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L^p-APPROXIMATION OF HOLOMORPHIC FUNCTIONS ON A CLASS OF CONVEX DOMAINS

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Abstract

Let Ω be a member of a certain class of convex ellipsoids of finite/infinite type in $\mathbb{C}^2$. In this paper, we prove that every holomorphic function in $L^p(\Omega)$ can be approximated by holomorphic functions on $\bar{\Omega}$ in $L^p(\Omega)$-norm, for $1 \leq p < \infty$. For the case $p = \infty$, the continuity up to the boundary is additionally required. The proof is based on L^p bounds in the additive Cousin problem.

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1. Introduction and main theorem

Let $\Omega \subset \mathbb{C}^2$ be a bounded domain, with smooth boundary $b\Omega$. The smoothness means that Ω admits a smooth, global defining function ρ on a neighbourhood of $\bar{\Omega}$ in the sense that $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ and $\nabla\rho \neq 0$ on $b\Omega = \{z \in \mathbb{C}^2 : \rho(z) = 0\}$, and $\nabla\rho \perp b\Omega$.

The main purpose of this paper is to study the L^p global approximation question: *Can every holomorphic function in $L^p(\Omega)$ be approximated by holomorphic functions on $\bar{\Omega}$ in $L^p(\Omega)$-norm, for $1 \leq p \leq \infty$?*

This problem is simple and classical when Ω is a domain in the complex plane (see, for example, [13] or [5]). In higher dimensions, it is a difficult problem because the boundary behaviour of domains in $\mathbb{C}^n$ for $n \geq 2$ is more complicated than in $\mathbb{C}$. Lieb [11] and Kerzman [9] independently obtained the first significant results by applying the L^p-estimates for the Henkin solution of the $\bar{\partial}$ equation to give a positive answer to the problem on strongly pseudoconvex domains. Their method provides a connection between the approximation problem and the additive Cousin problem in several complex variables (see [6]). Via this argument, Cole and Range [3] extended the results on A-measure in Henkin [7] to relatively compact, strongly pseudoconvex subdomains of complex manifolds.

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We would like to extend the result of Kerzman and Lieb to more general domains in $\mathbb{C}^2$. Unfortunately, the Henkin solutions are not available on weakly pseudoconvex domains (even of finite type) as shown in [10]. Therefore, we consider a more restricted class of convex domains on which we can establish the Henkin solutions.

Let Ω be a smooth, bounded domain in $\mathbb{C}^2$, with defining function ρ such that for any $p \in b\Omega$, there exist a neighbourhood $U_p = B(p, \delta)$ of p, a function F_p and coordinates $z_p = (z_{p,1}, z_{p,2})$ with the origin at p and such that

$$\Omega \cap U_p = \{z_p = (z_{p,1}, z_{p,2}) \in \mathbb{C}^2 : \rho(z_p) = F_p(|z_{p,1}|^2) + r_p(z_p) < 0\} \tag{1.1}$$

or

$$\Omega \cap U_p = \{z_p = (z_{p,1}, z_{p,2}) \in \mathbb{C}^2 : \rho(z_p) = F(x_{p,1}^2) + r_p(z_p) < 0\}, \tag{1.2}$$

where $z_{p,j} = x_{p,j} + iy_{p,j}$, with $x_{p,j}, y_{p,j} \in \mathbb{R}$, $j = 1, 2$, and $i = \sqrt{-1}$. We also assume that the functions $F_p : \mathbb{R} \to \mathbb{R}$ and $r_p : \mathbb{C}^2 \to \mathbb{R}$ satisfy:

(i) $F_p(0) = 0$;
(ii) $F_p'(t), F_p''(t), F_p'''(t)$ and $(F_p(t)/t)'$ are nonnegative on $(0, \delta)$;
(iii) $r_p(0) = 0$ and $\partial r_p/\partial z_{p,2} \neq 0$;
(iv) r_p is convex.

The class of such domains includes the following two well-known examples.

Example 1.1. If $F_P(t^2) = t^{2m}$ at the point $P \in b\Omega$, then $\Omega \cap U_P$ is convex of finite type $2m$ at P. In particular, when $m = 1$, Ω is strictly convex or, equivalently, strongly pseudoconvex at P.

Example 1.2. If $F_P(t^2) = 2\exp(-1/t^\alpha)$ for $0 < \alpha < 1$ or $F_P(t^2) = 2\exp(-1/t|\ln t|^\alpha)$ for $\alpha > 2$ at the point $P \in b\Omega$, then $\Omega \cap U_P$ is of infinite type at P.

Let $H^\infty(\Omega)$ be the weak-star closure of the algebra of functions that are continuous on $\bar{\Omega}$ and holomorphic in Ω. The following is our main result.

Theorem 1.3 (Global L^p approximation theorem). *Assume either of the following conditions hold:*

(i) *Ω is defined by (1.1) and there is a $\delta > 0$ such that*

$$\int_0^\delta |\ln F_P(t^2)|\, dt < \infty \quad \textit{for all } P \in b\Omega;$$

(ii) *Ω is defined by (1.2) and there is a $\delta > 0$ such that*

$$\int_0^\delta |\ln(t) \ln F_P(t^2)|\, dt < \infty \quad \textit{for all } P \in b\Omega.$$

Then, each holomorphic function $f \in L^p(\Omega)$ can be approximated in $L^p(\Omega)$-norm by holomorphic functions $\{f^\tau\}_{\tau \in (0,\tau_0)}$ on $\bar{\Omega}$ (as $\tau \to 0^+$), for some small τ_0, and for $1 \leq p < \infty$.

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Moreover, if the holomorphic function f only belongs to $H^\infty(\Omega) \cap C(\bar{\Omega})$, we also obtain a family of holomorphic functions $\{f^\tau\}_{\tau\in(0,\tau_0)}$ on $\bar{\Omega}$ so that:

(a) $\|f^\tau\|_{H^\infty(\Omega)} \lesssim \|f\|_{H^\infty(\Omega)}$ *for all* $\tau \in (0, \tau_0)$*;*
(b) $f^\tau \to f$ *in* $L^p(\Omega)$*-norm as* $\tau \to 0^+$*, for all* $1 \le p < \infty$*;*
(c) $f^\tau \to f$ *uniformly on* $\bar{\Omega}$ *as* $\tau \to 0^+$.

Here and in what follows, the notations $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant and $\approx$ means the combination of $\lesssim$ and $\gtrsim$.

In [4], the authors provide an example to show that the approximation theorem does not hold in general on smoothly bounded pseudoconvex domains. In 1978, Bedford and Fornaess [1] established the theorem on weakly pseudoconvex domains with real analytic boundary in $\mathbb{C}^2$. More generally, Beatrous and Range [2] obtained the result on weakly pseudoconvex domains in $\mathbb{C}^n$ under the additional condition that the closure of the domain is holomorphically convex.

The paper is organised as follows. In Section 2, we solve the additive Cousin problem on Ω. Section 3 is devoted to proving the global L^p approximation theorem.

2. The solution of the additive Cousin problem

THEOREM 2.1. *Assume the conditions on Ω in Theorem 1.3 hold. Let $V_j = U_j \cap \Omega$, where $\{U_j\}_{j=0,1,\dots,N}$ is an open covering of $\bar{\Omega}$. Then we can find a finite positive constant C such that the following property holds.*

If the holomorphic functions g_{ij} on $V_i \cap V_j$ satisfy

$$\begin{aligned} g_{ij} &= -g_{ji}, \\ g_{ij} + g_{jk} + g_{ki} &= 0, \end{aligned} \tag{2.1}$$

for all $i, j, k = 0, 1, \dots, N$, then there are holomorphic functions g_j on V_j, for $j = 0, 1, \dots, N$, such that

$$\begin{aligned} g_j - g_i &= g_{ij} \quad \text{on } V_i \cap V_j, \\ \|g_j\|_{L^p(V_j)} &\lesssim M_p(\{g_{ij}\}) \quad \text{for } 1 \le p \le \infty, \end{aligned}$$

where $M_p(\{g_{ij}\}) = \max\{\|g_{ij}\|_{L^p(V_i\cap V_j)} : i, j = 0, 1, \dots, N\}$.

PROOF. The proof comprises two steps. The first is to construct functions $v_j \in C^\infty(V_j)$, $j = 0, 1, \dots, N$, which satisfy $v_j - v_i = g_{ij}$ on $V_i \cap V_j$, for all $i, j = 0, 1, \dots$. The second is to change these nonholomorphic functions into holomorphic functions by using the following theorem.

THEOREM 2.2 [8, Theorem 1.2]. *If there exists $\delta > 0$ and either of the conditions* (i) *or* (ii) *in Theorem 1.3 hold, then for any $\bar{\partial}$-closed $(0, 1)$-form ϕ in $L^p(\Omega)$ with $1 \le p \le \infty$, the Henkin kernel solution u on Ω satisfies $\bar{\partial}u = \phi$ and*

$$\|u\|_{L^p(\Omega)} \lesssim \|\phi\|_{L^p(\Omega)}.$$

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Step 1. On $\bar{\Omega}$, we choose a partition of unity $\{\chi_j\}_{j=0,1,\ldots,N}$, where the χ_j are smooth functions with compact support in U_j for $j = 0, 1, \ldots, N$ and $\sum_{j=0}^N \chi_j = 1$ on $\bar{\Omega}$. Set

$$v_j = \sum_{\nu=0}^{N} \chi_\nu g_{\nu j}.$$

From the local finiteness of $\{V_j\}$, the functions v_j, $j = 0, 1, \ldots, N$, are smooth on V_j and, by the Minkowski inequality,

$$\|v_j\|_{L^p(V_j)} \lesssim M_p(\{g_{ij}\}). \tag{2.2}$$

Moreover,

$$v_j - v_i = \sum_{\nu=0}^{N} \chi_\nu g_{\nu j} - \sum_{\nu=0}^{N} \chi_\nu g_{\nu i} = \sum_{\nu=0}^{N} \chi_\nu (g_{\nu j} - g_{\nu i}) = \sum_{\nu=0}^{N} \chi_\nu g_{ij} = g_{ij},$$

where we have used (2.1) to replace $g_{\nu j} - g_{\nu i}$ by g_{ij}. Note that the functions v_j, $j = 0, 1, \ldots, N$, are not holomorphic. However, since $\bar{\partial} g_{ij} = 0$ on $V_i \cap V_j$, then

$$\bar{\partial} v_i = \bar{\partial} v_j \quad \text{on } V_i \cap V_j \text{ for all } i, j = 0, 1, \ldots, N. \tag{2.3}$$

Step 2. The above identity (2.3) implies that there is a smooth, globally well-defined $(0, 1)$-form ϕ on Ω, which is locally equal to $\bar{\partial} v_j$ on V_j, for $j = 0, 1, \ldots, N$.

Since $\bar{\partial} v_j = \sum_{\nu=0}^N (\bar{\partial} \chi_\nu) g_{\nu j}$, it follows that

$$\|\phi\|_{L^p_{0,1}(\Omega)} \le \sum_{j=0}^{N} \|\bar{\partial} v_j\|_{L^p(V_j)} \lesssim M_p(\{g_{ij}\}).$$

Since $\bar{\partial}\phi = 0$, by Theorem 2.2, there is a function u satisfying $\bar{\partial} u = \phi$ on Ω and

$$\|u\|_{L^p(\Omega)} \lesssim \|\phi\|_{L^p(\Omega)} \lesssim M_p(\{g_{ij}\}), \tag{2.4}$$

for $1 \le p \le \infty$. Now, on each V_j, for $j = 0, 1, \ldots, N$, we define

$$g_j = v_j - u,$$

so $\bar{\partial} g_j = \bar{\partial} v_j - \bar{\partial} u = \bar{\partial} v_j - \phi = 0$ on V_j. Thus, each function g_j is holomorphic in V_j, for $j = 0, 1, \ldots, N$. Moreover,

$$g_j - g_i = (v_j - u) - (v_i - u) = v_j - v_i = g_{ij} \quad \text{on } V_i \cap V_j.$$

Finally, (2.2) and (2.4) imply

$$\|g_j\|_{L^p(V_j)} \lesssim M_p(\{g_{ij}\}) \quad \text{for } 1 \le p \le \infty.$$

This completes the proof. □

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3. Proof of the global L^p approximation theorem

For convenience, we recall a preparation lemma which was proved in [3, 9] and [12] on arbitrary smooth domains.

Let $\{U_j, j = 1, \dots, N\}$ be an open covering of $b\Omega$ by neighbourhoods U_j of boundary points $P_j \in b\Omega$ such that there is a constant $\tau_0 > 0$ for which

$$z + \tau\mu_j \in \Omega \quad \text{for all } z \in \bar{\Omega} \cap U_j \quad \text{and} \quad 0 < \tau < \tau_0.$$

Here μ_j is the unit inner normal to $b\Omega$ at P_j. We choose $\chi_j \in C_0^\infty(U_j)$, $\chi_0 \in C_0^\infty(\Omega)$, so that $\sum_{j=0}^N \chi_j = 1$ on a neighbourhood $\widetilde{\Omega}$ of $\bar{\Omega}$. For $0 < \tau < \tau_0$, we choose $\eta(\tau) > 0$ (in fact, $\lim_{\tau\to 0^+} \eta(\tau) = 0$) such that

$$\Omega_{\eta(\tau)} := \{z \in \mathbb{C}^2 : \rho(z) < \eta(\tau)\} \subset \widetilde{\Omega} \cap \bigg(\bigcup_{j=0}^N U_j^\tau\bigg),$$

where $U_0^\tau = \Omega$ and $U_j^\tau = \{w - \tau\mu_j : w \in U_j \cap \Omega\} \cap U_j$, for $j = 1, \dots, N$. Moreover, when τ_0 is sufficiently small, $\{U_j^\tau : j = 0, 1, \dots, N\}$ is a covering of $\bar{\Omega}$, the L^p estimates for the Henkin solutions to the $\bar{\partial}$ equations on $\Omega_{\eta(\tau)}$ are independent of τ, and

$$\operatorname{supp}\chi_j \cap \bar{\Omega}_{\eta(\tau)} \subset U_j^\tau \quad \text{for all } 0 < \tau < \tau_0 \quad \text{and} \quad j = 0, 1, \dots, N.$$

LEMMA 3.1. *Suppose that $1 \le p \le \infty$ and $f \in L^p(\Omega)$ is holomorphic on Ω. For $0 < \tau < \tau_0$, define $f_0^\tau = f$ and*

$$f_j^\tau(z) = f(z + \tau\mu_j) \quad \textit{for } j = 1, \dots, N.$$

Then the following statements hold:

(a) *f_j^τ is holomorphic on U_j^τ and $L^p(U_j^\tau)$-integrable for $j = 0, 1, \dots, N$;*
(b) *$\lim_{\tau\to 0^+} f_j^\tau = f$ pointwise on $\Omega \cap U_j$;*
(c) *$\lim_{\tau\to 0^+} \|f_j^\tau - f\|_{L^p(U_j\cap\Omega)} = 0$ if either $1 \le p < \infty$, or $p = \infty$ and $f \in C(\bar{\Omega})$.*
(d) *Define $g_{ij}^\tau = f_j^\tau - f_i^\tau$ on $U_i^\tau \cap U_j^\tau$ and*

$$M_p^\tau(\{g_{ij}^\tau\}) = \max\{\|g_{ij}^\tau\|_{L^p(U_i^\tau\cap U_j^\tau)} : i, j = 0, 1, \dots, N\}.$$

Then

$$\lim_{\tau\to 0^+} M_p^\tau(\{g_{ij}^\tau\}) = 0 \quad \textit{if } 1 \le p < \infty, \textit{ or if } p = \infty \textit{ and } f \in C(\bar{\Omega}),$$

and

$$M_\infty^\tau(\{g_{ij}^\tau\}) \lesssim \|f\|_{L^\infty(\Omega)} \quad \textit{if } f \in L^\infty(\Omega).$$

PROOF OF THEOREM 1.3. The main idea is to apply the construction of the additive Cousin problem. Set

$$V_j^\tau = U_j^\tau \cap \Omega_{\eta(\tau)} \quad \text{for } 0 < \tau < \tau_0.$$

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Applying Theorem 2.2 to the holomorphic functions g_{ij}^τ on $V_i^\tau \cap V_j^\tau$, we obtain holomorphic functions g_j^τ on V_j^τ for $j = 0, 1, \ldots, N$, which satisfy

$$g_j^\tau - g_i^\tau = g_{ij}^\tau \quad \text{on } V_i^\tau \cap V_j^\tau \tag{3.1}$$

and

$$\|g_j^\tau\|_{L^p(V_j^\tau)} \lesssim M_p^\tau(\{g_{ij}^\tau\}). \tag{3.2}$$

The constant C implied in (3.2) is independent of τ since the L^p estimates of the Henkin solution and the partition of unity $\{\chi_j\}$ are independent of τ. By the definition of the f_j in Lemma 3.1 and (3.1),

$$f_j^\tau - g_j^\tau = f_i^\tau - g_i^\tau \quad \text{on } V_i^\tau \cap V_j^\tau.$$

Therefore, we can find a globally well-defined function f^τ which is holomorphic on $\Omega_{\eta(\tau)}$ such that

$$f^\tau = f_j^\tau - g_j^\tau \quad \text{on } V_j^\tau \tag{3.3}$$

and

$$\|f - f^\tau\|_{L^p(\Omega)} \le \sum_{j=1}^{N} \|f - f_j^\tau\|_{L^p(U_j \cap \Omega)} + (N+1) C M_p^\tau(\{g_{ij}^\tau\}).$$

Combining this estimate with Lemma 3.1,

$$\lim_{\tau \to 0^+} \|f - f^\tau\|_{L^p(\Omega)} = 0$$

if either $1 \le p < \infty$, or $p = \infty$ and f extends continuously to $\bar{\Omega}$.

Finally, if $f \in H^\infty(\Omega) \cap C(\bar{\Omega})$, then (3.3), (3.2) and Lemma 3.1 also imply

$$\|f^\tau\|_{L^\infty(\Omega)} \lesssim \|f\|_{L^\infty(\Omega)} \quad \text{uniformly in } \tau \in (0, \tau_0).$$

Since H^∞ is a subset of $L^p(\Omega)$-functions which are holomorphic on Ω for $1 \le p < \infty$, the limit

$$\lim_{\tau \to 0^+} \|f - f^\tau\|_{L^p(\Omega)} = 0$$

also holds for any $1 \le p < \infty$. This completes the proof. □

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