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Corrigenda

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N. MARTIN. 'Cobordism of homology manifolds'

A. W. Jessop (10) has pointed out that the proof of Lemma 4·5 is wrong as the ND condition enabling the inductive use of Theorem 4·3 will not hold in the case where $r = 2m+3$.

In what follows Theorem 4·3 and Lemma 4·5 are restated with slightly different conditions but with essentially similar conclusions. References to diagrams are those in the original. Paper references are those of the original together with others listed at the end here.

THEOREM 4·3. *Let M^m, N^n be proper submanifolds of an $ND(r)$ manifold Q^q where M is a PL manifold, N is a homology manifold, $q-4 \geqslant r \geqslant m+3$, $q \geqslant n+3$ and $\partial N \perp F/L$ for a normal D^{q-m} homotopy $ND(\text{codim}\,(q-r))$ bundle F over ∂M in ∂Q (L a PL-cell complex. Then there exists*

$$(W;\, M\times I,\, V)_{\text{rel}\,\partial}\colon (Q;\, M, N)_{(F,\,L)} \xrightarrow{\;t\;} (P;\, M, U;\, E, K),$$

such that W is an $ND(r+1)$-manifold and E is a D^{q-m} homotopy $ND(\text{codim}\,(q-r))$-bundle over K, a PL-cell complex extending L.

The proof of Theorem 4·3 from Lemma 4·5 goes through unchanged apart from the observation that we no longer require ND conditions on the complements of F and E as the corresponding conditions have been removed from Lemma 4·5 also. The indications to proof by induction on q are removed as the restated 4·5 stands proved independently in all dimensions.

LEMMA 4·5. *Let Σ^q be a 1-connected (and so $(q-1)$-connected) $ND(r)$-homology sphere. Let S^m be a PL-sphere and Σ^n be a homology sphere both PL embedded in Σ^q with $q-4 \geqslant r \geqslant m+3$ and $q \geqslant n+3$. Suppose also that $\Sigma^n \perp F/L$ for a normal homotopy $ND(\text{codim}\,(q-r))$-bundle F over a PL-cell subdivision L of S^m. Then Σ^q spans a 1-connected (and so contractible) $ND(r+1)$ homology ball B^{q+1} containing a PL-ball D^{m+1} spanning S^m and a contractible homology ball C^{n+1} spanning Σ^n, both properly embedded in B^{q+1} such that $C \perp E/K$ for a normal homotopy $ND(\text{codim}\,(q-r))$-bundle E over a PL-cell subdivision K of D extending L and with $E|L = F$.*

Proof. The ball B^{q+1} is constructed in several stages, each corresponding to peeling a layer from B.

Stage 1. The peeling off of the first layer will amount to saying that we could have assumed that F was the product bundle $L\times[-1,1]^{q-m}$. Certainly since all normal bundles of spheres in spheres are trivial, there is a homotopy $ND(\text{codim}\,(q-r))$-bundle isomorphism $G\colon F \simeq_h L\times[-1,1]^{q-m}$. Consider the homology manifold $\Sigma^q\times I \cup G$, whose boundary components are $\Sigma^q\times\{0\}$ and $\Sigma^q_1 = cl[\Sigma^q\times\{1\} - F\times\{1\}] \cup \bar{G} \cup L\times[-1,1]^{q-m}$ (see Fig. 3). $\Sigma^q\times I$ and G are certainly both $ND(r+1)$-manifolds (by 1·2·1 (*e*) and 2·4) and they have been glued together across a copy of F which is (by 2·4) an $ND(r)$-manifold so by part of the proof of 1·2·2 $\Sigma^q\times I\cup G$ is an $ND(r+1)$-space. To show it is an $ND(r+1)$-manifold then only requires knowing that Σ_1^q is an $ND(r)$-manifold. Let σ^k be a k-simplex in a triangulation of Σ_1^q in which $\bar{G} \cup L \times [-1,1]^{q-m}$ and $cl[\Sigma^q \times \{1\} - F \times \{1\}]$ are full subcomplexes. We need to know that $lk(\sigma^k, \Sigma_1^q)$ is $(r-k-2)$-connected, or as we are dealing with homology manifolds we need to know that for $k \leqslant r-3$, $lk(\sigma^k, \Sigma_1^q)$ is 1-connected. If σ^k is not in $\bar{F} \times \{1\}$ then $lk(\sigma^k, \Sigma_1^q)$ is homeomorphic to either $lk(\sigma^k, \Sigma^q \times \{1\})$ or $lk(\sigma^k, \partial G)$, whichever is relevant, and since both Σ^q and ∂G are $ND(r)$-manifolds the relevant link is 1-connected. So we need only consider the case of σ^k in $\bar{F} \times \{1\}$. Identify $\Sigma^q \times \{1\}$ with Σ^q. Then as Σ^q is $ND(r)$, $lk(\sigma^k, \Sigma^q)$ is 1-connected so by the Van Kampen theorem if

$$i: lk(\sigma^k, \bar{F}) \to lk(\sigma^k, cl[\Sigma^q - F])$$

is the inclusion map, since F is $ND(r)$ the normal closure of $i_* \pi_1(lk(\sigma^k, \bar{F}))$ is the whole of $\pi_1(lk(\sigma^k, cl[\Sigma^q - F]))$. Applying this to the problem of computing $\pi_1(lk(\sigma^k, \Sigma_1^q))$ via the Van Kampen theorem, gluing $cl[\Sigma^q \times \{1\} - F \times \{1\}]$ to $\bar{G} \cup L \times [-1,1]^{q-m}$ then implies that $lk(\sigma^k, \Sigma_1^q)$ is 1-connected as $\bar{G}$ is $ND(r)$. Thus Σ_1^q is an $ND(r)$-manifold and $\Sigma^q \times I \cup G$ is an $ND(r+1)$-manifold.

We also need to know that Σ_1^q is 1-connected but this follows since $r \geqslant m+3$ and $\Sigma^q - F$ is a deformation retraction of $\Sigma^q - S^m$ in that case. So $\Sigma^q - F$ is 1-connected and Van Kampen may again be applied.

We now add a collar $\Sigma_1^q \times I$ to complete the first layer. By the analogue of proposition 4·3 in (2) for homotopy ND-bundles G is a D^{q-m} homotopy bundle over $L \times I$ which we may assume has a zero cross-section. Thus inside the first layer we have contained $S^m \times I$ subdivided as $L \times 3I$ and having a normal bundle

$$(F \times I) \cup G \cup ((L \times [-1,1]^{q-m}) \times I)$$

over it and properly contained (see Fig. 4). Since $\Sigma^n \perp F/L$, $\Sigma^n \cap S^m$ is a cell subcomplex of L. Thus we can then consider the polyhedron $\Sigma^n \times I \cup G(\Sigma^n \cap S^m)$ in $\Sigma^q \times I \cup G$ which is an h-cobordism between Σ^n and a second homology sphere Σ_1^n, say, in Σ_1^q. Thus in our first layer we have an h-cobordism $(\Sigma^n \times I) \cup G(\Sigma^n \cap S^m) \cup (\Sigma_1^n \times I)$ which by construction is block transverse to the normal bundle

$$((F \times I) \cup G \cup ((L \times [-1,1]^{q-m}) \times I))/(L \times 3I).$$

This completes Stage 1 and shows that we may, without any loss of generality, suppose that F was in fact the trivial bundle over L. We therefore make this supposition and revert to the original notation for the next stage.

Stage 2. The peeling off of the second layer will amount to saying that we could have assumed that Σ^q was a genuine PL sphere.

Let $T = cl[\Sigma^q - L \times [-1,1]^{q-m}]$. Then T is an homology q-manifold with PL-manifold boundary and, since $r \geqslant m+3$, T is 1-connected. We may also suppose that T is an $ND(r)$-manifold. (Identify the product bundle with $L \times [-\frac{1}{2}, \frac{1}{2}]^{q-m}$ instead to force ∂T to be collared in T and then use an argument similar to one in Stage 1 to check that the new T is an $ND(r)$-space.)

Let θ be the abelian group of equivalence classes of oriented homology 3-spheres under the equivalence relation of $PL\,H$-cobordism and addition by connected sum. Then, by duality,

$$H^4(T, \partial T; \theta) \cong H_{q-4}(T; \theta) \cong H_{q-4}(S^{q-m-1}; \theta)$$

which vanishes unless $m = 3$. (The case $q = 4$ is trivial.)

For the moment suppose $m \neq 3$, then the vanishing of $H^4(T, \partial T; \theta)$ means that there is no obstruction to resolving T to be a PL-manifold relative to its boundary (12). T contains a proper n-dimensional submanifold $U = T \cap \Sigma^n$ of codimension $q-n \geqslant 3$ and so by the relative resolution theorem of (13) there exists an acyclic resolution $f: T_1 \to T$ of T to a PL-manifold T_1 such that T_1 contains a proper submanifold U_1 for which $f|U_1: U_1 \to U$ is a collapsible resolution, $f|\partial T_1$ is a PL-homeomorphism onto ∂T and T_1 is 1-connected. In addition by (14) the simplicial mapping cylinder $C(f)$ of f is an $ND(r+1)$-manifold. Also via $f|U_1$ we may identify a proper submanifold V of $C(f)$ which is an h-cobordism between U and U_1. Let $\Sigma_2^q = L \times [-1, 1]^{q-m} \cup_{f|T_1} T_1$. Then Σ_2^q is a PL, 1-connected homology q-sphere and as $q \geqslant 7$, Σ_2^q is PL-homeomorphic to S^q. Also Σ_2^q contains a submanifold $\Sigma_2^n = ((L \times [-1, 1]^{q-m}) \cap \Sigma^n) \cup_{f|\partial U_1} U_1$ which is an homology n-sphere. Thus $(L \times [-1, 1]^{q-m}) \times I \cup C(f)$ is the second layer of B^{q+1}, it being 1-connected and $ND(r+1)$.

The case $m = 3$ must be treated differently as the relevant obstruction group no longer vanishes. As $q \geqslant n+3$ the dimension of $S^m \cap \Sigma^n = n+m-q \leqslant m-3 = 0$. Let $S^m \cap \Sigma^n = \{v_1, \ldots, v_k\}$ a finite set of vertices of L (this set may of course be empty). As the normal bundle of S^m is the product bundle we may subdivide L if necessary to ensure that the closed stars $N_i = s\bar{t}(v_i, L)$ of v_i in L are pairwise disjoint. Let

$$X = cl\left[\Sigma^q - \bigcup_{i=1}^{k} N_i \times [-1, 1]^{q-m}\right].$$

Then as in the case of T above, X is 1-connected and we may assume X is an $ND(r)$-manifold. ∂X is a PL-manifold. A calculation as for T shows that the obstruction to resolving X to a PL-manifold relative to ∂X vanishes. As before let $U = X \cap \Sigma^n$ and now let $Y = cl[S^m - \bigcup_{i=1}^k N_i]$ then by (13, 14) there exists an acyclic resolution $g: X_1 \to X$ of X to a PL-manifold X_1 where X_1 contains submanifolds U_1 and Y_1 and such that $g|\partial X_1$ is a PL homeomorphism of ∂X_1 onto ∂X, $g|U_1$ is a collapsible resolution of U_1 onto U, $g|Y_1$ is a collapsible resolution of Y_1 onto Y, X_1 is 1-connected and the simplicial mapping cylinder of g, $C(g)$, is an $ND(r+1)$-manifold. Note that as $g|Y_1$ is a collapsible resolution, Y_1 and $C(g|Y_1)$ are both PL-manifolds.

Next $C(g|Y_1)$ is a proper submanifold of $C(g)$ so that $Y_1 \cup C(g|\partial Y_1)$ has a product normal bundle in $\partial C(g)$ (on identifying ∂X_1 with ∂X via g). By using this product structure and adding a collar to $\partial C(g)$ we may then extend the product bundle over $Y_1 \cup C(g|\partial Y_1)$ to a normal bundle of $C(g|Y_1)$ in $C(g)$ with $C(g|U_1)$ disjoint from the total space of this normal bundle. Then the second layer of B^{q+1} for the case $m = 3$ is $C(g) \cup \bigcup_{i=1}^k N_i \times [-1, 1]^{q-m} \times I$ an $ND(r+1)$ h-cobordism from Σ^q to Σ_2^q, say, containing $C(g|Y_1) \cup \bigcup_{i=1}^k N_i \times I$ a PL-manifold homeomorphic to $S^m \times I$,

$$C(g|U_1) \cup \bigcup_{i=1}^{k} \{v_i\} \times [-1, 1]^{q-m} \times I,$$

a homology n-sphere Σ_2^n, say, and a normal bundle over the $S^m \times I$ to which it is transverse. Σ_2^q is constructed to be a PL homology q-sphere and to be 1-connected so it is homeomorphic to S^q. Thus we have $S^m \times \{1\}$ a PL m-sphere with a normal bundle over it in a PL q-sphere. This is therefore a PL block bundle $(q-m \geqslant 3)$ and is trivial. We may therefore choose a trivilization of it and so regard it as a product bundle. Again we revert to the initial notation and now assume that without loss of generality, Σ^q is a PL q-sphere and F is the product bundle $L \times [-1, 1]^{q-m}$.

Stage 3. As in Stage 2 let $T = cl[\Sigma^q - F]$ and $U = cl[\Sigma^n - F]$. In $[-1, 1]^{q-m}$ let c_+, c_- be the two points $(1, 0, \ldots, 0)$, $(-1, 0, \ldots, 0)$ respectively. Then $L \times (c_+\}$ and $L \times \{c_-\}$ are PL m-spheres in ∂T with $\partial U \perp L \times \{c_\pm\} \times [-1, 1]^{q+m-1}/L \times \{c_\pm\}$ in ∂T. As $q - m \geqslant 7$, S^m is unknotted in Σ^n so $L \times \{c_+\}$ and $L \times \{c_-\}$ will bound disjoint properly embedded $(m+1)$-discs D_+ and D_- respectively in T. So the triple $(T; D_+ \cup D_-, U)$ is one in which the ambient manifold is PL and the two submanifolds are transverse on the boundary. Then by (9) using the cone transversality of (11) there is a small ambient isotopy of T, fixed on ∂T making U transverse to $D_+ \cup D_-$. The track of the ambient isotopy together with $F \times I$ induces the third layer of B^{q+1}. This layer is $ND(q+1)$ as it is a PL-manifold.

Stage 4. The fourth layer is just $\Sigma^q \times I$ but inside it we identify a D^{m+1} spanning S^m with a normal bundle with respect to which $\Sigma^n \times I$ transverse.

In $\Sigma^q \times I$ define $D^{m+1} = (S^m \times \Gamma) \cup (D_+ \times \{\frac{1}{2}\})$ where $\Gamma \subset [0, 1] \times [0, \frac{1}{2}]$ is the graph of the linear map $\gamma(t) = \frac{1}{2}t$ and the first factor $[0, 1]$ is contained in the first factor of $[-1, 1]^{q-m}$. As $\Gamma \approx I$, D^{m+1} has a cell subdivision $K_+ \cup (L \times \Gamma) = K$, say, where K_+ is a PL cell-division of D_+ extending L_+ $(= L)$ on ∂D_+ over which is defined the normal bundle E_+, say, with respect to which U is now transverse. There is then a normal homotopy $ND(\text{codim}\,(q-r))$ bundle E to D^{m+1} in $\Sigma^q \times I$ over K defined by

$$E|K_+ = E_+ \times [\tfrac{1}{4}, \tfrac{3}{4}]$$

and $E|L \times \Gamma$ is a normal PL block bundle over $L \times \Gamma$ in $L \times [-1, 1]^{q-m} \times [0, 1]$ (see Fig. 5). This PL block bundle clearly exists and *a fortiori* is a homotopy $ND(\text{codim}$ $(q-r))$-bundle. To complete this stage observe that since

$$\Sigma^n \perp (E_+ \cup E_- \cup F)/(K_+ \cup K_- \cup L \times [-1, 1]) \quad \text{in} \quad \Sigma^q \times \{0\}$$

then $\Sigma^q \times I \perp E/K$ in $\Sigma^q \times I$.

Stage 5. It remains to fill in the centre of B. This is done by putting in the cone on $\Sigma^q \times \{1\}$.

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