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Non-autonomous higher-order Moreau's sweeping process: Well-posedness, stability and Zeno trajectories

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In this article, we study the higher-order Moreau's sweeping process introduced in [1], in the case where an exogenous time-varying function $u(\cdot)$ is present in both the linear dynamics and in the unilateral constraints. First, we show that the well-posedness results (existence and uniqueness of solutions) obtained in [1] for the autonomous case, extend to the non-autonomous case when $u(\cdot)$ is smooth and piece-wise analytic, after a suitable state transformation is done. Stability issues are discussed. The complexity of such non-smooth non-autonomous dynamical systems is illustrated in a particular case named the higher-order bouncing ball, where trajectories with accumulations of jumps are exhibited. Examples from mechanics and circuits illustrate some of the results. The link with complementarity dynamical systems and with switching differential-algebraic equations is made.

Key words: 34A60, 34A38, 34H15, 93D05, 93C30

1 Introduction

The sweeping processes are a well-known class of differential inclusions, introduced by Moreau [22,23], and which has had numerous extensions since then (see, e.g., [3,15,20,30] and references therein). The so-called *higher-order sweeping process* (HOSP) was introduced in [1]. The primary objective of the HOSP is to settle a dynamical formalism, which provides a mathematical framework for a state or state-control unilaterally constrained dynamical system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + Eu(t) \\ 0 \leqslant w(t) = Cx(t) + Fu(t) \end{cases} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u : \mathbb{R}_+ \mapsto \mathbb{R}^p$ is an input or exogenous disturbance, $\lambda(t) \in \mathbb{R}^m$ is a Lagrange multiplier, $w(t) \in \mathbb{R}^m$ is an 'output' signal, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $E \in \mathbb{R}^{n\times p}$, $F \in \mathbb{R}^{m\times p}$. These systems are square because λ and w necessarily have the same dimension, λ being a Lagrange multiplier associated with the constraint $w(t) \geqslant 0$ for all $t \geqslant 0$. The complete analysis of the HOSP (including time-discretization) is done in the autonomous case [i.e., $u(\cdot) = 0$ in (1.1)] in [1] by embedding (1.1) into a specific *Distribution Differential Inclusion* (DDI) that is an extension of the second-order sweeping process, which is tailored to Lagrangian systems. The applications may be found in optimal control with state inequality constraints [10] as well as in feedback control of circuits [1, Section 6]. The analysis is also close to viability studies [4, 17] if one thinks of λ as an exogenous input (and not as a Lagrange multiplier associated with the inequality constraint). Within this framework the term $B\lambda + Eu(\cdot)$ may be considered as an input whose distributional part is λ while its function part is $u(\cdot)$.

Consider for instance the system $\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = x_3(t)$, $\dot{x}_3(t) = u(t)+\lambda(t)$, $0 \leqslant x_1(t)$. Suppose that $u(\cdot)$ is a bounded function, and that $x_2(0) < 0$, $x_1(0) = 0$, $x_3(0) = 0$. If $\lambda(\cdot)$ is a bounded function, then $x_1(t)$ becomes negative in a right neighbourhood of $t = 0$. The only solution to keep the non-negativity of $x_1(\cdot)$, is to make $x_2(\cdot)$ jump to a non-negative value at $t = 0$. This implies that $x_3(\cdot)$ must be a Dirac measure with atom at $t = 0$, and λ is the derivative of a Dirac measure. Consequently the dynamical system has to be interpreted in a suitable way to make sense, as it involves in a natural way distributions of higher degree, and the state jumps have to be incorporated in the analysis. Finally, in-between state jumps, one has to guarantee that the trajectories do not leave the admissible domain $w(t) \geqslant 0$, when the system 'slides' on the boundary $w(t) = 0$. Again this can be done with a suitable choice of the multiplier λ, which acts as a 'force' that keeps the state inside the admissible domain.

The HOSP allows to treat all system's modes ('sliding contact', state jumps, free-motion inside the admissible domain) in a single dynamical formalism. It indicates how to design λ to assure the positive invariance of the moving polyhedral set $\Phi_u \stackrel{\Delta}{=} \{x \in I\!R^n \mid Cx + Fu(t) \geqslant 0\}$. Consequently it is of interest to characterize in a precise way the nature of the solutions. The functional framework for the autonomous HOSP is carefully introduced in [1], where solutions are a subclass of Schwartz distributions constructed from functions of local special bounded variation. Whether or not the solutions possess accumulations of state jumps is an important feature. In this article, it is shown with specific input functions $u(\cdot)$ that left-accumulations of state jumps (i.e., accumulations on the left of some time) may occur in the non-autonomous HOSP. The basic idea is to consider systems which are 'higher order bouncing balls'. More specifically, one may view them as a chain of integrators with a constant input, and a specific 'impact law', which acts on the derivatives of the constrained signal $w(\cdot)$. In the mechanical bouncing ball system, $w(\cdot)$ is the continuous position of the ball, whereas $\dot{w}(\cdot)$ is its discontinuous velocity.

This article is organized as follows. Section 2 is devoted to the analysis of a state transformation, which allows to recast (1.1) into a suitable canonical form for the subsequent existence and uniqueness of solutions analysis. Section 3 recalls the HOSP framework in which (1.1) is embedded, and states the well-posedness of the non-autonomous HOSP. In Section 4, the relationships between the HOSP and complementarity systems, as well as switching DAEs, are explained. In Section 5, we study a particular case of the non-autonomous HOSP (named the higher-order bouncing ball for obvious analogy with Mechanics), and we show that the so-called restitution coefficients play a crucial role in the dynamical behaviour. Section 6 deals with the existence of equilibria, stability, and positive invariance. Conclusions end the article in Section 7, and some auxiliary mathematics are in the Appendix.

Notations and definitions: The indicator function of a set $\Phi \subset I\!R^n$ is defined as $\psi_\Phi(x) = 0$ if $x \in \Phi$, $\psi_\Phi(x) = +\infty$ if $x \notin \Phi$. When Φ is closed, non-empty and convex, so is $\psi_\Phi(\cdot)$ and its

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subdifferential $\partial\psi_\Phi : I\!R^n \rightrightarrows I\!R^n$ is a set-valued mapping, equal to the normal cone $N_\Phi(x)$ to Φ at x. We have $N_\Phi(x) = \{0\}$ if x is in the interior of Φ. A matrix M is positive definite ($M > 0$) if and only if $x^T M x > 0$ for all $x \neq 0$. The projection of a vector $x \in I\!R^n$ on Φ, in the metric defined by $M = M^T > 0$, is denoted as $\mathrm{proj}_M[\Phi; x]$. The lexicographical inequality: $(x_1, x_2, \ldots, x_n) \succcurlyeq 0$ means that either all $x_i = 0$, or the first non-zero $x_i > 0$. Let n and r be integers. The $n \times n$ identity matrix is I_n, $0^n = (0, 0, \ldots, 0)^T \in I\!R^{n\times 1}$, $0_n = (0, 0, \ldots, 0) \in I\!R^{1\times n}$, $0_{n\times r} \in I\!R^{n\times r}$ is the zero $n \times r$ matrix. Let M be a matrix with n rows, then $M_{\bar{r}} \in I\!R^{r\times n}$ denote the first r rows of M, $M_{\underline{p}}$ its last p rows. For a matrix M, $M_{i\bullet}$ is its ith row, $M_{\bullet i}$ is its ith column. A square matrix $M \in I\!R^{n\times n}$ is a Stieltjes matrix if [16, Definition 3.11.1]: $M = M^T$, $M_{ij} \geqslant 0$ for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $i \neq j$, and M is a P-matrix (hence, it is positive definite since it is symmetric). If M is a Stieltjes matrix, then M^{-1} is symmetric non-negative, i.e. the entries $(M^{-1})_{ij}$ are non-negative for all i, j. For a square matrix M, $\lambda_{\max}(M)$ denotes its largest eigenvalue, $\lambda_{\min}(M)$ denotes its smallest eigenvalue.

2 State transformation into a canonical form

The analysis of the autonomous HOSP with $u(\cdot) = 0$ is made in [1] from a specific state-space representation which allows to settle a suitable functional set of potential solutions that are Schwartz' distributions. In this section, we show how to extend this canonical transformed dynamics for (1.1), which will be useful for the well-posedness analysis. Let $m = 1$ and let the transfer function $C(sI_n - A)^{-1}B \neq 0$, $s \in \mathbb{C}$. Then, there exists $1 \leqslant r \leqslant n$, that is the relative degree between w and λ. In other words, $CA^{i-1}B = 0$ for all $1 \leqslant i \leqslant r-1$ and the scalar $CA^{r-1}B \neq 0$. Let us assume that $u(\cdot)$ is r-times differentiable, and let us denote $\mathcal{U}(t) = (u(t)^T, \dot{u}(t)^T, ..., u^{(r-1)}(t)^T)^T \in I\!R^{rp}$, and $\mathcal{W}(t) = (u(t)^T, \dot{u}(t)^T, \ldots, u^{(r)}(t)^T)^T \in I\!R^{(r+1)p}$. Let us perform the extended state transformation[1]:

$$z = Wx + T\mathcal{U} \tag{2.1}$$

with $z = \begin{pmatrix} \bar{z} \\ \xi \end{pmatrix}$, $\bar{z} = (z_1, z_2, \ldots, z_r)^T$, $\xi \in I\!R^{n-r}$, and where

$$z_i(t) = CA^{i-1}x(t) + \sum_{j=0}^{i-2} CA^j E u^{(i-2-j)}(t) + Fu^{(i-1)}, \tag{2.2}$$

with $2 \leqslant i \leqslant r$, $z_1(t) = w(t) = Cx(t) + Fu$. Notice that $\dot{z}_i = z_{i+1}$, $1 \leqslant i \leqslant r-1$. Due to the existence of a relative degree between w and λ, there exists a matrix $W \in I\!R^{n\times n}$, which is full-rank and such that [27]

$$WB = \begin{pmatrix} 0^{r-1} \\ CA^{r-1}B \\ 0^{n-r} \end{pmatrix} \in I\!R^n, \quad CW^{-1} = \begin{pmatrix} 1 & 0_{n-1} \end{pmatrix} \in I\!R^{1\times n} \tag{2.3}$$

[1] 'Extended' refers here to the fact that this transformation involves both the state and the exogenous term.

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$$WAW^{-1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0_{n-r} \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & \dots & 0 & 1 & 0_{n-r} \\ d_1 & d_2 & d_3 & \dots & d_r & d_\xi^T \\ B_\xi & 0^{n-r} & 0^{n-r} & \dots & 0^{n-r} & A_\xi \end{pmatrix}, \tag{2.4}$$

where $A_\xi \in \mathbb{R}^{(n-r)\times(n-r)}, B_\xi \in \mathbb{R}^{(n-r)\times 1}$, and $(d^T, d_\xi^T) = (CA^rW^{-1})^T$ with $d = (d_1, \dots, d_r)^T$. Moreover, from the definition of the variables z_i in (2.2), we have

$$T = \begin{bmatrix} F & 0_p & 0_p & \dots & \dots & 0_p & 0_p \\ CE & F & 0_p & \dots & 0_p & 0_p & 0_p \\ CAE & CE & F & \dots & 0_p & 0_p & 0_p \\ \vdots & & & & & \vdots & \vdots \\ \vdots & & & & & F & 0_p \\ CA^{r-2}E & CA^{r-3}E & \dots & \dots & CAE & CE & F \\ 0_{(n-r)\times p} & 0_{(n-r)\times p} & \dots & \dots & \dots & 0_{(n-r)\times p} & 0_{(n-r)\times p} \end{bmatrix} \in \mathbb{R}^{n\times rp}. \tag{2.5}$$

The matrix W being full-rank, the state transformation is bijective and $x = W^{-1}(z - T\mathcal{U})$. This allows one to transform (1.1) into the following canonical form:

$$\begin{cases} \dot z_1(t) = z_2(t) \\ \dot z_2(t) = z_3(t) \\ \vdots \\ \dot z_{r-1}(t) = z_r(t) \\ \dot z_r(t) = CA^rW^{-1}z(t) - CA^rW^{-1}T\mathcal{U}(t) + CA^{r-1}B\lambda + \sum_{i=0}^{r-1} CA^iEu^{(r-1-i)}(t) + Fu^{(r)} \\ \dot\xi(t) = A_\xi\xi(t) + B_\xi z_1(t) + G_\xi\mathcal{U}(t) \\ \\ 0 \leqslant w(t) = z_1(t) \end{cases} \tag{2.6}$$

for some matrix $G_\xi \in \mathbb{R}^{(n-r)\times rp}$ such that $G_\xi\mathcal{U}(t) = -\underline{W_{n-r}}AW^{-1}T\mathcal{U}(t) + \underline{W_{n-r}}Eu(t) + \underline{T_{n-r}}\dot{\mathcal{U}}(t)$. This stems from the fact that $\dot\xi = \underline{W_{n-r}}\dot x + \underline{T_{n-r}}\dot{\mathcal{U}}$. Noting that $\underline{T_{n-r}} = 0$, we obtain $\dot\xi = \underline{W_{n-r}}AW^{-1}z - \underline{W_{n-r}}AW^{-1}T\mathcal{U} + \underline{W_{n-r}}B\lambda + \underline{W_{n-r}}Eu$, and using (2.3) and (2.4) yields the result (in particular $\underline{W_{n-r}}B = 0_{(n-r)\times m}$). In Systems and Control the $\xi-$dynamics is called the *zero dynamics*. As we shall see later, the zero-dynamics plays a crucial role in the system's behaviour when trajectories evolve on the boundary of Φ_u. Notice that adding the term $T\mathcal{U}$ in the state transformation is necessary to obtain the chain of integrators in (2.6), but the zero-dynamics depends in general on $u(\cdot)$ and its derivatives. Let us denote $\sum_{i=0}^{r-1} CA^iEu^{(r-1-i)}(t) + Fu^{(r)} - CA^rW^{-1}T\mathcal{U}(t) = \bar G\mathcal{W}(t)$ for a suitable constant row vector $\bar G \in \mathbb{R}^{1\times(r+1)}$. One can then rewrite the rth line of (2.6) as

$$\dot z_r(t) = CA^rW^{-1}z(t) + CA^{r-1}B\lambda + \bar G\mathcal{W}(t) \;\; (= d^T\bar z + d_\xi^T\xi + CA^{r-1}B\lambda + \bar G\mathcal{W}(t)), \tag{2.7}$$

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where $\bar{G} \in I\!R^{1\times(r+1)p}$, and the dynamics in (2.6) more compactly as

$$\dot{z}(t) = WAW^{-1}z(t) + WB\lambda(t) + H\mathcal{W}(t) \tag{2.8}$$

where $H \in I\!R^{n\times(r+1)p}$ is a suitable matrix obtained from $\bar{G}\mathcal{W}$ and $W_\xi\mathcal{U}$ by grouping the terms of equal derivation index in $\mathcal{U}$ and $\mathcal{W}$. Obviously we may also calculate (2.8) from

$$\dot{z}(t) = WAW^{-1}z(t) + WB\lambda(t) + WEu(t) + T\dot{\mathcal{U}}(t) - WAW^{-1}T\mathcal{U}(t) \tag{2.9}$$

so that $H\mathcal{W} = WEu + T\dot{\mathcal{U}} - WAW^{-1}T\mathcal{U}$. Notice that one may have $T = 0$ but this does not mean that the transformed dynamics is independent of $u(\cdot)$. We still assume that $m = 1$ and that $e_i \geqslant 0$, $1 \leqslant i \leqslant r$.

Example 2.1 *Let us consider the system with $n = 4$, $p = 1$,*

$$A = \begin{pmatrix} 2 & 7 & 3-2\alpha & -2\beta+2 \\ -1 & -3 & -1+\alpha & \beta-1 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix},$$

where $\alpha, \beta \in I\!R$, $F = 0$ and $E = (1\ 1\ 1\ 1)^T$. We have $CB = 0$ and $CAB = 1$, and hence $r = 2$. The transformed dynamics may be obtained from (2.9) as

$$\begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = \lambda(t) - z_1(t) - z_2(t) + \alpha\xi_1(t) + \beta\xi_2(t) + (5-2\alpha)u(t) + 3\dot{u}(t) \\ \dot{\xi}_1(t) = \xi_2(t) + u(t) + 2\dot{u}(t) \\ \dot{\xi}_2(t) = \xi_1(t) + z_1(t) + 3u(t) \\ 0 \leqslant w(t) = z_1(t) \end{cases} \tag{2.10}$$

In this example, one has $W = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $T = (0\ 3\ 2\ 0)^T$. The derivative of $u(\cdot)$ enters the $\xi-$dynamics, which in turn enters the $\bar{z}$-dynamics in $\dot{z}_2(t)$.

Other examples can be found in Section 4.3. It is worth remarking that the $\xi-$dynamics may enter the $\bar{z}$-dynamics as shown in (2.10). Therefore, the well-posedness analysis has to be made from the most general case in (2.6).

3 Embedding into the HOSP

In this section, we briefly recall the mathematical tools and formalisms, which are necessary to construct the HOSP, a particular DDI. First, the set of solutions is introduced, then the DDI, the state-jump mapping and the link with complementarity are presented. The complete developments can be found in [1, Sections 2–4].

3.1 The space of distributional solutions

Let $I = [\alpha, \beta]$, $\alpha \in I\!R$, $\beta \in I\!R \cup \{+\infty\}$, be a real non-degenerate interval. We denote as $\mathcal{T}_n(I)$ the set of distributions of degree $n+1$, which are generated by RCSLBV functions on

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I [see (B 1) and (B 2) in Appendix B], whose successive derivatives possess an absolutely continuous part (denoted as $[\cdot]$), that is also RCSLBV on I. More precisely, the right derivative of the absolutely continuous component $[h]$ of $h(\cdot)$ is denoted as $\hat{h}^{(1)} \stackrel{\Delta}{=} \frac{d^+[h]}{dt}(t) = \lim_{\sigma\to 0^+} \frac{[h](t+\sigma)-[h](t)}{\sigma}$. The set of such functions is denoted as $\mathcal{F}_\infty(I;\mathbb{R}) = \cap_{k\in\mathbb{N}}\mathcal{F}_k(I;\mathbb{R})$, with

$$\mathcal{F}_k(I;\mathbb{R}) = \{h \in \mathcal{F}_{k-1}(I;\mathbb{R}) \mid \hat{h}^{(k)} \stackrel{\Delta}{=} \frac{d^+}{dt}[\hat{h}^{(k-1)}] \in RCSLBV(I;\mathbb{R})\}. \tag{3.1}$$

In particular $\mathcal{F}_0(I;\mathbb{R}) = RCSLBV(I;\mathbb{R})$, and $\mathcal{F}_1(I;\mathbb{R}) = \{h \in \mathcal{F}_0(I;\mathbb{R}) \mid \hat{h}^{(1)} \in RCSLBV(I;\mathbb{R})\}$. Furthermore, $\hat{h}^{(1)} = [\hat{h}^{(1)}] + \mathcal{J}_{\hat{h}^{(1)}}$ (see (B 1)), and $\mathcal{F}_2(I;\mathbb{R}) = \{h \in \mathcal{F}_1(I;\mathbb{R}) \mid \hat{h}^{(2)} \stackrel{\Delta}{=} \frac{d^+}{dt}[\hat{h}^{(1)}] \in RCSLBV(I;\mathbb{R})\}$.

If the distribution $T \in \mathcal{T}_n(I)$ and is generated by a function $F \in \mathcal{F}_\infty(I;\mathbb{R})$, it has a 'function' part denoted as $\{T\}(\cdot) = [\hat{F}^{(n)}](\cdot)$, and a 'measure' part denoted as $\ll T \gg$ such that $\langle \ll T \gg, \varphi\rangle = \int_{-\infty}^{+\infty} \varphi \, d[\hat{F}^{(n-1)}]$, for all $\varphi \in C_0^\infty(I)$. $\mathbf{D}$ denotes the distributional derivative, and dz denotes the Stieltjes or differential measure generated by a function z of local bounded variation (see Appendix B), while $C_0^\infty(I)$ is the space of real-valued $C^\infty(I)$ mappings with compact support contained in $]\alpha,\beta[$. Thus, given $n \in \mathbb{N}$, $\mathcal{T}_n(I)$ denotes the set of all Schwartz' distributions such that there exists a function $F \in \mathcal{F}_\infty(I;\mathbb{R})$ such that $T = \mathbf{D}^n F$. We have $\mathcal{T}_0(I) = \mathcal{F}_\infty(I;\mathbb{R})$. Let n be the smallest integer such that $T \in \mathcal{T}_n(I)$, we set the degree of T as

$$\deg(T) = \begin{cases} n+1 \text{ if } n \geqslant 1 \\ 1 \text{ if } n = 0 \text{ and } E_0(\{T\}) \neq \emptyset \\ 0 \text{ if } n = 0 \text{ and } E_0(\{T\}) = \emptyset \end{cases} \tag{3.2}$$

where $E_0(f)$ denotes the set of points of discontinuity of the function f. Distributions of degree $n = 0$ are continuous functions in $\mathcal{F}_\infty(I;\mathbb{R})$, those of degree $n = 1$ are discontinuous functions in $\mathcal{F}_\infty(I;\mathbb{R})$. The nth derivative of the Dirac measure, $\delta_t^{(n)}$ ($t \in I$) is of degree $n+2$. This concept of solutions is an extension of the case of Non-smooth Mechanics, where positions are locally absolutely continuous, velocities are RCLBV, accelerations are the differential measures of the velocities [20].

Finally, let us define the set $\mathcal{T}_\infty(I) = \cup_{n\in\mathbb{N}}\mathcal{T}_n(I)$: a Schwartz distribution belongs to $\mathcal{T}_\infty(I)$ if there exist $n \in \mathbb{N}$ and $F \in \mathcal{F}_\infty(I;\mathbb{R})$ such that $T = \mathbf{D}^n F$. This set contains Bohl distributions, which are used elsewhere for the analysis of Linear Complementarity Systems (LCS) [14]. Within this functional framework for solutions, the system's state is allowed to be discontinuous with accumulations of discontinuity times; however, the set of state-jump times is countable. For examples of functions in $\mathcal{F}_\infty(I;\mathbb{R})$ and distributions in $\mathcal{T}_\infty(I)$, see [1, Examples 1, 2, 3].

3.2 The distribution differential inclusion

Let us first recall that, in order to simplify the presentation we shall continue to assume that $m = 1$. In [1, Remarks 3, 7, 20] it is indicated how the material extends to the multivariable (MIMO) case $m \geqslant 2$, when $CA^{r-1}B$ is a Stieltjes matrix. We shall come back on the MIMO case in Section 4.1. Let K be a non-empty closed convex subset of

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$\mathbb{R}$. We denote by $T_K(x)$ the tangent cone of K at $x \in \mathbb{R}$ defined by

$$T_K(x) = \overline{\text{cone}}(K - \{x\}) \tag{3.3}$$

where cone$(K - \{x\})$ denotes the cone generated by $K - \{x\}$ and $\overline{\text{cone}}(K - \{x\})$ denotes the closure of cone$(K - \{x\})$, i.e. $\overline{\text{cone}}(K - \{x\}) = \overline{\text{cone}(K - \{x\})}$. The definition in (3.3) allows us to take into account constraints violations. Note that $T_{\mathbb{R}_+}(x) = \begin{cases} \mathbb{R} & \text{if} \quad x > 0 \\ \mathbb{R}_+ & \text{if} \quad x \leqslant 0 \end{cases}$ and $T_{\mathbb{R}}(x) = \mathbb{R}$. Let us now set $\Phi \triangleq \mathbb{R}_+$. For $z \in \mathbb{R}^r$, we set $Z_i = (z_1, z_2, \ldots, z_i)$, $(1 \leqslant i \leqslant r)$. By convention, we set $Z_0 = 0$ and $T^0_\Phi(Z_0) = \Phi$, and we define $T^1_\Phi(Z_1) = T_\Phi(z_1)$, $T^2_\Phi(Z_2) = T_{T^1_\Phi(Z_1)}(z_2)$, $T^r_\Phi(Z_r) = T_{T^1_\Phi(Z_{r-1})}(z_r)$ (so that $Z_r = \bar{z}$). To summarize

$$T^i_\Phi(Z_i) = T_{T^{i-1}_\Phi(Z_{i-1})}(z_i), \text{ for all } 1 \leqslant i \leqslant r. \tag{3.4}$$

It follows that $z_1 > 0 \Rightarrow T^i_\Phi(Z_i) = \mathbb{R}$ and $\partial\psi_{T^i_\Phi(Z_i)}(\cdot) = \{0\}$ for all $1 \leqslant i \leqslant r$. Moreover assume that there exists $j \in \{1, r-1\}$ such that $z_1 \leqslant 0, \ldots, z_j \leqslant 0$, and $z_{j+1} > 0$. Then, $T^0_\Phi(Z_0) = T^1_\Phi(Z_1) = \ldots = T^j_\Phi(Z_j) = \mathbb{R}_+$, and $T^{j+1}_\Phi(Z_{j+1}) = \ldots = T^{r-1}_\Phi(Z_{r-1}) = \mathbb{R}$. Let us finally remind that $\partial\psi_{\mathbb{R}}(x) = \{0\}$ for all reals x, $\partial\psi_{\mathbb{R}_+} = \{0\}$ if $x > 0$, $\partial\psi_{\mathbb{R}_+}(0) = \mathbb{R}^-$.

We now pass to the DDI. Let $T > 0$, $T \in \mathbb{R} \cup \{+\infty\}$ be given and set $I = [0, T[$. Let $z_0^T = (\bar{z}_0^T, \xi_0^T)$ be given in $\mathbb{R}^n$ with $\bar{z}_0 \in \mathbb{R}^r$ and $\xi_0 \in \mathbb{R}^{n-r}$. We also introduce a set $(e_1, \ldots, e_r)$ of r real numbers named *restitution coefficients* from an obvious analogy with Mechanics. The choice of these coefficients depends on the application (for instance in Mechanics the restitution applied to the velocity belongs to $[0, 1]$ because of the kinetic energy dissipation and kinematic consistency [12]). Let us denote

$$\zeta_i(t) = \frac{\{z_i\}(t^+) + e_i\{z_i\}(t^-)}{1 + e_i}, \quad 1 \leqslant i \leqslant r. \tag{3.5}$$

The DDI formalism is as follows:

Problem HOSP$(z_0;I)$: Find $z_1, \ldots, z_r \in \mathcal{T}_\infty(I)$ and $\xi_i \in \mathcal{T}_\infty(I)$ $(1 \leqslant i \leqslant n-r)$, satisfying the distributional differential equations:

$$\begin{cases} \mathbf{D}z_1 - z_2 = 0 \\ \mathbf{D}z_2 - z_3 = 0 \\ \mathbf{D}z_3 - z_4 = 0 \\ \vdots \\ \mathbf{D}z_{r-1} - z_r = 0 \\ \mathbf{D}z_r - CA^rW^{-1}\{z\} - \bar{G}\mathcal{W} = CA^{r-1}B\lambda \\ \mathbf{D}\xi = A_\xi\xi + B_\xi z_1 + G_\xi\mathcal{U}, \end{cases} \tag{3.6}$$

$$\begin{aligned} \lambda = \; & (CA^{r-1}B)^{-1}\left[\sum_{i=1}^{r-1} \mathbf{D}^{(r-i)} \ll \mathbf{D}z_i - \{z_{i+1}\} \gg\right] + \ll \mathbf{D}z_r - CA^rW^{-1}\{z\} \gg \\ & -(CA^{r-1}B)^{-1}\bar{G}\mathcal{W}, \end{aligned} \tag{3.7}$$

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and satisfying the Measure Differential Inclusion (MDI) on $]0,T[$:

$$\begin{cases} d\{z_1\} - \{z_2\}(t)dt \in -\partial\psi_{\Phi}(\zeta_1(t)), \\ d\{z_2\} - \{z_3\}(t)dt \in -\partial\psi_{T^1_{\Phi}(\{Z_1\}(t^-))}(\zeta_2(t)), \\ \vdots \\ d\{z_{r-1}\} - \{z_r\}(t)dt \in -\partial\psi_{T^{r-2}_{\Phi}(\{Z_{r-2}\}(t^-))}(\zeta_{r-1}(t)), \\ (CA^{r-1}B)^{-1}[d\{z_r\} - CA^rW^{-1}\{z\}(t)dt - \bar{G}\mathcal{W}dt] \in -\partial\psi_{T^{r-1}_{\Phi}(\{Z_{r-1}\}(t^-))}(\zeta_r(t)), \end{cases} \tag{3.8}$$

and the initial conditions:

$$\begin{cases} \{z_1\}(0^+) - z_1(0^-) \in -\partial\psi_{\Phi}(\zeta_1(0)), \\ \{z_2\}(0^+) - z_2(0^-) \in -\partial\psi_{T^1_{\Phi}(Z_1(0^-))}(\zeta_2(0)), \\ \vdots \\ \{z_{r-1}\}(0^+) - z_{r-1}(0^-) \in -\partial\psi_{T^{r-2}_{\Phi}(Z_{r-2}(0^-))}(\zeta_{r-1}(0)), \\ (CA^{r-1}B)^{-1}[\{z_r\}(0^+) - z_r(0^-)] \in -\partial\psi_{T^{r-1}_{\Phi}(Z_{r-1}(0^-))}(\zeta_r(0)), \\ \{\xi\}(0^+) = \xi_0. \end{cases} \tag{3.9}$$

(see Remark 3.1 about the initial data z_0 definition). The rationale behind the expression of λ in (3.7), is that this is a distribution whose degree depends on which of the state components z_i, $1 \leqslant i \leqslant r$, are discontinuous. The fact that the state-space representation uses a chain of integrators, explains the term between brackets in (3.7). For instance, a jump in z_1 will propagate through the differentiations and induce a distribution of degree 2 in z_2 (a Dirac measure), or degree 3 in z_3, of degree r in z_r, and finally of degree $r+1$ in λ. For more details see [1, Equations (32)–(40), Example 7]. In the MDI formalism (3.8), one considers only the measure parts of the distributions in order to give a meaning to the inclusions into normal cones. In view of (3.6), (3.7) and (3.8), it is legitimate to name the HOSP a *DDI*.

The rationale behind the choice for the normal cones to the tangent cones in the right-hand sides of the differential inclusions in (3.8), is that this guarantees, as we will see in the next sections, that any solution of the HOSP satisfies $z_1(t)(= w(t) = Cx(t) + Fu(t)) \geqslant 0$ for all $t > 0$ (i.e., except possibly initially on the left of $t = 0$) even if some of the derivatives of $z_1(\cdot)$ tend to make it leave this admissible domain. One can view this in the n-dimensional state space, as trajectories 'grazing' the admissible domain boundary $w = 0$ with a certain degree of tangency [that corresponds to the number of null derivatives of $z_1(\cdot)$]: the selections inside the normal cones as defined in (3.8) secure that if the first non-zero derivative has a negative sign on the left of t, then it jumps to a non-negative value on the right of t. Actually, the construction of the normal cones sequence imposes the lexicographical inequality $\{\bar{z}\}(t^+) \succcurlyeq 0$. The fact that the measures $dv_i \overset{\Delta}{=} d\{z\}_i - \{z_{i+1}\}dt$, $1 \leqslant i \leqslant r-1$, do not appear in the right-hand side of (2.8) [only λ does, see WB in (2.3)] stems from the fact that these measures are present in the MDI formalism only to take into account state re-initializations.

The set-valued functions in the right-hand side of (3.9) naturally extend the second-order sweeping process right-hand side (called Moreau's set [12]), and hence it is justified

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to name (3.6)–(3.9) a HOSP, though higher-relative-degree sweeping process could be more appropriate.

Remark 3.1 *If z denotes a solution of Problem* **HOSP**$(z_0;I)$*, we will write by convention that* $\{\bar{z}\}(0^-) = \bar{z}_0$, $\{\xi\}(0^-) = \xi_0$. *Then, the relations in (3.8) formulated on* $]0, T[$ *together with the initial conditions in (3.9) reduce to the relations in (3.8) formulated on* $I = [0, T[$.

The solutions of the Problem **HOSP**$(z_0;I)$ are distributions of a certain degree as the next proposition shows.

Proposition 3.1 [1, Proposition 3] *Let* $(z_1, \ldots, z_r, \xi) \in (\mathcal{T}_\infty(I))^n$ *be a solution of Problem* **HOSP**$(z_0;I)$. *Then,* $\deg(z_i) \leqslant i$ *for all* $1 \leqslant i \leqslant r$*, and* $z_1 = \{z_1\} \in \mathcal{F}_\infty(I;\mathbb{R})$, $\xi = \{\xi\} \in (\mathcal{F}_\infty(I;\mathbb{R}))^{n-r} \cap (C^0(I;\mathbb{R}))^{n-r}$.

The proof follows the same arguments as in [1, Example 1] and it is intuitively clear from (3.6), since $z_2 = Dz_1$, $z_3 = Dz_2 = D^2 z_1$, etc., while z_1 and ξ are functions of time (with z_1 possibly discontinuous).

3.3 The state jump mapping

Another way to write the MDI in (3.8) is as follows: find $z_1, \ldots, z_r, \xi_1, \ldots, \xi_{n-r} \in \mathcal{F}_\infty(I;\mathbb{R})$ such that

$$\begin{cases} dz_1 = z_2(t)dt + dv_1 \\ dz_2 = z_3(t)dt + dv_2 \\ \vdots \\ dz_{r-1} = z_r(t)dt + dv_{r-1} \\ dz_r = CA^r W^{-1} z(t)dt + \bar{G}\mathcal{W}(t)dt + CA^{r-1}B\, dv_r \\ d\xi = (A_\xi \xi(t) + B_\xi z_1(t) + G_\xi \mathcal{U}(t))dt, \end{cases} \tag{3.10}$$

where dv_i denotes Radon measures, dz_i ($1 \leqslant i \leqslant r$) is the differential measure generated by z_i, and the measures dv_i satisfy the inclusions [see (3.8)]:

$$dv_i \in -\partial\psi_{T_\Phi^{i-1}(\{Z_{i-1}\}(t^-))}(\zeta_i(t)) \quad \text{on } I, \text{ for all } 1 \leqslant i \leqslant r. \tag{3.11}$$

Roughly speaking, we retain only the measure part of the DDI, and the multiplier λ is replaced in the MDI formalism by the measure dv_r. It makes sense then to write inclusions into normal cones as in (3.8) or (3.11) since measures are signed while distributions are not (see [1, Section 2] for the rigorous mathematical meaning of the inclusions in (3.11)). The relationship between λ and dv_r is further understood by the fact that outside the atoms of the measures dv_i, $1 \leqslant i \leqslant r$, each measure satisfies $dv_i = \chi_i(t)dt$ for some function $\chi_i \in \mathcal{F}_\infty(I;\mathbb{R})$, and $\lambda = \chi_r(t)$.

The following holds [1, Propositions 4 and 5, Remark 15 (iii)].

Proposition 3.2 *Let* $m = 1$ *and* $CA^{r-1}B > 0$. *Let* z *be a solution of the Problem* **HOSP**$(z_0; I)$ *in (3.6)–(3.9). One has*

$$\begin{cases} \{z_i\}(t^+) - \{z_i\}(t^-) \in -\partial\psi_{T_\Phi^{i-1}(\{Z_{i-1}\}(t^-))}(\zeta_i(t)) & \text{for all} \quad 1 \leqslant i \leqslant r-1 \\ \{z_r\}(t^+) - \{z_r\}(t^-) \in -(CA^{r-1}B)\, \partial\psi_{T_\Phi^{r-1}(\{Z_{r-1}\}(t^-))}(\zeta_r(t)) & \text{for} \quad i = r \end{cases} \tag{3.12}$$

if and only if

$$\{z_i\}(t^+) = -e_i\{z_i\}(t^-) + (1+e_i)\mathrm{proj}\left[T_\Phi^{i-1}(\{Z_{i-1}\}(t^-)); \{z_i\}(t^-)\right]. \tag{3.13}$$

Moreover:

$$dv_i(\{t\}) = d\{z_i\}(\{t\}) - \{z_{i+1}\}(t)dt(\{t\}) = d\{z_i\}(\{t\}) = \{z_i\}(t^+) - \{z_i\}(t^-)$$

for all $1 \leqslant i \leqslant r-1$, *and*

$$\begin{aligned} (CA^{r-1}B)\, dv_r(\{t\}) &= d\{z_r\}(\{t\}) - CA^r W^{-1}\{z\}(t)dt(\{t\}) = d\{z_r\}(\{t\}) \\ &= \{z_r\}(t^+) - \{z_r\}(t^-). \end{aligned} \tag{3.14}$$

It follows that if $T_\Phi^i(\{Z_i\}(t^-)) = I\!R_+$ and if $\{z_{i+1}\}(t^-) < 0$, then $\{z_{i+1}\}(t^+) = -e_{i+1}\{z_{i+1}\}(t^-)$ so that sign$(\{z_{i+1}\}(t^+))$ =sign(e_{i+1}). We infer that $\{z_{i+1}\}(t^-) \geqslant 0 \Rightarrow e_{i+1} \geqslant 0$: only non-negative coefficients e_i bring the trajectories back in the admissible domain when a grazing trajectory tends to leave it. We therefore *choose* in the following $e_i \geqslant 0$ for all $1 \leqslant e_i \leqslant r$ (we shall see in Section 5 that further bounds may be imposed on e_i to better characterize the system's behaviour, see Lemma 5.1). In the case $m = 1$, both expressions in (3.12) are the same as long as $CA^{r-1}B > 0$; however, when $m \geqslant 2$ the expression for $i = r$ is crucial.

As alluded to above, in the HOSP formalism the positivity of the multiplier λ, that makes sense only if λ is a measure, is replaced by the positivity of the measure dv_r, which is the 'measure part' of the distribution λ. The definition of dv_r is clear from (3.14): it is defined from the discontinuity in the function part of z_r.

3.4 Existence and uniqueness of solutions

Following [1], let us first define the class of solutions.

Definition 3.1 Let $0 \leqslant a < b \leqslant T \leqslant +\infty$ be given. We say that a solution $z \in (\mathcal{T}_\infty([0,T[))^n$ of Problem **HOSP**$(z_0; [0,T[)$ is regular on $[a,b[$ if for each $t \in [a,b[$, there exists a right neighbourhood $[t, t+\sigma[$ $(\sigma > 0)$ such that the restriction of $\{z\}$ to $[t, t+\sigma[$ is analytic.

Regular solutions do not hamper the existence of Zeno behaviour with possible left accumulations of state jump times; however, they preclude accumulations on the right. Let us state the following fundamental assumption on the input $u(\cdot)$.

Assumption 3.1 *The function* $u : I\!R_+ \to I\!R^p$ *is a smooth, piece-wise analytic and bounded function, such that* $\sup_{s\in[0,T]} \|H\mathcal{W}(s)\|^2 \leqslant \alpha e^{\beta T}$ *for some* $\alpha > 0$, $\beta \in I\!R$.

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Such an assumption may appear quite stringent. However, the analyticity of the data has been shown to be necessary for the uniqueness of solutions of mechanical systems with unilateral constraints and impacts, see, e.g., [5, 9, 25, 28], preventing the appearance of infinity of state jumps on the right of some accumulation points. It is therefore a common assumption for systems with state unilateral constraints.

Theorem 3.2 *(Global existence and uniqueness in the class of regular solutions) Let* $CA^{r-1}B > 0$ *and Assumption 3.1 hold. For each* $z_0 \in \mathbb{R}^n$*, Problem* $\mathbf{HOSP}(z_0; [0, +\infty[)$ *has at least one regular solution* z *such that:* **i)** $z_1 \equiv \{z_1\} \geqslant 0$ *on* $[0, +\infty[$, **ii)** $||\{z\}(t)|| \leqslant \max(z_{\max}(t), e^{At}(||z_0 + \beta t))$ *for all* $t \in [0, +\infty[$, **iii)** *If* z^* *is a regular solution of problem* $\mathbf{HOSP}(z_0; [0, T^*[)$, $0 < T^* \leqslant +\infty$*, then* $\langle z^*, \varphi\rangle = \langle z, \varphi\rangle$ *for all* $\varphi \in C_0^\infty([0, T^*[; \mathbb{R}^n)$.

The proof of Theorem 3.2 can be found in [13, Section 4]. On the one hand, the complete proof is rather long; on the other hand, it happens that the proof of several key results is not changed compared with the autonomous case. Therefore, in the well-posedness proof, only the results of [1, Sections 4.3, 4.4, 4.6] need to be modified in order to comply with the non-autonomous case and $e_i > 0$ (only the case $e_i = 0$ is treated in [1]). It is still assumed that $m = 1$; however, the results still hold in the multivariable case $m \geqslant 2$, provided that the vector relative degree is $\bar{r} = (r, r, \ldots, r)^T$ for some $0 \leqslant r \leqslant n$, and the so-called decoupling matrix $CA^{r-1}B$ is a Stieltjes matrix. We shall give more details on the case $m \geqslant 2$ in Section 4.1.

4 The link with complementarity systems and switching DAEs

There is a close link between the DDI in (3.6)–(3.8) and complementarity systems. This stems from the MDI in (3.10). We may write each measure dv_i as

$$dv_i = \chi_i(t)dt + d\mathcal{J}_i, \tag{4.1}$$

where $\chi_i \in \mathcal{F}_\infty(I; \mathbb{R})$ and $d\mathcal{J}_i$ is an atomic measure with countable set of atoms generated by a right-continuous jump function $\mathcal{J}_i$.

Theorem 4.1 [1, Theorem 1, Remark 14] *Let* $m = 1$*, and* z *be a solution of Problem* $\mathbf{HOSP}(z_0; I)$*. Then, for each* $t \in I$*, we have*

$$0 \leqslant z_1(t^+) \perp dv_r(\{t\}) \geqslant 0. \tag{4.2}$$

The measure dv_i *is atomic, consequently* $\chi_i(t) = 0$ a.e. $t \in I$*, for all* $1 \leqslant i \leqslant r-1$*, while* $\chi_r(t) \in -\partial\psi_{T_\Phi^{r-1}(\{Z_{r-1}\}(t^-))}(\{z_r\}(t^+))$, a.e. $t \in I$*. Thus*

$$0 \leqslant z_1(t^+) \perp \chi_r(t) \geqslant 0, \text{ a.e. } t \in I. \tag{4.3}$$

This means that if $z_1(0) \geqslant 0$, then $z_1(t) \geqslant 0$ for all future times. However, the left limits of the derivatives of $z_1(\cdot)$ may take wrong signs at any time: the HOSP right-hand side takes care of bringing them back to non-negative values *via* the impact law (3.13), see Proposition 3.2. Here, we recover also the case of Mechanics because (4.3) implies that if

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$z_1(0) \geqslant 0$, then $z_1(t) \geqslant 0$ for all $t \in I$ and $dv_1 = 0$: the function $z_1(\cdot)$ is time continuous, and $z_2(\cdot)$ (the velocity) can have jumps. The fact that measures dv_i are atomic for all $1 \leqslant i < r-1$ follows from (3.10).

Using (3.6) and (3.7), the original dynamics (1.1) can therefore be written as the complementarity system with distributional solutions:

$$\begin{cases} \mathbf{D}x = A\{x\} + B\lambda + Eu \\ 0 \leqslant w(t) = Cx(t) + Fu(t) \perp \chi_r(t) \geqslant 0 \;\text{ for any } t \text{ that is not an atom of } dv_i,\ 1 \leqslant i \leqslant r. \\ 0 \leqslant w(t^+) \perp dv_r(\{t\}) \geqslant 0, \text{for all } t \in I. \end{cases} \tag{4.4}$$

The remaining part of this section is devoted also to show that the HOSP may be seen as a system of switching Differential Algebraic Equations (DAEs), where the switches are ruled by complementarity conditions (hence, are state dependent), as in LCS [14]. Other types of switching DAEs have been studied in [29], where switching times are exogenous and do not accumulate in finite time.

4.1 The canonical state space representation

The canonical transformation yielding (2.6) extends to $m \geqslant 2$, $m < n$, for systems with vector relative degree $\bar{r} = (r, r, .., r)^T \in \mathbb{R}^m$, which satisfy $CA^{i-1}B = 0$ for all $1 \leqslant i \leqslant r-1$, and $CA^{r-1}B \in \mathbb{R}^{m\times m}$ is non-singular. We shall adopt the notations: $\bar{z}_i \stackrel{\Delta}{=} (z_i^1, z_i^2, \ldots, z_i^r)^T \in \mathbb{R}^r$, $1 \leqslant i \leqslant m$, $\xi \in \mathbb{R}^{n-mr}$, $z^1 \stackrel{\Delta}{=} (w_1, w_2, \ldots, w_m)^T = (z_1^1, z_2^1, \ldots, z_m^1)^T \in \mathbb{R}^m$, $z^i \stackrel{\Delta}{=} (z_1^i, z_2^i, \ldots, z_m^i)^T \in \mathbb{R}^m$, $1 \leqslant i \leqslant r$, the zero dynamics vector $\xi \in \mathbb{R}^{n-mr}$, the state vector given by $z = (\bar{z}_1^T, \bar{z}_2^T, \ldots, \bar{z}_m^T, \xi^T)^T \in \mathbb{R}^n$, $B_\xi \in \mathbb{R}^{(n-mr)\times m}$, $A_\xi \in \mathbb{R}^{(n-mr)\times(n-mr)}$, $WB \in \mathbb{R}^{n\times m}$, $CW^{-1} \in \mathbb{R}^{m\times n}$. In case $m = 1$, we have denoted $z_i \stackrel{\Delta}{=} z_1^i$ for $1 \leqslant i \leqslant r$. The canonical form then reads:

$$\begin{cases} \dot{z}_i^1(t) = z_i^2(t) \\ \dot{z}_i^2(t) = z_i^3(t) \\ \vdots \\ \dot{z}_i^{r-1}(t) = z_i^r(t) \\ \dot{z}_i^r(t) = (CA^rW^{-1})_{i\bullet}z(t) + (CA^{r-1}B)_{i\bullet}\lambda(t) + (\bar{G}\mathcal{W}(t))_i \\ \dot{\xi}(t) = A_\xi\xi(t) + B_\xi z^1 + G_\xi\mathcal{U}(t), \end{cases} \quad 1 \leqslant i \leqslant m, \tag{4.5}$$

where $(CA^rW^{-1})_{i\bullet} = C_iA^rW^{-1} \in \mathbb{R}^{1\times n}$, $(CA^{r-1}B)_{i\bullet} = C_iA^{r-1}B \in \mathbb{R}^{1\times m}$, $1 \leqslant i \leqslant m$. The system in (4.5) can be rewritten compactly as

$$\begin{cases} \dot{z}^1(t) = z^2(t) \\ \dot{z}^2(t) = z^3(t) \\ \vdots \\ \dot{z}^{r-1}(t) = z^r(t) \\ \dot{z}^r(t) = CA^rW^{-1}z(t) + CA^{r-1}B\lambda(t) + \bar{G}\mathcal{W}(t) \\ \dot{\xi}(t) = A_\xi\xi(t) + B_\xi z^1 + G_\xi\mathcal{U}(t), \end{cases} \tag{4.6}$$

with $\bar{G} \in \mathbb{R}^{m\times(r+1)p}$. The set Φ in (3.4) can be generalized to $\Phi_m = \mathbb{R}^m_+$, $T_{\Phi_m}(Z^i) = \times_{k=1}^m T_\Phi^i(Z_k^i)$ [1, Remark 7], and from [26, Proposition 3.1.10], we obtain: $N_{T_{\Phi_m}(Z^i)}(z^{i+1}) =$

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$\times_{k=1}^{m} N_{T^i_\Phi(Z^i_k)}(z^i_k)$, with $Z^i \stackrel{\Delta}{=} (z^{1,T}, z^{2,T}, \ldots, z^{i,T})^T$, $Z^i_k \stackrel{\Delta}{=} (z^1_k, z^2_k, \ldots, z^i_k)^T$. The tangent and the normal cones in the right-hand side of (3.8) can be calculated, see an example in [13, Section F]. We denote $dv^i = (dv^i_1, dv^i_2, \ldots, dv^i_m)^T$ and we still impose (3.11) for each dv^i, $1 \leqslant i \leqslant r$, with $dv^i = dz^i - z^{i+1}$ [see (3.10)]. The state jump rule in (3.13) is unchanged for all $1 \leqslant i \leqslant r-1$, replacing z_i by z^i. For $i = r$ we obtain using (3.12):

$$\{z^r\}(t^+) = -e_r\{z^r\}(t^-) + (1+e_r)\mathrm{proj}_{(CA^{r-1}B)^{-1}}\left[T^{r-1}_{\Phi_m}(\{Z^{r-1}\}(t^-)); \{z^r\}(t^-)\right]. \tag{4.7}$$

The generalization of the MDI in (3.10) and (3.11) is as follows:

$$\begin{cases} dz^1 - z^2(t)dt \in -\partial\psi_{T^0_{\Phi_m}(\{Z^0\}(t^-))}(\zeta^1(t)) \\ dz^2 - z^3(t)dt \in -\partial\psi_{T^1_{\Phi_m}(\{Z^1\}(t^-))}(\zeta^2(t)) \\ \vdots \\ dz^{r-1} - z^r(t)dt \in -\partial\psi_{T^{r-2}_{\Phi_m}(\{Z^{r-2}\}(t^-))}(\zeta^{r-1}(t)) \\ dz^r - CA^rW^{-1}z(t)dt - \bar{G}\mathcal{W}(t)dt \in -(CA^{r-1}B)\,\partial\psi_{T^{r-1}_{\Phi_m}(\{Z^{r-1}\}(t^-))}(\zeta^r(t)) \\ d\xi = (A_\xi\xi(t) + B_\xi z_1(t) + G_\xi\mathcal{U}(t))dt. \end{cases} \tag{4.8}$$

As indicated in [1, Remarks 15, 17] [13, Section 5.2], Proposition 3.2 continues to hold when $CA^{r-1}B = (CA^{r-1}B)^T > 0$. Provided that $CA^{r-1}B$ is a Stieltjes matrix, Theorem 3.2, continues to hold also [1, Remark 20 ii)].

Another peculiarity of the MIMO case is that $CA^{r-1}B$ is usually non-diagonal, implying couplings between the variables at impacts. Indeed assume that $dv^r_j(\{t\}) \neq 0$ at some t, for some $1 \leqslant j \leqslant m$. From $\{z^r\}(t^+) - \{z^r\}(t^-) = CA^{r-1}B\; dv^r(\{t\})$, it follows that some variables $\{z^r_i\}$, $i \neq j$, may jump. A mechanical example is treated in details in [13].

Remark 4.1 *In the HOSP framework we allow for different restitution coefficients e_i for each variable z_i (in the SISO case), or each vector z^i (in the MIMO case). However, we do not allow one coefficient $e_{i,j}$* per *component z^i_j,* i.e. *we take $e_{i,j} = e_i$ for all $1 \leqslant j \leqslant m$, $1 \leqslant i \leqslant r$. This is the same in Mechanics where Moreau's impact rule has one global coefficient, while other models may consider one coefficient for each constraint [12].*

4.2 Complementarity switching DAEs

Due to the constraint imposed on dv_r, the complementarity in Theorem 4.1 continues to hold in the MIMO case. Let $z_1(\tau^+) = 0$ on some interval $]t, t+\sigma]$, $\sigma > 0$. On $]t, t+\sigma]$, the function $z_1(\cdot)$ is analytic, and we have $z_1(\tau) = z_2(\tau) = \cdots = z_r(\tau) = 0$. From (4.3), we have $\lambda(\tau) = \chi_r(\tau) \geqslant 0$ on $]t, t+\sigma[$, and $\dot{z}_r(\tau^+) \geqslant 0$ also. It can also be shown that $\dot{z}_r(\tau^+) > 0 \Rightarrow \chi_r(\tau) = 0$. Thus, $0 \leqslant \dot{z}_r(\tau^+) \perp \lambda(\tau) = \chi_r(\tau) \geqslant 0$ holds[2]. Using (2.7) this gives rise to the *contact Linear Complementarity Problem (LCP)* with unknown $\lambda(\tau)$:

$$0 \leqslant \lambda(\tau) \perp \dot{z}_r(\tau^+) = w^{(r)}(\tau^+) = CA^rW^{-1}\begin{pmatrix} 0 \\ \xi(\tau) \end{pmatrix} + CA^{r-1}B\lambda(\tau) + \bar{G}\mathcal{W}(\tau) \geqslant 0, \tag{4.9}$$

[2] Remind that $\chi_r(\cdot)$ is right-continuous, as well as $\dot{z}_r(\cdot)$, being $z_r(\cdot)$ in $\mathcal{F}_\infty(I; I\!R)$.

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for all $\tau \in]t, t+\sigma[$. Provided that $CA^{r-1}B$ is a P-matrix (which is guaranteed if it is a Stieljes matrix), the contact LCP in (4.9), which holds on time intervals where $z^1 = 0$, has a unique solution $\lambda(t)$ for any $\xi(t)$ and $\mathcal{W}(t)$, and it has 2^m modes corresponding to $\dot{z}_i^r(\tau^+) > 0$ and $\lambda_i(\tau) = 0$ (detachment from $z_i^1 = 0$), or $\dot{z}_i^r(\tau^+) = 0$ and $\lambda_i(\tau) > 0$ (contact remains active at $z_i^1 = 0$). Let us now assume that $z_i^1(\tau) = 0$ for all $i \in \mathcal{I} \subseteq \{1, m\}$, $z_1^j(\tau) > 0$ for all $j \in \bar{\mathcal{I}} = \{1, m\} \setminus \mathcal{I}$, and all $\tau \in]t, t+\sigma]$, $\sigma > 0$. We denote $\mathrm{card}(\mathcal{I}) = m'$, $\mathrm{card}(\bar{\mathcal{I}}) = \bar{m}$, $\bar{m} + m' = m$ [3], $\lambda_{\mathcal{I}} = (\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{m'}})^T$ with $i_j \in \mathcal{I}$, $z_{\mathcal{I}}^i$ is defined similarly as $z_{\mathcal{I}}^i = (z_{i_1}^i, z_{i_2}^i, \ldots, z_{i_{m'}}^i)^T$. We have $\lambda_{\mathcal{I}}(\tau) \geqslant 0$, $z_{\mathcal{I}}^1(\tau) = 0$, and $\lambda_{\bar{\mathcal{I}}}(\tau) = 0$, $z_{\bar{\mathcal{I}}}^1(\tau) > 0$, on $]t, t+\sigma]$. Thus, we obtain from (4.5):

$$\dot{z}_{\mathcal{I}}^r(\tau) = (CA^r W^1)_{\mathcal{I}\bullet} z(\tau) + (CA^{r-1}B)_{\mathcal{I}\mathcal{I}} \lambda_{\mathcal{I}}(\tau) + (\bar{G}\mathcal{W}(\tau))_{\mathcal{I}} \quad (=0), \tag{4.10}$$

where $(CA^r W^1)_{\mathcal{I}\bullet} \in \mathbb{R}^{m' \times n}$ is made of the rows of $CA^r W^1$ indexed in $\mathcal{I}$, $(CA^{r-1}B)_{\mathcal{I}\mathcal{I}} \in \mathbb{R}^{m' \times m'}$ is the principal submatrix of $CA^{r-1}B$ obtained by deleting rows and columns indexed in $\bar{\mathcal{I}}$, from which we infer the contact LCP for active constraints in $\mathcal{I}$ on $]t, t+\sigma[$:

$$0 \leqslant \lambda_{\mathcal{I}}(\tau) \perp (CA^r W^{-1})_{\mathcal{I}\bullet} z(\tau) + (CA^{r-1}B)_{\mathcal{I}\mathcal{I}} \lambda_{\mathcal{I}}(\tau) + (\bar{G}\mathcal{W}(\tau))_{\mathcal{I}} \geqslant 0. \tag{4.11}$$

Let us recall that if $CA^{r-1}B$ is a P-matrix, so is any of its principal submatrices and so are their inverses. Thus, the contact LCPs as in (4.11) always have a unique solution, whatever the number of active constraints. By assumption we have on $]t, t+\sigma[$: $\lambda_{\mathcal{I}}(\tau) = -(CA^{r-1}B)_{\mathcal{I}\mathcal{I}}^{-1} \left[(CA^r W^{-1})_{\mathcal{I}\bullet} z(\tau) + (\bar{G}\mathcal{W}(t))_{\mathcal{I}}\right] \geqslant 0$. Similarly to the case $m = 1$, the LCP in (4.11) rules the possible detachments from the active constraints at $\tau = t + \sigma$. It is noteworthy that $\lambda_{\bar{\mathcal{I}}}(\tau)$ may influence $z_{\bar{\mathcal{I}}}^i$. Indeed we have:

$$\begin{aligned} \dot{z}_{\bar{\mathcal{I}}}^r(\tau) &= (CA^r W^{-1})_{\bar{\mathcal{I}}\bullet} z(\tau) + (CA^{r-1}B)_{\bar{\mathcal{I}}\mathcal{I}} \lambda_{\mathcal{I}}(\tau) + (\bar{G}\mathcal{W}(\tau))_{\bar{\mathcal{I}}} \\ &= [(CA^r W^{-1})_{\bar{\mathcal{I}}\bullet} - (CA^{r-1}B)_{\bar{\mathcal{I}}\mathcal{I}} CA^{r-1}B)_{\mathcal{I}\mathcal{I}}^{-1} (CA^r W^{-1})_{\mathcal{I}\bullet}] z(\tau) \\ &\quad + (\bar{G}\mathcal{W}(\tau))_{\bar{\mathcal{I}}} - (CA^{r-1}B)_{\bar{\mathcal{I}}\mathcal{I}} (CA^{r-1}B)_{\mathcal{I}\mathcal{I}}^{-1} (\bar{G}\mathcal{W}(t))_{\mathcal{I}}, \end{aligned} \tag{4.12}$$

where $(CA^{r-1}B)_{\bar{\mathcal{I}}\mathcal{I}} \in \mathbb{R}^{(m-m') \times m'}$ is the submatrix of $CA^{r-1}B$ obtained from the rows indexed in $\bar{\mathcal{I}}$ and the columns indexed in $\mathcal{I}$. It represents the couplings between the unilateral constraints, and it makes the DAE vector field depend on the contact LCP solution. The contact DAE in a semi-explicit form is made of the dynamics in (4.6) with the constraint $z_{\mathcal{I}}^1 = 0$. It follows that this is a DAE with (vector) index $(r+1, r+1, \ldots, r+1)^T \in \mathbb{R}^m$.

Therefore, the HOSP is a DDI which switches between 2^m DAEs with index $r+1$, and with vector fields as in (4.10) and (4.12), for varying index sets $\mathcal{I}$ and $\bar{\mathcal{I}}$. We might name it complementarity switching DAEs. It is noteworthy that if the contact LCP implies a switch between a DAE associated with the index set of active constraints $\mathcal{I}_1$, and another DAE with index set of active constraints $\mathcal{I}_2$, then one may have (a) $\mathcal{I}_1 \subset \mathcal{I}_2$, (b) $\mathcal{I}_1 \supset \mathcal{I}_2$ (c) $\mathcal{I}_1 \neq \mathcal{I}_2$ with neither (a) nor (b). Case (a) means that active constraints are added possibly implying jumps in $\bar{z}$ solution of the MDI (see calculations in Section 5 on a particular case), case (b) means that some active constraints are deactivated (according to solutions of the contact LCP in (4.11) at time $t + \sigma$), case (c) is the more general

[3] Both m' and $\bar{m}$ are simple functions of time and state, along the HOSP solutions.

situation with a switching event accompanied with state jumps and detachments ruled by the contact LCP. These various transitions are taken into account in the framework of tracking control of complementarity Lagrangian systems in [21].

Remark 4.2 *The complementarity switching DAEs can also be viewed in the original x-dynamics (1.1), with DAEs in semi-explicit form. It corresponds to modes $w_i(x,u) = 0$ and $w_j(x,u) > 0$, $i, j \in \{1,\dots,m\}$. With each mode is associated a multiplier solution of the contact LCP, that modifies the DAE right-hand side. Getting back to the discrepancies with respect to switching DAEs as in [29]: in the HOSP the choice of the DAEs vector fields is dictated in part by the complementarity problem in (4.11).*

4.3 HOSP with time-dependent-switching state feedback

Let us assume that on $[0,\tau_1[$, $\tau_1 \geqslant \delta > 0$, we apply the feedback $u_1(x,t) = K_1 x + v_1(t)$, and on $[\tau_1,\tau_2[$, $\tau_2 \geqslant \tau_1 + \delta$ for some $\delta > 0$, we apply $u_2(x,t) = K_2 x + v_2(t)$, for some matrices K_1, K_2 and exogenous signals $v_1(\cdot)$, $v_2(\cdot)$ satisfying Assumption 3.1. This gives rise to the switched closed-loop dynamics [instead of (1.1)]:

$$
\begin{aligned}
&(a)\quad \begin{cases} \dot{x}(t) = (A+EK_1)x(t) + B\lambda + Ev_1(t) \\ 0 \leqslant w_1(t) = (C+FK_1)x(t) + Fv_1(t) \quad \text{for all } t \in [0,\tau_1[\\ x(0^-) = x_0, \end{cases} \\
&(b)\quad \begin{cases} \dot{x}(t) = (A+EK_2)x(t) + B\lambda + Ev_2(t) \\ 0 \leqslant w_2(t) = (C+FK_2)x(t) + Fv_2(t) \end{cases} \text{for all } t \in [\tau_1,\tau_2[.
\end{aligned} \tag{4.13}
$$

Let us assume that each subsystem $(A+EK_i, B, C+FK_i)$, $i = 1,2$, has a vector relative degree $\bar{r}_i = (r_i, r_i, \dots, r_i)^T$, so that one can associate, with each of them, a canonical representation as in (2.6) after the transformation $z = W_i x + T_i\mathcal{U}_i$, $i = 1,2$. Using Corollary 3.2, we can prove the well-posedness of the HOSP associated with the quadruple $[A+EK_1, B, C+FK_1, v_1(t)]$ on $[0,\tau_1[$ for any x_0. Given any $x(\tau_1^-)$, the well-posedness of the HOSP associated with the quadruple $(A+EK_2, B, C+FK_2, v_2(t))$ on $[\tau_1,\tau_2[$ can be proved also. If τ_1 is equal to a state jump time for subsystem (4.13) (a), then one can apply the state reinitialization mapping (3.13), (4.7) and (2.1) to this subsystem, to obtain $z_{s_1}(\tau_1^+)$ and $x(\tau_1^+) = W_1^{-1}(z_{s_1}(\tau_1^+) - T_1\mathcal{U}_1(\tau_1))$, where $z_{s_1} = (\bar{z}_{s_1}^T, \xi_{s_1}^T)^T$ denotes here the state variable of the canonical state representation (2.6) of subsystem (4.13) (a). Then in a second step one can apply the state reinitialization mapping (3.13), (4.7) and (2.1) for the subsystem (4.13) (b), considering $z_{s_2,0} \stackrel{\Delta}{=} W_2 x(\tau_1^+) + T_2\mathcal{U}_2(\tau_1) = W_2 W_1^{-1}(z_{s_1}(\tau_1^+) - T_1\mathcal{U}_1(\tau_1)) + T_2\mathcal{U}_2(\tau_1)$ as an initial condition, compute $z_{s_2}(\tau_1^+)$ and obtain a reinitialized state $x(\tau_1^{++}) = W_2^{-1}(z_{s_2}(\tau_1^+) - T_2\mathcal{U}_2(\tau_1))$ for subsystem (4.13) (b), where the upperscript $++$ indicates that the jump mapping has been applied twice at $t = \tau_1$. If τ_1 is not equal to a state jump time for subsystem (4.13) (a), then one can initialize subsystem (4.13) (b) with $x(\tau_1)$, with which is associated $z_{s_2}(\tau_1^-) \stackrel{\Delta}{=} W_2^{-1}(x(\tau_1) - T_2\mathcal{U}_2(\tau_1))$. Therefore, we see that the HOSP state jump rule, furnishes a natural way to switch between the two subsystems in (4.13), which are themselves complementarity switching DAEs. One can then define a sequence of switching times $\{\tau_i\}_{i\geqslant 0}$, $\tau_{i+1} \geqslant \tau_i + \delta$. We thus have proved the following:

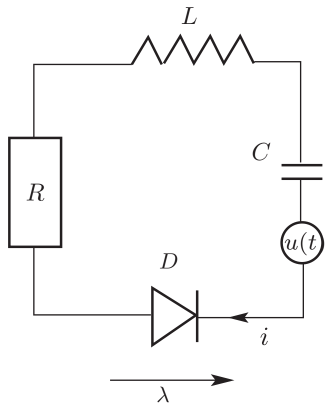

FIGURE 1. A circuit with an ideal diode and a voltage source.

Proposition 4.1 *Consider the unilaterally constrained system in (1.1). Let us define the sequence* $\{\tau_i\}_{i\geqslant 0}$, $\tau_{i+1} \geqslant \tau_i + \delta$, $\delta > 0$ *of switching times, and define* $u = K_{i+1}x + v_{i+1}$ *on* $[\tau_i, \tau_{i+1})$. *Assume that each subsystem* $(A + EK_i, B, C + FK_i)$ *has a vector relative degree* $\bar{r}_i = (r_i, \ldots, r_i)^T$, *and that* $(C + FK_i)(A + EK_i)^{r_i-1}B$ *is a Stieltjes matrix for each i. Assume further that the functions* $v_i : \mathbb{R}_+ \to \mathbb{R}^{p_i}$, *satisfy Assumption 3.1. Then, the switching HOSP system admits a global, unique regular solution in the sense of Definition 3.1.*

Remark 4.3 *There is a fundamental difference between the exogenous switching times in this section, and the complementarity switching times (*i.e., *the switching times due to the complementarity conditions): the latter admit accumulation times (where the crossing of the accumulation time is taken care of by the multiplier* λ*), while the former need some dwell time* $\delta > 0$. *An open problem is to design state-dependent switching times (other than the complementarity ones) between HOSPs as in (4.13). This however creates serious difficulties, like how to avoid finite accumulations or even an infinity of switching times at one instant, or continuation after such accumulations, or sliding modes,* etc.

Example 4.1 *Let us provide an academic example from circuits with ideal diodes. Let us consider the simple circuit in Figure 1. Its dynamics is given by* [12, Example 5.16]*:*

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{R}{L}x_2(t) - \frac{1}{LC}x_1(t) - \frac{1}{L}\lambda(t) + \frac{1}{L}u(t) \\ 0 \leqslant w(t) = -x_2(t) \perp \lambda(t) \geqslant 0, \end{cases} \tag{4.14}$$

where $x_1(t)$ *is the charge of the capacitor,* $x_2(t) = i(t)$ *is the current through the circuit,* $u(t)$ *is a voltage source. We first propose two dynamic feedback controllers:*

$$\textbf{(a)} \begin{cases} u_1(x, \lambda, t) = \lambda - Lx_3 + v_1(t) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = \lambda(t) \end{cases} \qquad \textbf{(b)} \begin{cases} u_2(x, \lambda, t) = \lambda + Lx_3 - x_4(t) + v_2(t) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = \lambda(t). \end{cases} \tag{4.15}$$

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The corresponding canonical dynamics are given by

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{R}{L}x_2(t) - \frac{1}{LC}x_1(t) \\ \qquad -x_3(t) + \frac{v_1(t)}{L} \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = \lambda(t) \\ w(t) = -x_2(t) \end{cases} \Leftrightarrow \begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = z_3(t) \\ \dot{z}_3(t) = -\frac{R}{L}z_3(t) - \frac{1}{LC}z_2(t) + \lambda(t) - \frac{\ddot{v}_1(t)}{L} \\ \dot{\xi}(t) = z_1(t) \\ z_1(t) = -x_2(t), \end{cases} \tag{4.16}$$

for (4.15) **(a)**, *and*

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{R}{L}x_2(t) - \frac{1}{LC}x_1(t) + x_3(t) \\ \qquad -\frac{x_4(t)}{L} + \frac{v_2(t)}{L} \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = \lambda(t) \\ w(t) = -x_2(t) \end{cases} \Leftrightarrow \begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = -\frac{R}{L}z_2(t) - \frac{1}{L}z_1(t) - F_\xi\xi(t) \\ \qquad -G_\xi z_1(t) + \frac{\lambda(t)}{L} - \frac{\dot{v}_2}{L} \\ \dot{\xi}(t) = A_\xi\xi(t) + B_\xi z_1(t) \\ z_1(t) = -x_2(t), \end{cases} \tag{4.17}$$

for (4.15) **(b)**, *where* $\xi(t) \in \mathbb{R}^2$, $F_\xi(sI_2 - A_\xi)^{-1}B_\xi + G_\xi = \frac{LCs^2+RCs-1}{Cs(s-L)}$, $s \in \mathbb{C}$, $(A_\xi, B_\xi, F_\xi, G_\xi)$ *a minimal representation, and* $x_4 = F_\xi\xi + G_\xi z_1$. *The proof is outlined in Appendix C. It is possible to apply Proposition 4.1 to a switching system with (4.16) and (4.17).*

5 The higher-order bouncing ball

In this section, we consider a very particular form of the 'input' term in (2.6) and of the chain of integrators, in order to mimic the mechanical bouncing ball. The focus is put on the existence of trajectories of the HOSP, which possess accumulations of state jumps (the solutions of the HOSP admit infinity of discontinuities in finite time, as functions of local bounded variations do). The results demonstrate that the dynamics of the HOSP may be quite complex when external excitation and non-zero restitution coefficients are considered. Notice that the exhibited complexity could not be shown in the autonomous framework of [1].

Definition 5.1 A left accumulation of state reinitialization times t_k at t, is an accumulation on the left of t. In other words, $\lim_{k\to+\infty} t_k = t$ with $t_k < t$ for all $k \geqslant 0$.

In Mechanics, the bouncing ball example shows that a constant input to a chain of two integrators, and a restitution $e \in (0, 1)$ can produce left accumulations of velocity jumps. The position is time-continuous and made of portions of paraboloids, velocities are piecewise linear functions of time, and the force input signal is constant. It is of interest to study whether or not a constant input in (2.6) can also induce left accumulations of state jumps, when $r \geqslant 3$. We denote $T_r(s) = \sum_{i=0}^{r-1} CA^iEs^{r-1-i} + Fs^r$, where $s \in \mathbb{C}$ is the Laplace variable, and we denote the Laplace transform as $\mathcal{L}[\cdot]$.

Lemma 5.1 *Consider* $r \geqslant 2$ *and* $m = 1$. *Let us consider the HOSP with external inputs and non-zero restitution coefficients* e_i *in (3.5), with all* $e_j = 0$ *except* $e_i > 0$ *for some*

$i \in \{2, \ldots, r\}$. Assume that $F = 0$, $CA^r = 0$ and $CA^{r-1}B > 0$, and that $\mathcal{L}[u(t)] = \frac{a}{T_r(s)}$, with $a < 0$, a constant and $T_r(s) \neq 0$. **(i)** *A unique regular solution exists globally for all $a < 0$ and any initial condition.* **(ii)** *Let $\{t_k\}$ be a strictly increasing sequence, with $\lim_{k\to+\infty} t_k < +\infty$. Then, a solution $z(\cdot)$ satisfying for all $k \geqslant 0$: $z_1(t_k) = 0$, $\Delta_k = t_{k+1} - t_k > 0$, $z_i(t_k^-) < 0$, $z_1(t) > 0$ on $]t_k, t_{k+1}[$, and $\lim_{k\to+\infty} z_j(t_k^+) = 0$ for all $j \in \{1, \ldots, r\}$, exists if and only if*

$$e_i \in \left]0, \frac{(i-1)!(r-i+1)!}{r! - (i-1)!(r-i+1)!}\right[. \tag{5.1}$$

Proof **(i)** The function $u(\cdot)$ is defined as the inverse Laplace transform $u(t) = \mathcal{L}^{-1}\left(\frac{a}{T_r(s)}\right)$. There exists a realization (M, N, P) of the transfer function $\frac{1}{T_r(s)}$ such that $\frac{1}{T_r(s)} = P(sI_{r-1} - M)^{-1}N$, $\dot{v}(t) = Mv(t) + Na$, $u(t) = Pv(t)$. Thus, $v(\cdot)$ is analytic and so is $u(\cdot)$. Corollary 3.2 applies.

(ii) *Only if* : From the data of the Lemma, we get $T_r(\frac{d}{dt})u(t) = a < 0$ so that from (2.6) the system is a chain of integrators with a constant input a. The equality of measures in (3.10) is given by

$$\begin{cases} dz_1 = z_2(t)dt + dv_1 \\ dz_2 = z_3(t)dt + dv_2 \\ \vdots \\ dz_{r-1} = z_r(t)dt + dv_{r-1} \\ dz_r = CA^{r-1}Bdv_r + a\,dt. \end{cases} \tag{5.2}$$

It is also worth noting that $z_i(t_k^-) < 0$ ($\Rightarrow z_i(t_k^+) > 0$) is necessary for the accumulation of 'bounces' to exist, since $z_j(t_k^+) = 0$ for all $j \neq i$, due to $e_j = 0$. Integrating on $]t_k, t[$ we obtain

$$z_1(t) = \sum_{j=0}^{r-1} z_{r-j}(t_k^+)\frac{(t-t_k)^{r-1-j}}{(r-1-j)!} + \frac{a}{r!}(t-t_k)^r \tag{5.3}$$

on $[t_k, t_{k+1}[$. Since $z_1(t_k) = 0$ for all k and all $e_j = 0$ except e_i, we get from (5.3)

$$z_i(t_k^+)\frac{\Delta_k^{i-2}}{(i-1)!} + \frac{a}{r!}\Delta_k^{r-1} = 0, \tag{5.4}$$

where we also used that $i \geqslant 2$. It follows that

$$\Delta_k = \left(-\frac{r!}{a}\frac{z_i(t_k^+)}{(i-1)!}\right)^{\frac{1}{r+1-i}}, \tag{5.5}$$

where we used that $0 \leqslant i-2 \leqslant r-2 < r-1$, and $\Delta_k > 0$ for all $k < +\infty$. Notice that the right-hand side of (5.5) is positive as $z_i(t_k^+) > 0$. Now, we have

$$z_{r-\alpha}(t_{k+1}^-) = z_i(t_k^+)\frac{\Delta_k^{\alpha-r+i}}{(\alpha-r+i)!} + \frac{a}{(\alpha+1)!}\Delta_k^{\alpha+1} \tag{5.6}$$

for $\alpha \in \{0, ..., r-i\}$. Taking $\alpha = r-i$, and recalling from Proposition 3.2 that $z_i(t_{k+1}^+) = -e_i z_i(t_{k+1}^-)$ when $z_i(t_{k+1}^-) < 0$, we obtain from (5.6) and (5.5)

$$z_i(t_{k+1}^+) = e_i \left(-1 + \frac{r!}{(i-1)!(r-i+1)!} \right) z_i(t_k^+), \tag{5.7}$$

from which we deduce that

$$z_i(t_{k+1}^+) = \left(e_i \left(-1 + \frac{r!}{(i-1)!(r-i+1)!} \right) \right)^{k+1} z_i(t_0^+), \quad k \geqslant 0. \tag{5.8}$$

One checks that provided $z_i(t_0^+) > 0$ then $z_i(t_{k+1}^+) > 0$ for all $1 \leqslant k < +\infty$ as $\frac{r!}{(i-1)!(r-i+1)!} > 1$, see Lemma A.1 in the Appendix. The necessary part is proved since (5.7) is a geometric series. It also follows from (5.5) and (5.7) that

$$\varDelta_k = -\frac{r!}{a(i-1)!} \bar{e}_i^k z_i(t_0^+), \tag{5.9}$$

with $\bar{e}_i = e_i \left(-1 + \frac{r!}{(i-1)!(r-i+1)!} \right)$. From (5.9), it follows that the total duration of the sequence of infinite impacts is bounded and equal to

$$\sum_{k=0}^{+\infty} \varDelta_k = \frac{-r!\, z_i(t_0^+)}{a(i-1)!} \frac{1}{1-\bar{e}_i}. \tag{5.10}$$

If : Let the system be initialized at $z(0^-)$ and let $z(0^+)$ be the solution of the state jump rule in Proposition 3.2. Then, from (5.3) there exists $t^* < +\infty$ such that for $t > t^*$ one has $z_1(t) < 0$ and $z_1(t^*) = 0$. Since the reasoning applies to any initial condition on $z(\cdot)$, one deduces that there exists a contact time for the initial data $z_j(0^-) = 0$ and $z_i(0^-) < 0$ at some bounded $t_0 > 0$. Using (5.8) with $k = 0$ it follows that $z_i(t_1^-) < 0$ and $z_i(t_1^+) > 0$. And so on. Therefore, (5.1) is sufficient for an accumulation to exist. $\square$

The Mechanical bouncing ball model implies $E = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = (1\ 0)$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so that $T_r(s) = \sum_{i=0}^{1} CA^i E s^{1-i} = 1$. In the case of a triple integrator $z_1^{(3)}(t) = u(t) + \lambda(t)$, one gets that either $e_2 = 0$ and $e_3 \in \left]0, \frac{1}{2}\right[$, or $e_3 = 0$ and $e_2 \in \left]0, \frac{1}{2}\right[$. If $i = r$ or $i = 2$ then $e_i \in \left]0, \frac{1}{r-1}\right[$. Since imposing $w(0) \geqslant 0$ implies that $w(\cdot)$ is continuous (as a consequence of (3.13) for $i = 1$, which implies that $\{z_1\}(t^+) = \{z_1\}(t^-)$ at all times $t \geqslant 0$ since $T_\varPhi^0 = \varPhi = I\!R_+$), one recovers the case of the bouncing ball where $r = i = 2$ so $e_2 \in]0, 1[$. If $i = \alpha + 1$ or $i = r - \alpha + 1$ for some α, then $e_i \in \left]0, \frac{1}{\frac{r!}{\alpha!(r-\alpha)!} - 1}\right[$ in (5.1). If $e_i \geqslant 0$ but it does not satisfy the constraint in (5.1), then the above accumulation may not exist ($e_i = 0$) or the solution may diverge ($e_i \geqslant \frac{(i-1)!(r-i+1)!}{r!-(i-1)!(r-i+1)!}$). Notice finally that after the finite time of state jump accumulation has been reached, the system may enter a phase of persistent contact with $\bar{z}(t) = 0$. Then, the multiplier $\lambda(t) = \chi_r(t)$ is a solution of the contact LCP (see Section 4) and takes care of maintaining the trajectories on the unilateral constraint boundary.

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6 Fixed points, stability, positive invariance

6.1 Existence and uniqueness of fixed points

Consider the MDI in (3.10) and (3.11) in a MIMO setting as (4.6), then fixed points $z^* = (\bar{z}^{*,T}, \xi^{*,T})^T$ have to satisfy the generalized equation:

$$\begin{cases} 0 = WAW^{-1}z^* + N\chi^*(t) + H\mathcal{W}(t) \\ \bar{\chi}^*(t) \in -\begin{pmatrix} \partial\psi_{\Phi_m}(z^{1,*}) \\ \partial\psi_{T_{\Phi_m}(z^{1,*})}(z^{2,*}) \\ \vdots \\ \partial\psi_{T^{r-1}_{\Phi_m}(z^{1,*},\ldots,z^{r-1,*})}(z^{r,*}) \end{pmatrix} \end{cases} \tag{6.1}$$

for all $t \geqslant 0$, $\bar{\chi} = (\chi_1, ..., \chi_r)^T \in I\!R^{rm}$, $\chi = (\bar{\chi}, 0_{n-rm})^T$ and

$$N = \begin{pmatrix} I_{r-1} & 0^{r-1} & 0_{(r-1)\times(n-r)} \\ 0_{r-1} & CA^{r-1}B & 0_{n-r} \\ 0_{(n-r)\times(r-1)} & 0^{n-r} & I_{n-r} \end{pmatrix}. \tag{6.2}$$

Example 6.1 *Consider the classical bouncing ball dynamics: $\dot{z}_1(t) = z_2(t)$, $\dot{z}_2(t) = u(t) + \lambda$, $0 \leqslant z_1(t) \perp \chi_2(t) \geqslant 0$, where $\lambda = \chi_2(t)$ outside impact times. We suppose $z_1(0) \geqslant 0$ so that $dv_1 = 0$. Fixed points are solutions of $\chi_2(t) = -u(t)$, $0 \leqslant z_1^* \perp \chi_2(t) \geqslant 0$. Suppose that $u(t) < 0$, then the unique fixed point is $z_1^* = z_2^* = 0$ with $\chi_2^*(t) = -u(t) > 0$ (this is the static equilibrium of the ball on the ground). This justifies why $\bar{\chi}^*$ may be time-dependent in (6.1) while the set-valued term is independent of time.*

Equilibria occur outside jump times; hence, $\bar{z}^* = (z^{1,*,T}, 0_m, \ldots, 0_m)^T$, $\chi_i^*(t) = 0$, $1 \leqslant i \leqslant r-1$, $\chi_r(t) \geqslant 0$. This allows us to rewrite equivalently (6.1) as

$$\begin{cases} (a)\ \ 0 \in \partial\psi_{\Phi}(z^{1,*}) \\ (b)\ \ 0 \in \partial\psi_{T^{i-1}_{\Phi_m}(z^{1,*},0,\ldots,0)}(0), \ \ 2 \leqslant i \leqslant r-1 \\ (c)\ \ 0 \in CA^rW^{-1}(z^{1,*}, 0_{r-1}, \xi^{*,T})^T + \bar{G}\mathcal{W}(t) + CA^{r-1}B\chi_r^*(t) \\ (d)\ \ 0 = A_\xi\xi^* + B_\xi z^{1,*} + G_\xi\mathcal{U}(t) \\ (e)\ \ \chi_r^*(t) \in -\partial\psi_{T^{r-1}_{\Phi_m}(z^{1,*},0^m,\ldots,0^m)}(0). \end{cases} \tag{6.3}$$

The inclusions in (6.3) (a) and (b) are trivially satisfied since the right-hand sides are cones. Let $m = 1$. We can study first two cases: (i) $z_1^* > 0 \Rightarrow \chi_r(t) = 0$ and fixed points are solutions of $d_1z_1^* + d_\xi^T\xi^* + \bar{G}\mathcal{W}(t) = 0$ and $A_\xi\xi^* + B_\xi z_1^* + G_\xi\mathcal{U}(t) = 0$. (ii) $z_1^* = 0 \Rightarrow \chi_r(t) \geqslant 0$, and fixed points are solutions of $0 \in d_\xi^T\xi^* + \bar{G}\mathcal{W}(t) + CA^{r-1}B\chi_r^*(t)$, $A_\xi\xi^* + G_\xi\mathcal{U}(t) = 0$, $\chi_r^*(t) \geqslant 0$ [we made use of (2.4) to simplify the equations]. Various conditions can be derived from these equations to study the existence and uniqueness of equilibrium points.

Proposition 6.1 *Suppose that $CA^{r-1}B > 0$, and A_ξ is full rank. Let $m = 1$, $d_1 - d_\xi^T A_\xi^{-1} B_\xi \neq 0$, and assume that $d_\xi^T A_\xi^{-1} G_\xi\mathcal{U}(t) - \bar{G}\mathcal{W}(t)$ is non-constant. Then, necessarily $z_1^* = 0$*

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(i.e., *the equilibrium, if it exists, necessarily occurs on the unilateral constraint boundary*). *On the boundary the equilibrium exists if and only if* $(CA^{r-1}B)^{-1}(d_\xi^T A_\xi^{-1} G_\xi \mathcal{U}(t) - \bar{G}\mathcal{W}(t)) \geqslant 0$.

Proof Let $z_1^* > 0$, then $\chi_r^*(t) = 0$ and using (6.3) (c) and (d) we obtain $z_1^* = (d_1 - d_\xi^T A_\xi^{-1} B_\xi)^{-1}(d_\xi^T A_\xi^{-1} G_\xi \mathcal{U}(t) - \bar{G}\mathcal{W}(t))$, which is not possible if the term between brackets is not constant. The second result follows from (6.3) (c)–(e), using (2.4). One obtains $(CA^{r-1}B)^{-1}(d_\xi^T A_\xi^{-1} G_\xi \mathcal{U}(t) - \bar{G}\mathcal{W}(t)) = \chi_r^*(t)$, while (c) (e) with $z_1^* = 0$ is equivalent to $\chi_r^*(t) \in \mathbb{R}_+$. The result follows. $\square$

Proposition 6.1 allows us to determine when the mixed LCP: $Ax^* + B\lambda^*(t) + Eu(t) = 0$, $0 \leqslant w^* = Cx^* + Fu(t) \perp \lambda^*(t) \geqslant 0$ has a unique solution $(x^*, \lambda^*(t))$ with $w^* = 0$, and $u(t)$ not identically zero (the exogenous term 'pushes' the system on $\mathrm{bd}(\Phi_u)$, in a sort of static equilibrium on a moving constraint). In (2.10), the matrix A_ξ is full rank. When $m \geqslant 2$, the rth row of WAW^{-1} in (2.4) is equal to $(D_1, D_2, \ldots, D_r, D_\xi)$ with $D_i \in \mathbb{R}^{m\times m}$, $1 \leqslant i \leqslant r$, $D_\xi \in \mathbb{R}^{m\times(n-rm)}$. Also, $\bar{G} \in \mathbb{R}^{m\times(r+1)p}$, $G_\xi \in \mathbb{R}^{(n-mr)\times rp}$ and $\Phi = (\mathbb{R}^+)^m$. Then, (6.3) (c)–(e) becomes

$$\begin{cases} D_1 z^{1,*} + D_\xi \xi^* + \bar{G}\mathcal{W}(t) + CA^{r-1}B\chi_r(t) = 0 \\ A_\xi \xi^* + B_\xi z^{1,*} + G_\xi \mathcal{U}(t) = 0 \\ \chi_r(t) \in -\partial\psi_{T_\Phi^{r-1}(z^{1,*},0^m,\ldots,0^m)}(0^m). \end{cases} \tag{6.4}$$

In general one may have equilibria with $z_i^{1,*} = 0$ and $z_j^{1,*} > 0$, $i \neq j$. We do not study further the generalized equation (6.4); however, conditions which guarantee that it has a unique solution are given in [13, Remark 10].

6.2 Stability and stabilization

The canonical form (4.6) is obtained by considering λ as an 'input' and w as an 'output', therefore its use for control with $u(\cdot)$ differs from the usual case where the canonical form is obtained with the input $u(\cdot)$. In addition one has to cope with the unilateral constraints and the state jumps. Let us analyze the asymptotic stability *via* passification. Let us consider the MIMO system in (4.6). We can write the associated MDI formalism as

$$dz = WAW^{-1}z(t)dt + \bar{N}dv + H\mathcal{W}(t)dt, \tag{6.5}$$

with $dv = (dv_1^T, \ldots, dv_r^T)^T \in \mathbb{R}^{rm}$, $dv_i \in \mathbb{R}^m$ [4], $\bar{N} = \begin{pmatrix} I_{m(r-1)} & 0_{(r-1)m\times m} \\ 0_{m\times m(r-1)} & CA^{r-1}B \\ 0_{(n-mr)\times m(r-1)} & 0_{(n-mr)\times m} \end{pmatrix} \in \mathbb{R}^{n\times mr}$, $CA^{r-1}B \in \mathbb{R}^{m\times m}$. We recall that $H\mathcal{W}(t) = \begin{pmatrix} 0^{m(r-1)} \\ \bar{G}\mathcal{W}(t) \\ G_\xi \mathcal{U}(t) \end{pmatrix}$, see (2.7) and (4.6), and that $\mathcal{W}(t)$ and $\mathcal{U}(t)$ involve $u^{(i)}(t)$ for $0 \leqslant i \leqslant r$. In the autonomous case, the stability of the trivial solution $z^* = 0$ is shown in [1, Proposition 10] under a positive real [11, Definition 2.29] condition of the triplet $(WAW^{-1}, \hat{N}, \hat{C})$, where when $m = 1$, $\hat{N} = \begin{pmatrix} \hat{\hat{N}} \\ 0_{(n-r)\times r} \end{pmatrix} \in \mathbb{R}^{n\times r}$, $\hat{\hat{N}} = \begin{pmatrix} I_{r-1} & 0^{r-1} \\ 0_{r-1} & CA^{r-1}B \end{pmatrix} \in \mathbb{R}^{r\times r}$ is a leading principal submatrix of N in (6.2), $\hat{C} = (I_r, 0_{r\times(n-r)})$, and thus defining a system with r inputs and r outputs. Then, the matrix

[4] Written here with some abuse of notation.

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$J = \begin{pmatrix} \hat{\tilde{N}}^{-1} & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & J_\xi \end{pmatrix}$, $0 < J_\xi = J_\xi^T \in \mathbb{R}^{(n-r)\times(n-r)}$, is a solution of the KYP Lemma LMI associated with the triplet $(WAW^{-1}, \hat{N}, \hat{C})$ [11, Lemma 3.1]. Let us extend this result to the MIMO case by letting $u = Ky + v$ for some $K \in \mathbb{R}^{p\times n}$ and $y = Lx$ is a measurable output and v the new input. The triplet (A, B, C) in (1.1) is transformed into the triplet $(\tilde{A}, B, \tilde{C}) \stackrel{\Delta}{=} (A + EKL, B, C + FKL)$, which does not necessarily have the same relative degree as (A, B, C). Letting $v = 0$ and still denoting the vector relative degree of $(\tilde{A}, B, \tilde{C})$ as $\bar{r} = (r, \ldots, r)^T \in \mathbb{R}^m$, we can use the canonical transformation to obtain:

$$\begin{aligned} dz &= W\tilde{A}W^{-1}z(t)dt + \tilde{N}dv, \quad \tilde{N} = \begin{pmatrix} \tilde{\tilde{N}} \\ 0_{(n-rm)\times rm} \end{pmatrix} \in \mathbb{R}^{n\times rm} \\ \tilde{\tilde{N}} &= \begin{pmatrix} I_{m(r-1)} & 0_{m(r-1)\times m} \\ 0_{m\times m(r-1)} & \tilde{C}\tilde{A}^{r-1}B \end{pmatrix} \in \mathbb{R}^{rm\times rm}, \end{aligned} \tag{6.6}$$

with $\tilde{C}\tilde{A}^{r-1}B \in \mathbb{R}^{m\times m}$. The next proposition states the stability of the origin for the dynamical system **HOSP**$(z_0, [0, T[)$, and extends the results of Section 4.5 in [1] to the MIMO case with feedback. Let $\mathcal{C} \stackrel{\Delta}{=} (I_{mr}, 0_{mr\times(n-rm)})$, $\mathcal{A} \stackrel{\Delta}{=} W\tilde{A}W^{-1}$, then the triplet $(\mathcal{A}, \tilde{N}, \mathcal{C})$ defines a system with rm inputs and rm outputs. The next proposition states conditions under which one can find a common Lyapunov function for the HOSP seen as switching DAEs, with possible state jumps and solutions being higher-degree distributions.

Proposition 6.2 *Suppose that the pair $(\mathcal{C}, \mathcal{A})$ is observable, the pair $(\mathcal{A}, \tilde{N})$ is controllable, and the matrix $\mathcal{A}$ is exponentially stable. If the transfer matrix $\mathcal{C}(sI_n - \mathcal{A})^{-1}\tilde{N}$ is strictly positive real, then the trivial solution of the (autonomous) HOSP is Lyapunov stable and globally attractive.*

Proof From the KYP Lemma for SPR systems [11, Lemma 3.11] there exists $\mathbb{R}^{n\times n} \ni \mathcal{P} = \mathcal{P}^T > 0$ such that $\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} < 0$ and $\mathcal{P}\tilde{N} = \mathcal{C}^T$. The second equality implies that $\mathcal{P} = \begin{pmatrix} P & 0_{rm\times(n-rm)} \\ 0_{(n-rm)\times rm} & P_\xi \end{pmatrix}$ with $P\tilde{\tilde{N}} = I_{rm}$. Thus, the strict positive realness implies that $\tilde{\tilde{N}} = P^{-1} > 0$, which in turn implies that $\tilde{C}\tilde{A}^{r-1}B > 0$. Let us now show that $\mathcal{P}$ defines a quadratic storage function for the autonomous HOSP (including state jumps), then derive a dissipation inequality. The transfer matrix being SPR, it is clear that on any interval (τ, s) not containing any state jump time, $s > \tau$, the dissipation equality $V(\{z\}(s)) - V(\{z\}(\tau)) = \int_{(\tau,s)} \underbrace{\frac{dv}{dt}(t)^T\mathcal{C}\{z\}(t)}_{=\chi_r(t)^T z^1(t)=0} dt + \int_{(\tau,s)} \{z\}(t)^T(\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A})\{z\}(t)dt$ holds with the storage function $V(\{z\}) = \frac{1}{2}\{z\}^T\mathcal{P}\{z\}$ [11, Example 4.65]. It is then sufficient to check that the storage function is non-increasing at state-discontinuity times. We can calculate that $V(t^+) - V(t^-) = (\{Z\}^r(t^+) + \{Z\}^r(t^-))^T\tilde{\tilde{N}}^{-1}(\{Z\}^r(t^+) - \{Z\}^r(t^-))$, where $\{Z\}^r = (\{z\}^1, \{z\}^2, \ldots, \{z\}^r)^T$ (see Section 4.1). Using (6.6), we have $\{Z\}^r(t^+) - \{Z\}^r(t^-) = \tilde{\tilde{N}}dv(\{t\})$; hence, we obtain $V(t^+) - V(t^-) = (\{Z\}^r(t^+) + \{Z\}^r(t^-))^T dv(\{t\}) = \sum_{i=1}^r \langle \{z\}^i(t^+) + \{z\}^i(t^-), \eta_i(t)\rangle$, with $\eta_i(t) \in -\partial\psi_{T^{i-1}_{\Phi_m}(\{Z\}^{i-1}(t^-))}(\zeta_i(t))$. We can rewrite $\{z\}^i(t^+) + \{z\}^i(t^-) = \zeta^i(t) - \frac{1-e_i}{1+e_i}\{z\}^i(t^-)$, and since $\{0\}$ belongs to the normal cone computed at any point of its domain of

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definition, it follows by maximal monotonicity of the normal cone mapping (to a convex set) that $\langle \{z\}^i(t^+) + \{z\}^i(t^-), \eta_i(t)\rangle \leqslant 0$ for all $1 \leqslant i \leqslant r$. Thus, Lyapunov stability is shown. Next, we have for all $s \geqslant 0$: $V(\{z\}(s)) - V(\{z\}(0)) \leqslant -\lambda_{\max}(-\mathcal{A}^T\mathcal{P} - \mathcal{P}\mathcal{A}) \int_{[0,s]} ||\{z\}(t)||^2 dt \leqslant -\frac{\lambda_{\max}(-\mathcal{A}^T\mathcal{P}-\mathcal{P}\mathcal{A})}{\lambda_{\min}(\mathcal{P})} \int_{[0,s]} V(\{z\}(t))dt$. Assume now that $V(t) \to V_\infty > 0$ as $t \to +\infty$. Thus, there exists $T < +\infty$ and $\delta > 0$ such that $V(t) \geqslant \delta$ for all $t \geqslant T$. We obtain for all $s > T$:

$$\begin{aligned} V(\{z\}(s)) &\leqslant V(\{z\}(0)) - \tfrac{\lambda_{\max}(-\mathcal{A}^T\mathcal{P}-\mathcal{P}\mathcal{A})}{\lambda_{\min}(\mathcal{P})} \left(\int_{[0,T]} V(\{z\}(t))dt - \int_{[T,s]} V(\{z\}(t))dt \right) \\ &\leqslant V(\{z\}(0)) - \tfrac{\lambda_{\max}(-\mathcal{A}^T\mathcal{P}-\mathcal{P}\mathcal{A})}{\lambda_{\min}(\mathcal{P})} \int_{[0,T]} V(\{z\}(t))dt - \tfrac{\lambda_{\max}(-\mathcal{A}^T\mathcal{P}-\mathcal{P}\mathcal{A})}{\lambda_{\min}(\mathcal{P})} \delta(s-T) \end{aligned} \tag{6.7}$$

Letting $s \to +\infty$ in both sides yields a contradiction, showing that $V(t) \to 0$ and consequently $\{z\}(t) \to 0$ as $t \to +\infty$, for any initial condition. □

Remark 6.1 *Continuity in the initial conditions usually does not hold in the MIMO case, a well-known fact in unilaterally constrained mechanical systems [5, 12]. This precludes in general the application of the Krasovskii–LaSalle invariance principle, except in cases where continuity holds* [18, Theorem 6.3.1].

Remark 6.2 *We do not investigate here conditions that guarantee the existence of K such that given (A, B, C) and L, then $(\mathcal{A}, \tilde{N}, \mathcal{C})$ is SPR (which is a novel kind of passification by feedback, where the passification of the operator $dv \mapsto \mathcal{C}z$ is done using the input $u(\cdot)$). The KYP Lemma yields a nonlinear matrix inequality in $\mathcal{P}$ and K. It is noteworthy that $(\mathcal{A}, \tilde{N}, \mathcal{C})$ may be SPR while (A, B, C) is not, e.g., for all systems with $r \geqslant 2$. We have exhibited a Lyapunov function and shown global stability of the origin, which proves that the origin is the unique fixed point under the above SPR condition. This could be shown directly as proved in* [13, Lemma 5].

6.3 Positive invariance

Nagumo's theorem states a sufficient condition for the positive invariance of sets $\mathcal{S}$ [8], for ODEs $\dot{x}(t) = f(x(t), t)$ with unique solutions, as $f(x, t) \in T_{\mathcal{S}}(x)$ for all $x \in \mathcal{S}$, where $T_{\mathcal{S}}(x)$ is the tangent cone (or in a more general setting the contingent cone [4]) to $\mathcal{S}$ at x. When the system is autonomous, the condition is also necessary [4, 17]. Its geometric interpretation is clear: on the boundary of $\mathcal{S}$ the vector field points inwards $\mathcal{S}$. For the second-order sweeping process applied to Lagrangian systems, Moreau's viability Lemma holds [12, Lemma 5.1] [24, Proposition 2.4], which states that if the initial position is admissible, then it is sufficient that the velocity belongs to the tangent cone linearization cone for almost all times, to guarantee that positions are admissible for all times. The HOSP is a differential inclusion which guarantees that $z^1(t^+) \in \Phi_m$ for any initial condition $\{z\}(0^-)$ (or, in the x-dynamics (1.1), the positive invariance of the set $\Phi_u = \{x \in \mathbb{R}^n \mid Cx + Fu \geqslant 0\}$). Moreau's viability Lemma is extended as follows.

Lemma 6.1 *Assume that z is a solution of HOSP$(z_0; I)$, that $z^1(0) \in \Phi_m$, and that $\{z^2\}(t) \in T_{\Phi_m}(z^1(t))$ for Lebesgue-almost all $t \in I$. Then, $z^1(t) \in \Phi_m$ for all $t \in I$.*

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Proof Suppose that there exists $t_1 > 0$ such that $z^1(t_1) \notin \Phi_m$, that is there exists a non-empty set $\mathcal{J} = \{j_1, j_2, ..., j_q\}$ such that $z^1_{j_i}(t_1) < 0$ for all $j_i \in \mathcal{J}$. By absolute continuity of $z^1(\cdot)$, we have $z^1_{j_i}(t_1) - z^1_{j_i}(0) = \int_{[0,t_1]} \{z^2_{j_i}\}(s)ds < 0$, since by assumption $z^1_{j_i}(0) \geqslant 0$. From the definition of the tangent cone, and the fact that $T_{\Phi_m}(Z^i) = \times_{k=1}^m T^i_{\Phi}(Z^i_k)$, it follows that almost everywhere $\{z^2_{j_i}\}(t) \in T_{\Phi}(z^1_{j_i}(t)) = I\!R_+$. Hence, a contradiction, and we infer that $\mathcal{J} = \emptyset$. □

Let us now prove the following result, which we may name the HOSP Viability Lemma of order 3, while Lemma 6.1 may be named the Viability Lemma of order 2.

Lemma 6.2 *Let z be a solution of HOSP$(z_0; I)$, with $z^1(\cdot)$, $\{z^2\}(\cdot)$ and $\{z^3\}(\cdot)$ absolutely continuous on $I = [0, T[$, $T > 0$. Assume that $z^1(0) \in \Phi_m$, $\{z^2\}(0) \in T_{\Phi_m}(z^1(0))$, and $\{z^3\}(t) \in T^2_{\Phi_m}(z^1(t), \{z_2\}(t))$ for all $t \in I$. Then $z^1(t) \in \Phi_m$ for all $t \in I$ and $\{z_2\}(t) \in T_{\Phi_m}(z^1(t))$ for all $t \in I$.*

Proof First of all, we remark that since $T_{\Phi_m}(Z^i) = \times_{k=1}^m T^i_{\Phi}(Z^i_k)$, we can do the proof for the case $m = 1$ in order to simplify the presentation. We can therefore adopt the notation for the SISO case in what follows. Let us consider two cases: **a)** $z_1(0) = 0$ with subcases **a1)** $\{z_2\}(0) = 0$, and **a2)** $\{z_2\}(0) > 0$, and **b)** $z_1(0) > 0$. Let us analyze case **b)**. If $z_1(t) > 0$ for all $t \in I$ then $T_{\Phi}(z_1(t)) = I\!R$ for all $t \in I$ and $\{z_2\}(t) \in T_{\Phi}(z_1(t))$ for all $t \in I$. If there exists $t_1 > 0$ such that $z_1(t_1) = 0$, then we can go to case **a)**, changing $t = t_1$ to $t = 0$. Let us therefore analyze case **a)**. In case **a2)**, we have $z_1(t) - z_1(0) = \int_0^t \{z_2\}(s)ds$ for all $t \in I$, by the absolute continuity. Also, $\{z_2\}(t) > 0$ in a right neighbourhood of $t = 0$, so that $z_1(t) > 0$ in this neighbourhood, and we are back to case **b)**. In case **a1)**, we have $z_1(0) = \{z_2\}(0) = 0$ so $\{z_3\}(0) \geqslant 0$. We consider two subcases **a11)** $\{z_3\}(0) = 0$, and **a12)** $\{z_3\}(0) > 0$. Let us start with **a12)**. In a right neighbourhood of $t = 0$, we have by continuity $\{z_3\}(t) > 0$, consequently $\{z_2\}(t) > 0$ and $z_1(t) > 0$ in this neighbourhood, since $\{z_2\}(t) - \{z_2\}(0) = \int_0^t \{z_3\}(s)ds$ and $z_1(t) - z_1(0) = \int_0^t \{z_2\}(s)ds$ for all $t \in I$. So we are back to case **b)**. We can now split case **a11)** into three subcases: **a111)** $\{z_4\}(0) = 0$, **a112)** $\{z_4\}(0) > 0$, **a113)** $\{z_4\}(0) < 0$. In case **a113)**, we have $\{z_3\}(t) - \{z_3\}(0) = \int_0^t \{z_4\}(s)ds$ so that $\{z_3\}(t) = \int_0^t \{z_4\}(s)ds < 0$ in a right neighbourhood of $t = 0$. However, by assumption $\{z_3\}(t) \in T^2_{\Phi}(z_1(t), \{z_2\}(t))$ for all $t \in I$, so in particular $\{z_3\}(0) \in T^2_{\Phi}(0, 0) = I\!R_+$ so that $\{z_3\}(0) \geqslant 0$ in a right neighbourhood of $t = 0$: this is a contradiction, and we infer that case **a113)** is impossible. Thus, only **a111)** and **a112)** are possible. In case **a112)**, we have $z_1(0) = \{z_2\}(0) = \{z_3\}(0) = 0$ and $\{z_4\}(0) > 0$. Again by integrating in a right neighbourhood of $t = 0$, we get $\{z_3\}(t) > 0$, $\{z_2\}(t) > 0$ and $\{z_1\}(t) > 0$ in this neighbourhood, and we are back to case **b)**. Case **a111)** is $z_1(0) = \{z_2\}(0) = \{z_3\}(0) = \{z_4\}(0) = 0$. As we saw in case **a113)** we cannot have $\{z_4\}(t) < 0$ in a right neighbourhood of $t = 0$. Thus we can only get $\{z_4(t)\} \geqslant 0$, i.e, either $\{z_4\}(t) = 0$ almost everywhere or not in a right neighbourhood of $t = 0$. Integrating again on $[0, t]$ for some small enough $t > 0$, we infer that in this neighbourhood $z_1(t) \in \Phi$ with either $z_1(t_1) = 0$ or $z_1(t_1) > 0$ for some $t_1 > 0$, and in both cases using similar arguments as above, we deduce that $\{z_2(t)\}(t) \in T_{\Phi}(z_1(t))$ for all $t \in [0, t_1]$. Since the solution is supposed to exist on I, the result is proved. □

Viability Lemmas of higher order could be proved in a similar way.

7 Conclusions

In this article, we have shown that the autonomous HOSP introduced in [1], can be extended when an exogenous term [a control, or a disturbance $u(\cdot)$] acts in the smooth dynamics as well as in the inequality function, and is well-posed provided that the exogenous term satisfies some analycity conditions. The link with complementarity systems and switching DAEs is studied. Stability and stabilization by state feedback issues are analyzed under positive real constraints. The so-called high-order bouncing ball illustrates how complex such differential inclusions (with distribution solutions) may be, with Zeno behaviours for transitions between constrained and unconstrained modes. It is noteworthy that the time-discretization of the HOSP, which is detailed in [1], also applies to the non-autonomous case studied above.

Topics of future investigations are numerous: relax the uniqueness of solutions, which is not a crucial property for stability; relax the Stieltjes property of the matrix $CA^{r-1}B$ in the MIMO case; analyze the MIMO case with vector relative degree $\bar{r} = (r_1, r_2, \ldots, r_m)^T$, with $r_i \neq r_j$, $i \neq j$, which would allow to switch between DAEs with different indices; find conditions such that switching feedback controllers allow the HOSP to coincide with given switching DAEs and/or LCS (said another way: try to recast larger classes of switching DAEs or LCSs into an HOSP framework); use switching strategies in the HOSP with time-dependent switching state feedback controllers, to study new stabilization strategies, with possibly non-monotonic Lyapunov functions; study cases of globally well-posed nonlinear HOSP (involving products of distributions) relying on non-standard analysis [6]; analyze the case of mixed unilateral and bilateral (equality) constraints; analyze switching between systems (HOSP) with varying state dimension (switching dynamic feedback controllers).

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Appendix A Auxiliary lemma

Lemma A.1 *Let $r \geqslant 2$ and $i \in \{2, \ldots, r\}$. Then, $\frac{r!}{(i-1)!(r-i+1)!} > 1$. Consequently one has $\frac{(i-1)!(r-i+1)!}{r!-(i-1)!(r-i+1)!} > 0$.*

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Proof A simple calculation shows that when $i = 2$ and $i = r$, then $\frac{r!}{(i-1)!(r-i+1)!} = r$. Let us consider first $r = 2\alpha+1$, $\alpha \in I\!N$, $\alpha \geqslant 1$, and $i \leqslant \frac{r}{2}+1 \Leftrightarrow i \leqslant \alpha+1$ since i and α are integers. Then, $r-i+1 \geqslant \alpha+1$. Thus, $r-i+1 \geqslant i$. Let us examine the term $(i-1)!(r-i+1)!$. Let us increase i to $i+1$. We get $i!(r-i)!$. Doing so the term $(i-1)!$ is multiplied by i while the term $(r-i+1)!$ is divided by $r-i+1$. Since $r-i+1 \geqslant i$ one finds that $i!(r-i)! \leqslant (i-1)!(r-i+1)!$. By induction it follows that the maximum value is attained at $i = 2$. The problem is symmetric in the sense that $(i-1)!(r-i+1)!$ has the same value for $i = k$ and $i = r-k+2$. By the symmetry one may now consider $r \geqslant i \geqslant \frac{r}{2}+1$ and conclude similarly. The reasoning can be applied for $i = 2, 3, \ldots, k$ with $k < \frac{r}{2}+1$, which shows that the maximum is indeed attained for $i = 2$ and $i = r$. In the even case with $r = 2\alpha$, $\alpha \in I\!N$, $\alpha \geqslant 1$, one gets $i < \frac{r}{2}+1 \Leftrightarrow i \leqslant \alpha$ so that $r-i+1 > \alpha$ and $r-i+1 > i$. A reasoning similar to the odd case applies and by symmetry one considers integers $r \geqslant i \geqslant \alpha+2$. For $i = \alpha+1$, one gets $\frac{r!}{(i-1)!(r-i+1)!} = \frac{r!}{\alpha!\alpha!} > 1$ for all $r \geqslant 2$. Therefore, $-1 + \frac{r!}{(i-1)!(r-i+1)!} > 0$, and the last statement follows. □

Appendix B Some mathematical definitions

The next notions may be found in [1,20]. Let I denote a non-degenerate real interval (not empty, nor reduced to a singleton).

- By $z \in BV(I; I\!R^n)$, it is meant that z is a $I\!R^n$-valued function of Bounded Variation if there exists a constant $C > 0$ such that for all finite sequences $t_0 < t_1 < \ldots. < t_N$ (N arbitrary) of points of I, we have $\sum_{i=1}^{N} \|z(t_i) - z(t_{i-1})\| \leqslant C$. Let J be a subinterval of I. The real number $\mathrm{var}(z, J) \stackrel{\Delta}{=} \sup \sum_{i=1}^{N} \|z(t_i) - z(t_{i-1})\|$, where the supremum is taken with respect to all the finite sequences $t_0 < t_1 < \ldots < t_N$ (N arbitrary) of points of J, is called the variation of z in J.

  Any BV function has a countable set of discontinuity points and is almost everywhere differentiable. A BV function defined on $[a, b] \subset I$ possesses left-limits in $]a, b]$ and right-limits in $[a, b[$. Moreover, the functions $t \mapsto z(t^+) \stackrel{\Delta}{=} \lim_{s\to t, s>t} z(s)$ and $t \mapsto z(t^-) \stackrel{\Delta}{=} \lim_{s\to t, s<t} z(s)$ are both BV functions.
- We denote by $LBV(I; I\!R^n)$ the space of functions of Locally Bounded Variation, i.e. of bounded variation on every compact subinterval of I.
- We denote by $RCLBV(I; I\!R^n)$ the space of Right-Continuous functions of Locally Bounded Variation. It is known that if $z \in RCLBV(I; I\!R^n)$ and $[a, b]$ denotes a compact subinterval of I, then z can be represented in the form:

$$z(t) = \mathcal{J}_z(t) + [z](t) + \zeta_z(t), \text{forall } t \in [a, b],$$

  where $\mathcal{J}_z$ is a jump function, $[z]$ is an absolutely continuous function and ζ_z is a singular function. Here, $\mathcal{J}_z$ is a jump function in the sense that $\mathcal{J}_z$ is right-continuous and given any $\varepsilon > 0$, there exist finitely many points of discontinuity $t_1, \ldots, t_N$ of $\mathcal{J}_z$ such that $\sum_{i=1}^{N} \|\mathcal{J}_z(t_i) - \mathcal{J}_z(t_i^-)\| + \varepsilon > \mathrm{var}(\mathcal{J}_z, [a, b])$, $[z]$ is an absolutely continuous function in the sense that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^{N} \|[z](\beta_i) - [z](\alpha_i)\| < \varepsilon$, for any collection of disjoint subintervals $]\alpha_i, \beta_i] \subset [a, b] (1 \leqslant i \leqslant N)$ such that $\sum_{i=1}^{N} (\beta_i - \alpha_i) < \delta$, and ζ_z is a singular function in the sense that ζ_z is a continuous and bounded variation function on $[a,b]$ such that $\dot{\zeta}_z = 0$ almost everywhere on $[a,b]$.

- By $z \in RCSLBV(I;\mathbb{R}^n)$ it is meant that z is a Right-Continuous function of Special Locally Bounded Variation, i.e., z is of bounded variation and can be written as the sum of a jump function and an absolutely continuous function on every compact subinterval of I. So, if $z \in RCSLBV(I;\mathbb{R}^n)$ then

$$z = [z] + \mathcal{J}_z, \tag{B 1}$$

where $[z]$ is a locally absolutely continuous function called the absolutely continuous component of z, and $\mathcal{J}_z$ is uniquely defined up to a constant by

$$\mathcal{J}_z(t) = \sum_{t \geqslant t_n} z(t_n^+) - z(t_n^-) = \sum_{t \geqslant t_n} z(t_n) - z(t_n^-), \tag{B 2}$$

where $t_1, t_2, \ldots, t_n, \ldots$ denote the countably many points of discontinuity of z in I. Notice that the notion of solutions that is used in [29] for switched DAEs (with exogenous switching times) has the same structure as (B 1) with a 'smooth' and a jump parts (without finite accumulations of jump instants).

Differential (or Stieltjes) measure. Let $z \in LBV(I;\mathbb{R}^n)$ be given. We denote by dz the Stieltjes or differential measure generated by z. For $a \leqslant b$, $a, b \in I$ one has $dz([a,b]) = z(b^+) - z(a^-)$, $dz([a,b[) = z(b^-) - z(a^-)$, $dz(]a,b]) = z(b^+) - z(a^+)$, $dz(]a,b[) = z(b^-) - z(a^+)$. In particular, we have $dz(\{a\}) = z(a^+) - z(a^-)$.

Appendix C Calculation of the canonical form (4.17)

Starting from the x-dynamics in (4.17), one calculates $w(s) = \frac{Cs^2 - CLs}{s^2(LCs^2+RCs-1)}\lambda(s) - \frac{Cs^3}{s^2(LCs^2+RCs-1)}v_2(s)$. Starting from the z_2-dynamics one finds that $w(s) = z_1(s) = \frac{Cs^2}{s^2(LCs^2+RCs-1)}\lambda(s) - \frac{Cs^3}{s^2(LCs^2+RCs-1)}v_2(s) - \frac{LCs}{s^2(LCs^2+RCs-1)}sx_4(s)$. Let $sx_4(s) = H(s)z_1(s) + G(s)v_2(s)$ for some transfer functions $H(s)$ and $G(s)$ to be calculated. We obtain $z_1(s) = \left(1 + \frac{LCH(s)}{LCs^2+RCs-1}\right)^{-1} \frac{C}{LCs^2+RCs-1}\lambda(s) - \left(1 + \frac{LCH(s)}{LCs^2+RCs-1}\right)^{-1} \frac{Cs+LCG(s)}{LCs^2+RCs-1}v_2(s)$. Equalling both expressions, we obtain $H(s) = \frac{1}{LC}\left(\frac{s(LCs^2+RCs-1)}{s-L} - \frac{(LCs^2+RCs-1)(s-L)}{s-L}\right) = \frac{LCs^2+RCs-1}{C(s-L)}$. Similar calculations yield $\left(1 + \frac{LCH(s)}{LCs^2+RCs-1}\right)^{-1} \frac{Cs+LCG(s)}{LCs^2+RCs-1} = \frac{C(s-L)s+C(s-L)LG(s)}{s(LCs^2+RCs-1)}$ while we want this expression to equal $\frac{Cs}{LCs^2+RCs-1}$. This yields $G(s) = 0$. Hence, $x_4(s) = \frac{LCs^2+RCs-1}{C(s-L)s}z_1(s)$. Since $\dot{z}_2(t) = -\frac{R}{L}z_2(t) - \frac{1}{LC}z_1(t) - x_4(t) - \frac{\dot{v}_2(t)}{L} + \frac{\lambda(t)}{L}$, the result follows.

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