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Smooth Mappings with Higher Dimensional Critical Sets

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Abstract. In this paper we provide lower bounds for the dimension of various critical sets, and we point out some differential maps with high dimensional critical sets.

1 Introduction

Let M^n, N^n be differential orientable manifolds with M compact and let ω_N be a volume form on N. We first observe that the critical set $C(f)$ of a differentiable mapping $f\colon M \to N$ is actually the set $V(f^*\omega_N)$ of zeros of the form $f^*\omega_N$. According to Church and Timourian [1], except for a number of special cases, those differentiable mappings from M to N which are not topological coverings have critical sets of dimension at least one. Obviously, the equality $\int_M f^*\omega_N = 0$, which is equivalent to $\deg(f) = 0$, implies that f is not a topological covering, since the absolute degree of a covering map is at least one, being the number of its sheets [3, p. 258]. In this paper we first show that the mappings of zero degree have actually higher dimensional critical sets and then adjust the zero codimension case to higher codimension cases for further developments. For the higher codimension case, the role of the form $f^*\omega_N$ will be played by forms of type $f^*\omega_N \wedge \theta$, where $f\colon M^m \to N^n, m > n$ and $\theta \in \Omega^{m-n}(M)$ are closed. The equality $\int_M f^*\omega_N \wedge \theta = 0$ will allow us to provide lower bounds for various combinations of the dimension of the critical set $C(f)$, the dimension of the set $B(f) := f(C(f))$ of critical values of f and the dimension of the set $U(f,g)$ of points $x \in M$, at which the fibers of f and g through x are not transversal (here $g\colon M^m \to P^{m-n}$ is another differentiable mapping). We are next concerned about some classes of maps for which the equality $\int_M f^*\omega_N \wedge \theta = 0$ holds for every closed differential form $\theta \in \Omega^{m-n}(M)$. The class of maps for which $f^*\omega_N$ is exact obviously satisfies the equality, and one of its subclasses consists of those maps which are homotopic with maps for which every point is critical.

2 Preliminary Results

Let M be a differential manifold and f, g be differential maps defined on M. We define $U(f,g)$ to be the set of points $p \in M$ which are simultaneously regular for f and g, and the fibers $f^{-1}(f(p))$, $g^{-1}(g(p))$ are untransversal. If θ is a differential form on M, we define $V(\theta)$ to be the vanishing set of θ. If the degree of θ coincide

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to the codimension of f, we also define $\mathcal{V}_f(\theta)$ as the set of regular points p of f at which θ vanishes along the fiber of f through p. Some relations among the sets $V(\theta)$, $\mathcal{V}_f(\theta)$, $U(f,g)$ and $C(f\times g)$ are proved first. We pay some special attention to the particular cases $\theta = f^*\omega_N$ and $\theta = f^*\omega_N \wedge \alpha$, where ω_N is a volume form on the orientable target manifold N of f and α is some closed form on the source manifold M of f. We then exploit these relations to get, on one hand, some lower bounds for the dimension of the critical set of f, and, on the other, some relations involving the dimensions of the previously mentioned sets.

Proposition 2.1 *Let M, N be differential manifolds such that* $\dim M \geq \dim N$ *and N is orientable. If ω_N is a volume form on N and $f\colon M \to N$ is a differentiable mapping, then $(f^*\omega_N)_p = 0$ if and only if p is a critical point of f. In other words $V(f^*\omega_N) = C(f)$.*

Proof Indeed we have successively,

$$\begin{aligned}(f^*\omega_N)_p = 0 &\Leftrightarrow (f^*\omega_N)_p(u_1,\dots,u_n) = 0 \text{ for all } u_1,\dots,u_n \in T_p(M)\\ &\Leftrightarrow (\omega_N)_{f(p)}((df)_p(u_1),\dots,(df)_p(u_n)) = 0\\ &\qquad\qquad\qquad\qquad \text{for all } u_1,\dots,u_n \in T_p(M)\\ &\Leftrightarrow (df)_p(u_1),\dots,(df)_p(u_n) \text{ are linearly dependent}\\ &\qquad\qquad\qquad\qquad \text{for all } u_1,\dots,u_n \in T_p(M)\\ &\Leftrightarrow \dim[\mathrm{Im}(df)_p] < \dim N\\ &\Leftrightarrow \mathrm{rank}_p f < \dim N\\ &\Leftrightarrow p \in C(f).\end{aligned}$$

■

Remark 2.2 If N is a compact, connected, orientable, n-dimensional manifold, $\varphi\colon N \to \mathbb{R}$ is a differentiable function, and ω_N is a volume form on N, then $\int_N \varphi\omega_N = 0$ implies that either $\varphi \equiv 0$, or $N\setminus\varphi^{-1}(0)$ is not connected, namely $\varphi^{-1}(0) = V(\varphi\omega_N)$ separates N. Consequently, the equality $\int_N \theta = 0$ implies that either $\theta = 0$, or $V(\theta)$ separates N, for every differential form $\theta \in \Omega^n(N)$.

Theorem 2.3 ([5, p. 48]) *Every connected differential manifold M^n is a Cantor manifold. More precisely, no subset of M of dimension $\leq n-2$ separates M, where $n =$* $\dim M$. *Consequently, every subset of M which separates M has dimension at least $n-1$.*

Theorem 2.4 *Let M^n, N^n $(n \geq 2)$ be compact, connected, differential manifolds with N orientable, and let $f\colon M \to N$ be a differentiable mapping.*

(i) *If M is orientable and* $\deg(f) = 0$, *then* $\dim[C(f)] \geq n-1$.
(ii) *If M is not orientable then* $\dim[C(f)] \geq n-1$.

Proof (i) If ω_N is a volume form on N, observe that $\deg(f) = 0$ if and only if $\int_M f^*\omega_N = 0$, which shows that either $C(f) = V(f^*\omega_N) = M$, or $V(f^*\omega_N)$ separates M. In both cases the conclusion $\dim[C(f)] \geq n-1$ follows immediately. Next, if $f^*\omega_N = d\alpha$ for some $\alpha \in \Omega^{n-1}(M)$, then obviously $\deg(f) = 0$.

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(ii) Consider the orientable double cover $p\colon \tilde{M} \to M$ and observe that $\deg(f \circ p) = 0$. Indeed, if ω_N is a volume form on N, then $f^*\omega_N$ is exact since $H^n(M) \cong 0$, that is, $f^*\omega_N = d\alpha$ for some $\alpha \in \Omega^{n-1}(M)$. Thus, we have successively $(f \circ p)^*\omega_N = p^*(f^*\omega_N) = p^*(d\alpha) = d(p^*\alpha)$. This shows that $\int (f \circ p)^*\omega_N = \int d(p^*\alpha) = 0$, or equivalently $\deg(f \circ p) = 0$ and, according to (i), $\dim[C(f \circ p)] \geq n-1$. Since p is a local diffeomorphism, it follows that $C(f) = p(C(f \circ p))$ and, according to Hodel [4], it follows that $\dim[C(f)] = \dim[p(C(f \circ p))] = \dim[C(f \circ p)] \geq n-1$. ∎

Corollary 2.5 *Let M^n, N^n $(n \geq 2)$ be compact, connected, orientable manifolds, let ω_N be a volume form on N, and let $f\colon M \to N$ be a differentiable mapping. If $f^*\omega_N$ is exact, then* $\dim[C(f)] \geq n-1$. *In particular, if f is homotopic to one map $g\colon M \to N$ having just critical points, then* $\dim[C(f)] \geq n-1$.

Proof We just need to show that $f^*\omega_N$ is exact whenever f is homotopic to one map $g\colon M \to N$ having just critical points. Indeed, in such a case, the homomorphisms $f^*, g^*\colon H^n_{DR}(M) \to H^n_{DR}(M)$ are equal. Consequently

$$[f^*\omega_N] = f^*([\omega_N]) = g^*([\omega_N]) = [g^*\omega_N] = 0,$$

the last equality following by using Proposition 2.1. This shows that $f^*\omega_N$ is, indeed, exact. ∎

In what follows we are going to provide an approach for the higher codimension case ($\dim M =: m > n := \dim N$), in which the role of the form $f^*\omega_N$ will be played by forms of type $f^*\omega_N \wedge \theta$, where $\theta \in \Omega^{m-n}(M)$ are closed. If $f\colon M^m \to N^n$, $(m > n)$ is a differentiable mapping and $\omega \in \Omega^{m-n}(M)$, consider the set $R(f) := M \backslash C(f)$ of regular points of f and

$$\mathcal{V}_f(\omega) := \{p \in R(f) \mid \omega_p(u_1, \dots, u_{m-n}) = 0, \text{ for all } u_1, \dots, u_{m-n} \in \ker(df)_p\}.$$

Observe that

$$\mathcal{V}_f(\omega) = \{p \in R(f)\colon (i^*_{f(p)}\omega)_p = 0\}, \text{ where } i_{f(p)}\colon f^{-1}(f(p)) \backslash C(f) \hookrightarrow M \backslash C(f)$$

is the inclusion map of the fiber $(f|_{R(f)})^{-1}(y) = f^{-1}(f(p)) \backslash C(f)$ of $f|_{R(f)}$ passing through p. In other words

$$\mathcal{V}_f(\omega) = \bigcup_{p \in R(f)} V(i^*_{f(p)}\omega).$$

Definition 2.6 Two submanifolds N_1, N_2 of a given finite dimensional differential manifold M are said to *intersect transversally* at $p \in N_1 \cap N_2$, written $N_1 \pitchfork_p N_2$ if $T_p(M) = T_p(N_1) + T_p(N_2)$. If N_1, N_2 do not intersect transversally at $p \in N_1 \cap N_2$, we use the notation $N_1 \not\pitchfork_p N_2$.

Proposition 2.7 *Let M, N, P be differential manifolds such that* $\dim(M) \geq \dim(N) + \dim(P)$. *If N and P are orientable and ω_N, ω_P are volume forms on N and P respectively, then the inclusion $\mathcal{V}_f(g^*\omega_P) \subseteq C(g) \cup U(f, g)$ holds, for every differentiable map $f\colon M \to N$, $g\colon M \to P$, where*

$$U(f, g) := \{x \in R(f) \cap R(g) \mid f^{-1}(f(x)) \not\pitchfork_x g^{-1}(g(x))\}.$$

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Proof Indeed, we have successively

$$\begin{aligned}\mathcal{V}_f(g^*\omega_P) &= (R(g) \cup C(g)) \cap \mathcal{V}_f(g^*\omega_P)\\ &= (R(g) \cap \mathcal{V}_f(g^*\omega_P)) \cup (C(g) \cap \mathcal{V}_f(g^*\omega_P))\\ &\subseteq \{x \in R(f) \cap R(g) \mid (i^*_{f(x)}(g^*\omega_P))_x = 0\} \cup C(g)\\ &= \{x \in R(f) \cap R(g) \mid ((g \circ i_{f(x)})\omega_P)^*_x = 0\} \cup C(g)\\ &= C(g) \cup \bigcup\{C(g \circ i_{f(x)}) \mid x \in R(f) \cap R(g)\}.\end{aligned}$$

Next, for $x \in R(f) \cap R(g)$, we first recall that

$$T_x(f^{-1}(f(x))) = \ker(df)_x,\ T_x(g^{-1}(g(x))) = \ker(dg)_x$$

and then observe that we have successively:

$$\begin{aligned}x \in C(g \circ i_{f(x)}) &\Leftrightarrow \operatorname{rank}_x(g \circ i_{f(x)}) < \dim(P)\\ &\Leftrightarrow \dim[\operatorname{Im}((dg)_x|_{\ker(df)_x})] < \dim(P)\\ &\Leftrightarrow \dim[\ker(df)_x] - \dim[\ker((dg)_x|_{\ker(df)_x})] < \dim(P)\\ &\Leftrightarrow \dim[\ker(df)_x] - \dim[\ker(df)_x \cap \ker(dg)_x] < \dim(P)\\ &\Leftrightarrow \dim[\ker(df)_x + \ker(dg)_x] < \dim(P) + \dim(\ker(dg)_x)\\ &\Leftrightarrow \dim[T_x(f^{-1}(f(x))) + T_x(g^{-1}(g(x)))] < \dim(M)\\ &\Leftrightarrow T_x(f^{-1}(f(x))) + T_x(g^{-1}(g(x))) \neq T_x(M)\\ &\Leftrightarrow f^{-1}(f(x)) \not\pitchfork_x g^{-1}(g(x))\\ &\Leftrightarrow x \in U(f,g).\end{aligned}$$

Therefore we get

$$\mathcal{V}_f(g^*\omega_P) \subseteq C(g) \cup \bigcup\{C(g \circ i_{f(x)}) \mid x \in R(f) \cap R(g)\} \subseteq C(g) \cup U(f,g). \qquad \blacksquare$$

Proposition 2.8 *Let $M^m, N^n, m \geq n$ be differential manifolds and $f\colon M \to N$ be a differentiable mapping. If N is oriented and $\theta \in \Omega^{m-n}(M)$, then $V(f^*\omega_N \wedge \theta) = C(f) \cup \mathcal{V}_f(\theta)$.*

Proof Let p be a regular point of f such that $(f^*\omega_N)_p \wedge \theta_p = 0$ and $u_1, \dots, u_m$ be a base of $T_p(M)$ such that $u_1, \dots, u_n$ span an n-dimensional complementary subspace of $\ker(df)_p$ and $u_{n+1}, \dots, u_m$ is a base of $\ker(df)_p$. Then we have successively

$$\begin{aligned}0 &= (f^*\omega_N)_p \wedge \theta_p(u_1, \dots, u_m)\\ &= \sum_{\substack{\sigma \in S_m\\ \sigma(1)<\cdots<\sigma(n)\\ \sigma(n+1)<\cdots<\sigma(m)}} (\operatorname{sgn}\sigma)(f^*\omega_N)_p(u_{\sigma(1)}, \dots, u_{\sigma(n)}) \times \theta_p(u_{\sigma(n+1)}, \dots, u_{\sigma(m)})\\ &= \sum_{\substack{\sigma \in S_m\\ \sigma(1)<\cdots<\sigma(n)\\ \sigma(n+1)<\cdots<\sigma(m)}} (\operatorname{sgn}\sigma)(\omega_N)_{f(p)}((df)_p(u_{\sigma(1)}), \dots, (df)_p(u_{\sigma(n)})) \times \theta_p(u_{\sigma(n+1)}, \dots, u_{\sigma(m)}).\end{aligned}$$

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But, if σ is not the identity permutation, then at least one of the vectors $u_{\sigma(1)}, \dots, u_{\sigma(n)}$ is among the vectors $u_{n+1}, \dots, u_m$, meaning that at least one of the vectors $(df)_p(u_{\sigma(1)}), \dots, (df)_p(u_{\sigma(n)})$ is zero. Thus, all terms of the above sum are zero except the one which corresponds to the identity permutation. Therefore we have that

$$\begin{aligned} 0 &= (\omega_N)_{f(p)}((df)_p(u_1), \dots, (df)_p(u_n)) \times \theta_p(u_{n+1}, \dots, u_m) \\ &= (f^*\omega_N)_p(u_1, \dots, u_n) \times \theta_p(u_{n+1}, \dots, u_m). \end{aligned}$$

But since p is a regular point, it follows that $\theta_p(u_{n+1}, \dots, u_m) = 0$, and the proof of the inclusion $V(f^*\omega_N \wedge \theta) \subseteq C(f) \cup \mathcal{V}_f(\theta)$ is complete. The other inclusion is obvious. ∎

Corollary 2.9 *Let M, N, P be differential manifolds such that $\dim(M) \geq \dim(N) + \dim(P)$ and N, P are orientable. If $f\colon M \to N, g\colon M \to P$ are differentiable maps and ω_N, ω_P are volume forms on N and P respectively, then the following relations hold:*

(i) *If $\theta \in \Omega^{m-n}(M)$, then $\dim V(f^*\omega_N \wedge \theta) \leq \dim C(f) + \dim \mathcal{V}_f(\theta)$;*
(ii) $C(f) \cup \mathcal{V}_f(g^*\omega_P) = C(g) \cup \mathcal{V}_g(f^*\omega_N) = V(f^*\omega_N \wedge g^*\omega_P) \subseteq C(f) \cup C(g) \cup U(f,g)$;
(iii) $\dim[V(f^*\omega_N \wedge g^*\omega_P)] \leq \dim[C(f)] + \dim[C(g)] + \dim[U(f,g)]$;
(iv) $V(f^*\omega_N \wedge g^*\omega_P) = C(f \times g)$*, where* $f \times g\colon M \to N \times P, (f \times g)(x) = (f(x), g(x))$.

Proof (i) Indeed, taking into account that $C(f)$ is a closed set, it follows, by means of (ii) and [5, Corollary 1, p. 32], that the following relations hold:

$$\begin{aligned} \dim[V(f^*\omega_N \wedge \theta)] &= \dim[C(f) \cup \mathcal{V}_f(\theta)] \\ &\leq \max\{\dim[C(f)], \dim[\mathcal{V}_f(\theta)]\} \\ &\leq \dim[C(f)] + \dim[\mathcal{V}_f(\theta)]. \end{aligned}$$

(ii) The relations follow immediately from Propositions 2.7 and 2.8.

(iii) Taking into account that $C(f)$ and $C(g)$ are closed sets, it follows, by means of statement (ii) and [5, Corollary 1, p. 32] that the following relations hold:

$$\begin{aligned} \dim[V(f^*\omega_N \wedge g^*\omega_P)] &= \dim[C(f) \cup C(g) \cup U(f,g)] \\ &\leq \max\{\dim[C(f) \cup C(g)], \dim U(f,g)\} \\ &\leq \dim[C(f) \cup C(g)] + \dim[U(f,g)] \\ &\leq \dim[C(f)] + \dim[C(g)] + \dim[U(f,g)]. \end{aligned}$$

(iv) In order to show the equality $V(f^*\omega_N \wedge g^*\omega_P) = C(f \times g)$ we first recall that $\pi_N^*\omega_N \wedge \pi_P^*\omega_P$, where $\pi_N\colon N \times P \to N, \pi_P\colon N \times P \to P$ are the projections, is a volume form on $N \times P$, and the following equalities hold:

$$\begin{aligned} (f \times g)^*(\pi_N^*\omega_N \wedge \pi_P^*\omega_P) &= (f \times g)^*(\pi_N^*\omega_N) \wedge (f \times g)^*(\pi_P^*\omega_P) \\ &= (\pi_N \circ (f \times g))^*\omega_N \wedge (\pi_P \circ (f \times g))^*\omega_P \\ &= f^*\omega_N \wedge g^*\omega_P. \end{aligned}$$

Thus, by using Proposition 2.1, we deduce that

$$C(f \times g) = V[(f \times g)^*(\pi_N^* \omega_N \wedge \pi_P^* \omega_P)] = V(f^* \omega_N \wedge g^* \omega_P). \qquad \blacksquare$$

3 Lower Bounds for the Dimension of Certain Critical Sets

In this section we give lower bounds for various combinations of the dimension of the critical set $C(f)$, the dimension of $B(f) = f(C(f))$, the dimension of the critical set $C(f \times g)$ and the dimension of $U(f, g)$, where $g\colon M \to P^{m-n}$ is another differentiable map.

Theorem 3.1 *Let M^m, N^n, P^{m-n} be compact orientable manifolds and let ω_N be a volume form on N. If $f\colon M \to N$ is a differentiable map such that $f^*\omega_N$ is an exact form, then the following inequalities hold:*

(i) $\dim[C(f)] + \dim[\mathcal{V}_f(\theta)] \geq m - 1$, *for all $\theta \in Z^{m-n}(M)$. Consequently, the inequality $\dim C(f) \geq m - \gamma_f - 1$ holds, where $\gamma_f := \min\{\dim \mathcal{V}_f(\theta) \mid \theta \in Z^{m-n}(M)\}$.*

(ii) $\dim[C(f)] + \dim[C(g)] + \dim[U(f, g)] \geq \dim C(f \times g) \geq m - 1$ *for every diffentiable map $g\colon M \to P$.*

Proof (i) Since $f^*\omega_N$ is exact, it follows that $f^*\omega_N = d\alpha$ for some $\alpha \in \Omega^{n-1}(M)$. This means that

$$\int_M f^*\omega_N \wedge \theta = \int_M d\alpha \wedge \theta = \int_M d(\alpha \wedge \theta) + (-1)^n \int_M \alpha \wedge d\theta = 0,$$

for each $\theta \in Z^{m-n}(M)$. Therefore $\dim V(f^*\omega_N \wedge \theta) \geq m - 1$ for all $\theta \in Z^{m-n}(M)$ namely $\dim C(f) + \dim \mathcal{V}_f(\theta) \geq \dim V(f^*\omega_N \wedge \theta) \geq m-1$ for any closed differential form $\theta \in Z^{m-n}(M)$. By considering the minimum with respect to the closed forms $\theta \in Z^{m-n}(M)$, one gets $\dim C(f) \geq m - \gamma_f - 1$.

(ii) The inequality $\dim[C(f)] + \dim[C(g)] + \dim[U(f, g)] \geq \dim C(f \times g)$ follows from Corollary 2.9(iii)(iv). Taking $\theta = g^*\omega_P$ for some volume form ω_P on P, one can see, as above, that the equality $\int_M f^*\omega_N \wedge g^*\omega_P = 0$ holds. This shows that $\dim[C(f \times g)] = \dim[V(f^*\omega_N \wedge g^*\omega_P)] \geq m - 1$. $\blacksquare$

Corollary 3.2 *If M^m is a compact, differential, orientable manifold such that $H^n_{dR}(M) \cong 0$ for some $n < m$, then $\dim[C(f)] \geq m - 1$ for every differentiable mapping $f\colon M \to N \times P$, where N^n, P^{m-n} are orientable differential manifolds.*

Proof Indeed, $f = (\pi_N \circ f) \times (\pi_P \circ f)$, and we only need to apply Theorem 3.1(ii), since the differential form $(\pi_N \circ f)^*\omega_N \in Z^n_{dR}(M) = B^n_{dR}(M)$ is obviously exact. $\blacksquare$

Example 3.3 If $m, n \geq 2$, then for every differentiable map $f\colon S^{m+n} \to S^m \times S^n$, the inequality $\dim[C(f)] \geq m + n - 1$ holds.

Corollary 3.4 *Let $M^m, N^n, m \geq n$ be compact, differential, orientable manifolds and $f\colon M \to N$ be a differentiable mapping such that $f^*\omega_N$ is exact for some volume form ω_N on N. If the fibers of $f|_{M \setminus C(f)}$ are orientable and there exists a closed form $\theta \in Z^{m-n}(M)$ such that the restriction of θ to each fiber of $f|_{M\setminus C(f)}$ is a volume form of that fiber, then the inequality $\dim C(f) \geq m - 1$ holds.*

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Proof Indeed, in this case $\gamma_f = 0$ because $\mathcal{V}_f(\theta) = \varnothing$. ∎

Let us observe that the requirements of Corollary 3.4 are quite strong and that they are not satisfied by certain mappings as follows from the following example.

Example 3.5 Consider the mapping $f\colon S^{n+k} \to T^n$, $f = \exp \circ p$, where

$$p\colon S^{n+k} \to \mathbb{R}^n \prec \mathbb{R}^{n+k+1},\ p(x_1, \dots, x_{n+k+1}) = (x_1, \dots, x_n)$$

and $\exp\colon \mathbb{R}^n \to T^n$ is the exponential mapping. Then $\gamma_f \geq k$. Indeed $C(f) = S^{n+k} \cap \mathbb{R}^n = S^{n-1}$, namely $\dim C(f) = \dim C(p) = n-1$. Consequently, for any closed differential form $\theta \in Z^k(S^{n+k})$, one gets, by using Theorem 3.1(i), the following relations

$$\begin{aligned} n+k-1 &\leq \dim C(f) + \dim \mathcal{V}_f(\theta) \\ &= n-1 + \dim \mathcal{V}_f(\theta) \\ &= n + \dim \mathcal{V}_f(\theta) - 1, \end{aligned}$$

or equivalently $\dim \mathcal{V}_f(\theta) \geq k$, for all $\theta \in Z^k(S^{n+k})$. Therefore, no closed form $\theta \in Z^k(S^{n+k})$ has the property that its restrictions to the fibers of the mapping f are volume forms, although those fibers are orientable, being k-dimensional spheres. Note that f is homotopic to a constant map, which obviously has only critical points.

An extreme situation is represented by fibrations.

Corollary 3.6 *Let M^m, N^n, $m > n \geq 2$ be compact, differential, orientable manifolds, and let ω_N be a volume form on N. If $f\colon M \to N$ is a differentiable fibration such that $f^*\omega_N$ is an exact form, then $\dim \mathcal{V}_f(\theta) \geq m-1$ for all $\theta \in Z^{m-n}(M)$, or equivalently $\gamma_f \geq m-1$. In particular, the inequality $\dim[C(g)] + \dim[U(f,g)] \geq m-1$ holds for every differentiable map $g\colon M \to P$, where P^{m-n} is an orientable manifold.*

Example 3.7 If $f_n\colon S^{2n-1} \to S^n$, where $n = 2, 4$ or 8, is the Hopf fibration, then for every $\theta \in Z^{n-1}(S^{2n-1})$, we have $\dim[\mathcal{V}_{f_n}(\theta)] \geq 2n-2$, or equivalently $\gamma_{f_n} \geq 2n-2$. Consequently, the inequality $\dim[C(g)] + \dim[U(f_n, g)] \geq 2n-2$ holds for every differentiable map $g\colon S^{2n-1} \to S^{n-1}$. In particular $\dim[U(f_2, g)] \geq 2$ for every Morse function $g\colon S^3 \to S^1$.

Remark 3.8 If M, N and f are as in Theorem 3.1, then $\gamma_f = 0$ if and only if $\dim \mathcal{V}_f(\theta) = 0$ for some $\theta \in Z^{m-n}(M)$. A candidate for such a θ is $*f^*\omega_N$, since $\mathcal{V}_f(*f^*\omega_N) = \varnothing$ whenever N is oriented and ω_N is a volume form on N. Unfortunately, the n-form $*f^*\omega_N$ is usually not closed but just co-closed. Another candidate for such a θ can be constructed for a mapping $f\colon M \to N$ whose proper restriction $M \backslash f^{-1}(B(f)) \to N \backslash B(f)$ has orientable fibers. Note that the mentioned restriction is, according to Ehresmann's fibration theorem, a locally trivial fibration, (see [2, p. 15]). In this respect consider an open covering $\{U_i\}_{i \in I}$ of $N \backslash B(f)$ such that $f^{-1}(U_i)$ is diffeomorphic with $U_i \times f^{-1}(y_i)$ for some $y_i \in U_i$, via a diffeomorphism

$$\varphi_i\colon f^{-1}(U_i) \to U_i \times f^{-1}(y_i) \text{ satisfying } p_i \circ \varphi_i = f,$$

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where $p_i: U_i \times f^{-1}(y_i) \to U_i$ is the projection. We also consider a volume form ω_i on $f^{-1}(y_i)$ as well as the mapping $q_i \circ \varphi_i$, where $q_i: U_i \times f^{-1}(y_i) \to f^{-1}(y_i)$ is the projection and observe that $(q_i \circ \varphi_i)^* \omega_i$ is a closed form on $f^{-1}(U_i)$. We next consider a partition of unity $\{f_i\}_{i \in I}$ subordinate to the covering $\{f^{-1}(U_i)\}_{i \in I}$ of $M \setminus f^{-1}(B(f))$ and define the differential form $\omega = \sum_{i \in I} f_i \omega_i \in \Omega^{m-n}(M \setminus f^{-1}(B(f)))$. Finally, we denote by θ_f a differential form obtained from ω by either extending it, if possible, to the whole manifold M, or by multiplying it with a real differentiable and nonnegative function ψ having the property $\psi^{-1}(0) = M \setminus f^{-1}(B(f))$, and observe that $\mathcal{V}_f(\theta_f) = \varnothing$. We also observe that θ_f is not unique and it may not be closed. Also the forms $i_q^* \theta_f$ may have zeros for some $q \in B(f)$, where $i_q: f^{-1}(q) \setminus C(f) \hookrightarrow M \setminus C(f)$ is the inclusion mapping.

Corollary 3.9 *Let $M^m, N^n, m \geq n$ be compact, differential, oriented manifolds and $f: M \to N$ be a differentiable map such that $f^* \omega_N$ is exact for some volume form ω_N on N. If θ_f is closed, then $2 \dim C(f) + \dim B(f) \geq n - 1$.*

Proof Indeed, by applying Morita's theorem [6, pp. 129,130] to the restriction

$$\overline{R(f) \cap f^{-1}(B(f))} \to f(\overline{R(f) \cap f^{-1}(B(f))}) =: Y \subseteq B(f), x \mapsto f(x),$$

we first observe that $\dim[R(f) \cap f^{-1}(B(f))] \leq \dim[C(f)] + \dim[B(f)] + m - n$. We next observe that $\mathcal{V}_f(\theta_f) = \mathcal{V}_f(\theta_f|_{R(f) \cap f^{-1}(B(f))})$, since $\mathcal{V}_f(\theta_f|_{M \setminus f^{-1}(B(f))}) = \varnothing$. Consequently, we have successively:

$$\begin{aligned} \gamma_f &\leq \dim(\mathcal{V}_f(\theta_f)) = \dim(\mathcal{V}_f(\theta_f|_{R(f) \cap f^{-1}(B(f))})) \\ &\leq \dim[R(f) \cap f^{-1}(B(f))] \\ &\leq \dim[C(f)] + \dim[B(f)] + m - n. \end{aligned}$$

The required inequality follows now by using Theorem 3.1. ■

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