Proceedings of the Edinburgh Mathematical Society (1983) **26**, 259–263 ©

THE DUALS OF GENERIC SPACE CURVES AND COMPLETE INTERSECTONS

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(Received 21st April 1982)

In a previous paper we discussed the duals of generic hypersurfaces: both smooth hypersurfaces in $\mathbb{R}^n$ and algebraic hypersurfaces in real or complex projective space $\mathbb{P}^n$. In this note we show how to extend the methods of [1] to cover the case of complete intersections in $\mathbb{P}^n$ and preface this with a brief discussion on the contact of space curves in $\mathbb{R}^3$ with planes. We shall use the notation of [1].

1. Space curves

Following [5] (see [8]) we can study the contact of a space curve $C \subset \mathbb{R}^3$ with planes by considering the family of height functions $H: C \times S \to \mathbb{R}, H(x,a) = x.a$ where S^2 is the unit sphere in $\mathbb{R}^3$ and . denotes the usual Euclidean inner product. The set $\Sigma = \{(x,a) \in C \times S^2 : H_a : C \to R$ has an $A_{\geqq 1}$ singularity$\}$ is the unit normal bundle to C. The image of Σ in $\mathbb{R} \times S^2$ under the map $(H,\pi)(x,a) = (H(x,a),a)$ is the dual of C, the set of all planes tangent to C. Here we are parametrising the planes in $\mathbb{R}^3$ by their unit normals and distance from 0. (This actually gives a double covering of the planes since (c,a) and $(-c,-a)$ determine the same plane.)

If we project Σ onto S^2 we obtain the Gauss map of C. The local structure of the dual and Gauss map will, by [1], generically be that of a Legendre (resp. Lagrange) mapping and is determined by the type of singularities of the functions H_a. Generically one expects A_1, A_2 and A_3 singularities and that these will be versally unfolded by the family. One can determine the geometric conditions which give rise to these singularities quite easily.

Proposition 1.1. *Assume the space curve C has nonzero curvature at P. Then a plane through P has order of contact* 1 *with C unless it contains the tangent line,* 2 *if it does, but is not the osculating plane at P and* 3 *if it is the osculating plane and the torsion of C at P is nonzero. If the torsion is zero but its derivative does not vanish, this order of contact is* 4. *In general, if the order of contact is $d \geqq 2$, the function H_a has a singularity of type A_{d-1}. This is universally unfolded by the family H in each of the above cases.*

Proof. These assertions are easily verified. For example taking the x_1, x_2, x_3 axes to be the tangent, normal and binormal at P, the curve is given locally as a function of arc length s as

$$x(s)=\left(s-\frac{\kappa^2}{6}s^3-\frac{\kappa\dot{\kappa}}{8}s^4+O(5),\frac{\kappa}{2}s^2+\frac{\dot{\kappa}}{6}s^3+\frac{1}{24}(\ddot{\kappa}-\kappa^3-\kappa\tau^2)s^4+O(5),\right.$$

$$\left.\frac{1}{6}\kappa\tau s^3+\frac{1}{24}(2\dot{\kappa}\tau+\kappa\dot{\tau})s^4+O(5)\right),$$

where $O(p)$ denotes a function which vanishes at $s=O$ to pth order. Short computations now give the result. (Note that $\kappa\neq O$ is a generic condition (see [2] Section 2 for example)).

The dual will have a cuspidal edge of points corresponding to the osculating planes of the curve, and swallowtails corresponding to osculating planes at points of zero torsion. The Gauss map has folds and cusps (respectively) at these points. (Note that this characterises points of zero torsion on a generic curve as those points inside every neighbourhood of which there exists a pair of distinct points sharing a common tangent plane.)

2. Complete intersections

One can specify a curve C in $\mathbb{P}^3$ as the zero set of two homogeneous polynomial equations $f_1(x_0,\ldots,x_4)=f_2(x_0,\ldots,x_4)=0$. We want to consider the contact of such a smooth curve C with the planes of $\mathbb{P}^3$. To ensure that C is smooth we shall ask that $\{f_i=0\}$ is nonsingular along C and that $\{f_1=0\}$ and $\{f_2=0\}$ meet transversally in C. Such curves are *complete intersections.* If f_1 is of degree d_1 and f_2 of degree d_2 clearly the complete intersections are parametrised by an open non-empty subset W of a product of two projective spaces

$$\mathbb{P}^{N_1}\times\mathbb{P}^{N_2}\left(N_j=\binom{d_j+3}{d_j}\right),$$

where $\mathbb{P}^{N_j}$ is the projective space of non zero homogeneous polynomials of degree d_j. As in [1] we want to prove that the dual of C, i.e. those planes tangent to C, has the local structure in the dual space $\check{\mathbb{P}}^3$ as described in Section 1. The key again is to show that for d_1, d_2 sufficiently large there are sufficiently many deformations of C in the space W to ensure versality of a certain contact map.

Since we are addressing ourselves to local matters we can work in an affine chart, say $x_0=1$, and write $f_i(1,x_1,x_2,x_3)$ as $f_i(x_1,x_2,x_3)$. Without loss of generality we may suppose that f_1 and f_2 vanish at $0\in\mathbb{C}^3$ and the tangent to $f_1=f_2=0$, the curve C, at 0 is the x_3-axis. We measure the contact of C with planes by considering the restriction of the family

$$H:\mathbb{C}^3\times\mathbb{C}^2\to\mathbb{C}$$

$$(x,a)\to x_1+a_2x_2+a_3x_3=H_a(x)$$

to $C\times\mathbb{C}^2$ near $x=0$, which we also denote by H. Parametrising C as $x(u)=(x_1(u),x_2(u),u)$ we see that H_a has a singularity at u if and only if

$$\frac{\partial x_1}{\partial u}+a_2\frac{\partial x_2}{\partial u}+a_3=0 \text{ at } u,$$

if and only if $\{x_1+a_2x_2+a_3x_3=H_a(u)\}$ is tangent to C at $x(u)$. So if $\Sigma=\{(x,a)\in C\times\mathbb{C}^2:(\partial H/\partial u)(x(u),a)=0\}$ the image of Σ under $(H,\pi):C\times\mathbb{C}^2\to\mathbb{C}\times\mathbb{C}^2$ (where $\pi(x,a)=a$) is part of the dual of C corresponding to points near 0. (Clearly the other part can be obtained by using the family $G:\mathbb{C}^3\times\mathbb{C}^2\to\mathbb{C}$ defined by $G(x,a)=a_1x_1+x_2+a_3x_3$.)

Now by a linear change of co-ordinates on $\mathbb{C}^3$ we may suppose that the tangent plane to $\{f_j=0\}$ at 0 is $x_j=0$. Applying the implicit function theorem to the equations $f_1=f_2=0$ we can find smooth functions $g_1(x_3),g_2(x_3)$ with $f_j(g_1(x_3),g_2(x_3),x_3)=0, j=1,2$, so that C is parametrised near 0 by $(g_1(u),g_2(u),u)$.

Indeed by applying the implicit function theorem to the map $\mathbb{C}^3\times W\to\mathbb{C}^2$ given by $(x,f)\to f(x)$ (where f has two components f'_1,f'_2) we can find smooth functions g_1,g_2 defined on some neighbourhood of $(0,f_1,f_2)\in\mathbb{C}\times W$ with $(g_1(u,f),g_2(u,f),u)$ parametrising the curve $\{f=0\}$ near $0\in\mathbb{C}^3$.

Proposition 2.1. *Let $\tilde{H}:\mathbb{C}\times\mathbb{C}^2\times W\to\mathbb{C}$ be the map $\tilde{H}(u,a,f)=g_1(u,f)+a_2g_2(u,f)+a_3u$ defined for u close to 0, f close to (f_1,f_2). If $d_1\geqq k$ then the k-jet extension $j_1^k\tilde{H}:\mathbb{C}\times\mathbb{C}^2\times W\to J_0^k(1,1)$, defined by $(u,a,f)\to k$-jet of $\tilde{H}(-,a,f)$ at u, is a submersion at $(0,a,(f_1,f_2))$, for any $a\in\mathbb{C}^2$. (Here we are taking k-jets without constant terms, which is the reason for the subscript 0 in the jet space.)*

Proof. It is clearly enough to consider the case $d_1=k$. We shall use the tangent vectors in the W space coming from the path $(f_1+sx_3^p,f_2)=f_s$, for $1\leqq p\leqq k$. If $g_i(x_3,s)$ are the corresponding families then $f_i(g_1(x_3,s),g_2(x_3,s),x_3)+\delta_{i1}sz_3^p\equiv 0$, and differentiating this identity with respect to s, and setting $s=0$ we obtain

$$\sum_{j=1}^{2}\frac{\partial f_i}{\partial x_j}(g_1(x_3),g_2(x_3),x_3)\frac{\partial g_j}{\partial s}(x_3,0)+\delta_{i1}x_3^p\equiv 0 \tag{1}$$

Using the path f_s the corresponding tangent vector in the jet space $J_0^k(1,1)$ is the k-jet of $(\partial\tilde{H}/\partial s)(u,a,f(0))$. But $\partial\tilde{H}/\partial s=(\partial g_1/\partial s)+a_2(\partial g_2/\partial s)$. Now $(\partial f_1/\partial x_1)(0)=c_1$ say with $c_1\neq 0$, and $(\partial f_1/\partial x_2)(0)=0$, so $(\partial f_1/\partial x_2)(g_1(x_3),g_2(x_3),x_3)$ can be written $x_3h_1(x_3)$. Similarly $(\partial f_2/\partial x_2)(0)=c_2\neq 0$ and $(\partial f_2/\partial x_1)(g_1(x_3),g_2(x_3),x_3)=x_3h_2(x_3)$. Using (1) when $i=2$ it follows that if $j^r(\partial g_2/\partial s)\neq 0$, then $j^{r-1}(\partial g_1/\partial s)\neq 0$. Taking r-jets of (1) when $i=1$, for $0\leqq r\leqq p$ we now find that $c_1\cdot j^p(\partial g_1/\partial s)=-x_3^p$, and so $j^k(\partial\tilde{H}/\partial s)=-(c_1)^{-1}x_3^p+0(p+1)$, where $0(p+1)$ denotes a polynomial of degree $\geqq p+1$. Since $c_1\neq 0$ the result now follows.

Of course working with the other family G, and the associated $\tilde{G}$ we will obtain a submersion provided $d_2\geqq k$, by using tangent vectors of the form $(f_1,f_2+sx_3^p)$. Now using the compactness of C, and Thom's fundamental transversality lemma one can deduce, just as in [1], the following theorem.

Theorem 2.2. *Provided d_1 and d_2 are $\geqq 4$ for (f_1,f_2) in an open dense subset of W the dual of $C=\{f_1=f_2=0\}$ is locally generic i.e. ignoring quasi global self intersections the dual has the local structure of the discriminant variety of an A_1, A_2 or A_3 singularity.*

Proof. We only need to consider the jet space $J^4(\mathbb{C},\mathbb{C})$ and transversality to the orbits of $0, x, x^2, x^3$ and x^4.

Exactly the same ideas will give similar results for arbitrary complete intersections. If in some affine chart the variety V is given by $f_1(x_1,\ldots,x_n)=\cdots=f_r(x_1,\ldots,x_n)=0$, with the tangent space at 0 to $f_i=0$ given by $x_i=0$ we can parametrise V locally by $(g_1(x_{r+1},\ldots,x_n),\ldots,g_r(x_{r+1},\ldots,x_n),x_{r+1},\ldots,x_n)$. Considering $\tilde{H}:\mathbb{C}^{n-r}\times\mathbb{C}^{n-1}\times W\to\mathbb{C}$ defined by $\tilde{H}(u,a,f)=g_1(u,f)+a_2g_2(u,f)\ldots+a_rg_r(u,f)+a_{r+1}u_{r+1}+\cdots+a_nu_n$ we find that the jet extension $j_1^k\tilde{H}$ is a submersion at $(0,a,(f_1,\ldots,f_n))$ for any $a\in\mathbb{C}^{n-1}$ if the degree of $f_1, d_1\geqq k$. This time one uses the paths $(f_1+s\phi(x_{r+1},\ldots,x_n),f_2,\ldots,f_r)$ where the ϕ's are homogeneous polynomials of degree p, $1\leqq p\leqq k$. One can deduce

Theorem 2.3. *For $n\leqq 6$ the duals of an open dense set of complete intersections in $\mathbb{P}^n$ have the local structure of the discriminant varieties of simple singularities provided the degrees d_i of the defining equations satisfy $d_i\geqq n$. For curves which are complete intersections the same holds for any n, and the simple singularities are of type A_p, $1\leqq p\leqq n$.*

Proof. This follows from the fact that for $k\geqq n, n>m, J_0^k(m,1)$ has a stratification by the orbits of simple singularities and strata of codimension $>m+n-1$, so since we can ensure transversality of each of the contact maps with the stratification the result follows. The assertion for curves follows because the only orbits of $J_0^k(1,1)$ are those of $0, x$ and type A_p, $1\leqq p\leqq k-1$.

Remarks 2.4.

(i) Clearly all of the above discussion also works for real complete intersections in real projective n space $\mathbb{RP}^n$.

(ii) In proving that the jet extension map $j_1^k\tilde{H}$ is a submersion we used very few of the tangent vectors to W at $(f_1,\ldots,f_r)$, indeed only some of those from the first component. Consequently one might hope that one could relax the conditions on the d_i and still obtain a submersion. For example perhaps Theorem 2.2 holds whenever $d_1, d_2\geqq 2$? (It is clearly false if either $d_i=1$.) Unfortunately although this may be the case the techniques used in this paper will not by themselves give a proof. For a computation along the lines of that given in Proposition 2.1 shows that (with the notation used there) when $a_2=0$ and $d_1=2$ the only tangent vectors one can obtain in $J_0^4(1,1)$ are those with initial terms $x_3, x_3^2, g_1, g_2, x_3g_1, x_3g_2, g_1^2, g_1g_2, g_2^2$. So if g_1 and g_2 vanish to sufficiently high order we do not obtain a submersion onto $J_0^4(1,1)$. In other words for $d_1=2$ (or indeed $d_1=3$) we need some further information on the contact of the curve with its tangent planes before our methods are to be of any use.

(iii) For $n>6$ for both smooth submanifolds of $\mathbb{R}^n$ and complete intersections in $\mathbb{P}^n$ one has to replace the stratification by orbits of simple singularities by

Looijengas canonical stratification because one encounters uncountably many orbits i.e. smooth moduli. Unfortunately apart from the simple orbits and some results on the simple elliptic families $\tilde{E}_6$ due to Wall [8] (and following Wall's ideas $\tilde{E}_7$ [3]) nothing is known about this stratification. In the algebraic case, where one is not interested in the Gauss map i.e. the bifurcation set but only in the dual i.e. the discriminant set of the singularity results of Looijenga, Wirthmuller and Damon ([5], [9], [4]) show that for many unimodular families the discriminants are all homeomorphic and one can obtain a corresponding extension of Theorem 2.3 for larger values of n.

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