder over some line segment[1] β contained in $\partial\Delta$. Second, if the point $p \in \partial\Sigma_\infty$ was a boundary point where $T_p\Sigma_\infty$ was vertical, then the boundary maximum principle would reach the same conclusion. The next claim shows that Σ_∞ meets $\pi^{-1}(\gamma)$ uniquely on γ, so $\Sigma_\infty \supseteq (\beta \times [0, \infty))$ is a contradiction.

Claim 5. $\Sigma_\infty \cap \pi^{-1}(\gamma) = \gamma$.

Proof of Claim 5. To prove this, we use the same barrier technique of Menezes [11]. Let γ_i be a smooth component of γ and let $p \in \gamma_i$ be any point. Consider L, the vertical plane of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ containing the tangent line to γ_i at p (this is well defined even for $p \in \partial\gamma_i$, since γ_i is smooth). As γ is convex, this implies that Δ is contained in the same connected component of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ defined by L, and so is Σ_∞.

For $c \geq 0$, let $u_c : \Delta \to \mathbb{R}$ be as in (4.6) and let $c_0 = \sup_\Delta u_0$. For $c_2 > c_1 > c_0$, consider a rectangle $R \subseteq L$, with boundary $\partial R = r_1 \cup r_2 \cup s_1 \cup s_2$ given by two parallel horizontal segments r_1 and r_2 and two vertical segments s_1 and s_2, such that $s_1 \subseteq \{z = c_1\}$ and $s_2 \subseteq \{z = c_2\}$, that projects into $\mathbb{R}^2 \rtimes_A \{0\}$ in a compact segment $r \ni p$ with endpoints $q_1 = \pi(s_1)$ and $q_2 = \pi(s_2)$, contained on the same half-space determined by $\{y = 0\}$ (the vertical plane containing α) and with q_2 outside Δ (see Figure 5).

Let $q_3 \in \pi(R)$ be a point interior to the projection of R that is not in Δ. Then, $\widetilde{q}_3 = \pi^{-1}(q_3) \cap r_2$ divides r_2 into two compact segments $r_3 \cup r_4$, with r_3 projecting entirely outside Δ and with $p \in \pi(r_4)$.

[1]If $\beta \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ is a smooth curve, the π-cylinder $\beta \times [0, \infty)$ is minimal if and only if β is a line segment: to see this, just use the foliation of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by vertical planes which are parallel to the vertical plane generated by the endpoints of β. It also follows from the more general formula $H(x, y, z) = k_g(x, y)e^{-z\mathrm{trace}(A)}$, where $k_g(x, y)$ denotes the geodesic curvature of β on the point $(x, y, 0)$. The proof of this formula is a simple computation.

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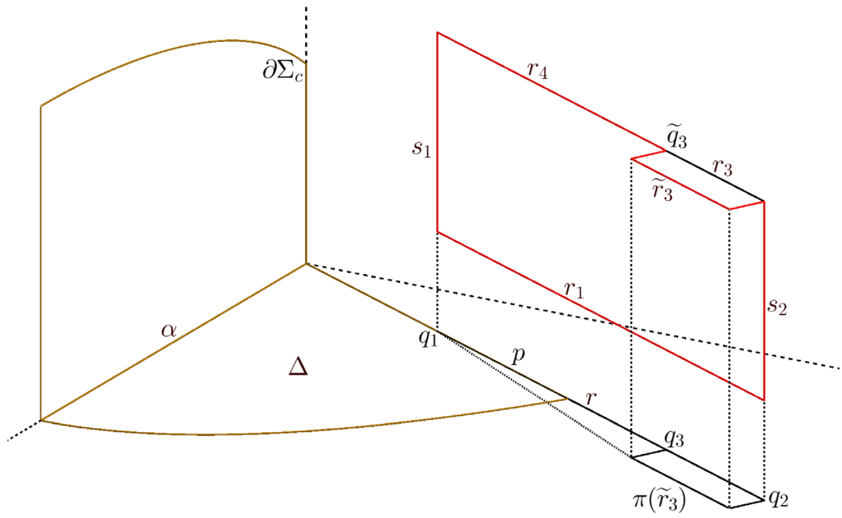

FIGURE 5. The construction of the barrier $\widetilde{R}$, by deforming ∂R over r_3.

As L it is transversal to a (horizontal) Killing field, it is stable. In particular, it follows from the useful criteria due to Fischer-Colbrie and Schoen [4, Theorem 1] (also proved in Proposition 1.32 of the book by Colding and Minicozzi II [2]) that R is strictly stable, and thus small perturbations of ∂R give rise to minimal surfaces with the perturbed boundary.

Change r_2 by making a parallel translation of r_3 (whose projection still does not intersect Δ) in the direction of the half-space that contains Σ_∞, joined by two small segments. Denote such a curve by $\widetilde{r}_3$, where $r_3 \cup \widetilde{r}_3$ bounds a small rectangle in the horizontal plane $\{z = c_2\}$. We assume that this perturbation is small and that its projection does not intersect Δ. Let $\widetilde{R}$ be a minimal surface of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ whose boundary is the perturbed rectangle $r_1 \cup \widetilde{r}_3 \cup r_4 \cup s_1 \cup s_2$. Such a surface is nowhere vertical and is contained in the convex hull of its boundary. In particular, it is contained in $\{z \geq c_1\}$ and in the same half space as Σ_∞ with respect to L.

It is easy to see that $\pi(\widetilde{R}) \cap \Delta \neq \emptyset$, otherwise $\widetilde{R} \cap R$ would have a interior contact point. Moreover, $\widetilde{R}$ is above u_0 on $\pi(\widetilde{R}) \cap \Delta$, by the construction of $\widetilde{R}$. Then, if $\Sigma_\infty \cap \pi^{-1}(\gamma_i) \neq \gamma_i$, we would have that $\Sigma_\infty \cap \widetilde{R} \neq \emptyset$, and thus, for some $\ell > 0$ there would be a first contact point between Σ_ℓ and $\widetilde{R}$. As $\partial\Sigma_\ell$ does not intersect the convex hull of $\partial\widetilde{R}$, it does not intersect $\widetilde{R}$. Moreover, $\partial\widetilde{R}$ cannot intersect Σ_ℓ, as this would imply that such a point would be in the plane L, so Σ_ℓ would have a vertical tangent plane. Such a contact point would be interior for both. Therefore we reach a contradiction that proves the claim. □

From Claim 5 and from the previous argument, we obtain that Σ_∞ is a π-graph, nowhere vertical, which finishes the proof of Theorem 4.1. □

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