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TOTAL TORSION OF CLOSED LINES OF CURVATURE

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In this article we investigate the total torsion of closed lines of curvature on a surface in $\mathbb{E}^3$ and obtain the following results.

(1) The total torsion of a closed line of curvature on a surface is $k\pi$, where k is an integer. Conversely, if the total torsion of a closed curve is $k\pi$ for an integer k, then the curve can appear as a line of curvature on a surface. In particular, if the total torsion of a closed curve is $2k\pi$, then it can appear as a line of curvature on a closed, oriented surface of genus 1.

(2) The total torsion of a closed line of curvature on an ovaloid is zero.

1. INTRODUCTION

Let C be a closed curve in three dimensional Euclidean space $\mathbb{E}^3$. One of the global properties of C is its total torsion T defined by the integral $T = \int_C \tau ds$, where s and τ are the arc length and the torsion of C, respectively. It is well known that for any real number r there is a closed curve C such that its total torsion T is equal to r. On the other hand, we have the following a theorem of Geppert [**3**] (see for instance, [**4**]).

THEOREM A. *The total torsion of a closed curve on a unit sphere is zero.*

In [**2**] Chen investigated the total torsion of a class of closed curves on a developable surface and proved the following theorem.

THEOREM B. *Let C be a closed curve on a developable surface M. If C is perpendicular to the rectilinear generators of M everywhere, then the total torsion of C is zero.*

In this article, we investigate the total torsion of closed lines of curvature on a surface and particularly on an ovaloid and obtain the following results.

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Theorem 1. *The total torsion of a closed line of curvature on a surface in $\mathbb{E}^3$ is $k\pi$, where k is an integer. Conversely, if the total torsion of a closed curve in $\mathbb{E}^3$ is $k\pi$ for an integer k, then the curve can appear as a line of curvature on a surface. In particularly, if the total torsion of a closed curve is $2k\pi$, then it can appear as a line of curvature on a closed, oriented surface of genus 1.*

Theorem 2. *The total torsion of a closed line of curvature on an ovaloid in $\mathbb{E}^3$ is zero.*

From our results, we can see that Theorem 1 and Theorem 2 generalise Theorem B and Theorem A, respectively, since the curve in Theorem B is actually a line of curvature on a developable surface and the sphere is an ovaloid on which any curve is a line of curvature.

In [**1**], Blaschke proposed 16 problems on the differential geometry of ovaloids, the first one among them is: *What can we say about the lines of curvature on an ovaloid?* Theorem 2 is a partial response to the problem.

2. Proofs of Theorems

Let $C : \mathbf{r} = \mathbf{r}(s), 0 \leqslant s \leqslant L$, be a closed curve on a surface M in $\mathbb{E}^3$, where s is the arc length of C. Suppose that C has no point with $\ddot{\mathbf{r}} = 0$. Denote the Frenet frame of C by $\{\mathbf{r}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$, where $\boldsymbol{\alpha} = \dot{\mathbf{r}}, \boldsymbol{\beta} = \ddot{\mathbf{r}}/|\ddot{\mathbf{r}}|$, and $\boldsymbol{\gamma} = \boldsymbol{\alpha} \times \boldsymbol{\beta}$. Denote the Darboux frame of C on M by $\{\mathbf{r}; \boldsymbol{\alpha}, \boldsymbol{\nu}, \mathbf{n}\}$, where $\mathbf{n} = \mathbf{n}(s) = \mathbf{n}\big(\mathbf{r}(s)\big)$ is the normal vector field on M along C and $\boldsymbol{\nu} = \mathbf{n} \times \boldsymbol{\alpha}$. Since the Darboux frame is orthonormal, we have the following equations:

$$
\text{(1)} \qquad \begin{cases} \dot{\boldsymbol{\alpha}} = & k_g \boldsymbol{\nu} + k_n \mathbf{n}, \\ \dot{\boldsymbol{\nu}} = -k_g \boldsymbol{\alpha} & + \tau_g \mathbf{n}, \\ \dot{\mathbf{n}} = -k_n \boldsymbol{\alpha} - \tau_g \boldsymbol{\nu}, & \end{cases}
$$

where the functions k_g and k_n are the geodesic curvature and normal curvature of C, respectively, and the function τ_g is the geodesic torsion of C.

Choosing $\big\{\boldsymbol{\nu}(s), \mathbf{n}(s)\big\}$ as the positive orientation of the plane spanned by $\boldsymbol{\nu}(s)$ and $\mathbf{n}(s)$, we define $\theta(s)$ to be the oriented angle from $\boldsymbol{\beta}(s)$ to $\mathbf{n}(s)$, then we have

$$
\text{(2)} \qquad \mathbf{n}(s) = \boldsymbol{\beta}(s)\cos\theta(s) + \boldsymbol{\gamma}(s)\sin\theta(s)
$$

and

$$
\text{(3)} \qquad \boldsymbol{\nu}(s) = \mathbf{n}(s) \times \boldsymbol{\alpha}(s) = \boldsymbol{\beta}(s)\sin\theta(s) - \boldsymbol{\gamma}(s)\cos\theta(s).
$$

From (1), (2) and (3), with the help of the Frenet equations, we find that the geodesic torsion τ_g of C satisfies the following equation.

$$
\text{(4)} \qquad \tau_g = \langle \dot{\boldsymbol{\nu}}, \mathbf{n} \rangle = \tau + \dot{\theta},
$$

where $\dot{\nu} = d\nu/ds$ and $\dot{\theta} = d\theta/ds$.

If C is a line of curvature on a surface M, then by the Rodrigues' formula, (that is, $\dot{\mathbf{n}} = -\kappa\boldsymbol{\alpha}$, where the function κ is the curvature of C), we have the following

$$\tau_g = -\langle \dot{\mathbf{n}}, \boldsymbol{\nu} \rangle = \langle \kappa\boldsymbol{\alpha}, \boldsymbol{\nu} \rangle = 0.$$

It is clear that the converse is also true. Thus we have the following (see for instance, [**5**]).

LEMMA 1. *A curve C on a surface M is a line of curvature on M if and only if the geodesic torsion τ_g of C vanishes.*

PROOF OF THEOREM 1: Let $C : \mathbf{r} = \mathbf{r}(s), 0 \leqslant s \leqslant L$, be a closed line of curvature on a surface M in $\mathbb{E}^3$, where s is the arc length of C. Then by Lemma 1 and (4) we have

$$\tau = -\dot{\theta}, \tag{5}$$

thus we have

$$T = \int_C \tau ds = -\int_0^L \dot{\theta} ds = \theta(0) - \theta(L). \tag{6}$$

Since C is closed, $\mathbf{r}(0) = \mathbf{r}(L)$ and $\mathbf{r}^{(k)}(0) = \mathbf{r}^{(k)}(L)$, for $k = 1, 2, 3$, where $\mathbf{r}^{(k)} = d^k\mathbf{r}/ds^k$. Consequently, we have

$$\boldsymbol{\alpha}(0) = \dot{\mathbf{r}}(0) = \dot{\mathbf{r}}(L) = \boldsymbol{\alpha}(L), \tag{7}$$

$$\boldsymbol{\beta}(0) = \frac{\ddot{\mathbf{r}}(0)}{|\ddot{\mathbf{r}}(0)|} = \frac{\ddot{\mathbf{r}}(L)}{|\ddot{\mathbf{r}}(L)|} = \boldsymbol{\beta}(L). \tag{8}$$

Since $\mathbf{n}(0)$ is equal to either $\mathbf{n}(L)$ or $-\mathbf{n}(L)$, we may have

$$T = \theta(0) - \theta(L) = k\pi, \tag{9}$$

for some integer k.

Conversely, suppose that C is a closed curve with total torsion $k\pi$, where k is an integer. Denote the Frenet frame of C by $\{\mathbf{r}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$. We define a unit vector field $\mathbf{n}(s)$ along C as follows.

$$\mathbf{n}(s) = \boldsymbol{\beta}(s)\cos\phi(s) + \boldsymbol{\gamma}(s)\sin\phi(s), \tag{10}$$

where $\phi(s)$ is defined by

$$\phi(s) = -\int_0^s \tau(s) ds. \tag{11}$$

Then we construct the following parameterised surface S_1.

$$\mathbf{r}_1 = \mathbf{r}_1(s,t) = \mathbf{r}(s) + t\mathbf{n}(s) \times \boldsymbol{\alpha}(s),\ |t| < \varepsilon,\ 0 \leqslant s \leqslant L. \tag{12}$$

Since we have $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(L)$ and $\boldsymbol{\beta}(0) = \boldsymbol{\beta}(L)$ such as above, then from (11) and the hypothersis, we have

$$\phi(0) = 0,\ \ \phi(L) = -\int_0^L \tau ds = -k\pi, \tag{13}$$

thus we obtain $\mathbf{n}(0) = \pm\mathbf{n}(L)$.

Now, since C is closed, there is a small positive number ε such that S_1 is a regular surface and $\mathbf{n}(s)$ is the normal unit vetor field of S_1 along C and also ϕ is equal to θ in (2). Differentiating (11), we obtain

$$\dot{\theta}(s) + \tau(s) = 0, \tag{14}$$

from which we may conclude that C is a line of curvature on M, according to (4) and Lemma 1.

Moreover if, in particular, C is a closed curve with total torsion $2k\pi$, where k is an integer, then we may construct the following parameterised surface S_2.

$$\begin{aligned} \mathbf{r}_2 = \mathbf{r}_2(s,t) = \mathbf{r}(s) + \delta\big[\mathbf{n}(s) + \mathbf{n}(s) \times \boldsymbol{\alpha}(s)\cos t + \mathbf{n}(s)\sin t\big],\\ 0 \leqslant t < 2\pi,\ 0 \leqslant s \leqslant L, \end{aligned} \tag{15}$$

where δ is a constant. Then since the total torsion of C is equal to $2k\pi$, we have with the help of (10) and (11)

$$\mathbf{r}_2(0,t) = \mathbf{r}_2(L,t),\ 0 \leqslant t < 2\pi.$$

Now it is easy to see that S_2 is a torus and C is on S_2 with $t = 3\pi/2$ if it is a regular surface.

To prove that S_2 is really a regular surface, we make the following straightforward computation.

$$\frac{\partial \mathbf{r}_2}{\partial s} = \boldsymbol{\alpha}(s) + \delta\frac{\partial}{\partial s}\big[\mathbf{n}(s) + \mathbf{n}(s) \times \boldsymbol{\alpha}(s)\cos t + \mathbf{n}(s)\sin t\big], \tag{16}$$

$$\frac{\partial \mathbf{r}_2}{\partial t} = \delta\big[-\mathbf{n}(s) \times \boldsymbol{\alpha}(s)\sin t + \mathbf{n}(s)\cos t\big], \tag{17}$$

$$\begin{aligned} d\frac{\partial \mathbf{r}_2}{\partial s} \times \frac{\partial \mathbf{r}_2}{\partial t} &= \delta\boldsymbol{\alpha}(s) \times \big[-\mathbf{n}(s) \times \boldsymbol{\alpha}(s)\sin t + \mathbf{n}(s)\cos t\big] \\ &\quad + \delta^2\Big(\frac{\partial}{\partial s}\big[\mathbf{n}(s) + \mathbf{n}(s) \times \boldsymbol{\alpha}(s)\cos t + \mathbf{n}(s)\sin t\big]\Big) \\ &\quad \times \big[-\mathbf{n}(s) \times \boldsymbol{\alpha}(s)\sin t + \mathbf{n}(s)\cos t\big] \\ &= \delta\big[-\mathbf{n}(s)\sin t + \boldsymbol{\alpha}(s) \times \mathbf{n}(s)\cos t\big] + \delta^2\mathbf{m}(s,t), \end{aligned} \tag{18}$$

where

$$(19)\quad \mathbf{m}(s,t) = \Big(\frac{\partial}{\partial s}\big[\mathbf{n}(s) + \mathbf{n}(s)\times\boldsymbol{\alpha}(s)\cos t + \mathbf{n}(s)\sin t\big]\Big) \times \big[-\mathbf{n}(s)\times\boldsymbol{\alpha}(s)\sin t + \mathbf{n}(s)\cos t\big].$$

Then we have

$$(20)\quad \begin{aligned}\Big|\frac{\partial\mathbf{r}_2}{\partial s}\times\frac{\partial\mathbf{r}_2}{\partial t}\Big|^2 &= \delta^2 + 2\delta^3\big[-\mathbf{n}(s)\sin t + \boldsymbol{\alpha}(s)\times\mathbf{n}(s)\cos t\big]\cdot\mathbf{m}(s,t) + \delta^4\mathbf{m}^2(s,t)\\ &= \delta^2\big[1 + \delta f(s,t) + \delta^2 g(s,t)\big],\end{aligned}$$

where

$$(21)\quad f(s,t) = 2\big[-\mathbf{n}(s)\sin t + \boldsymbol{\alpha}(s)\times\mathbf{n}(s)\cos t\big]\cdot\mathbf{m}(s,t),\ \ g(s,t) = \mathbf{m}^2(s,t).$$

Since C is closed, the functions $f(s,t)$ and $g(s,t)$ are all bounded, thus there is a small positive number δ such that

$$\Big|\frac{\partial\mathbf{r}_2}{\partial s}\times\frac{\partial\mathbf{r}_2}{\partial t}\Big|^2 > \frac{1}{2}\delta^2 > 0.$$

Consequently, S_2 is a torus, that is, a closed, oriented surface of genus 1.

Since C is a curve on S_2 with $t = 3\pi/2$, we have

$$(22)\quad \frac{1}{\delta}\Big(\frac{\partial\mathbf{r}_2}{\partial s}\times\frac{\partial\mathbf{r}_2}{\partial t}\Big)\Big|_{t=(3\pi/2)} = \frac{1}{\delta}\boldsymbol{\alpha}(s)\times\big[\delta\mathbf{n}(s)\times\boldsymbol{\alpha}(s)\big] = \mathbf{n}(s),$$

which implies that $\mathbf{n}(s)$ is the normal vector of S_2 along C. Since from above we know that $\mathbf{n}(s)$ is the normal vector of S_1 along C, too, the two sufaces S_1 and S_2 are tangent with each other along C. According to Joachimsthal's theorem, C must be the line of curvature on S_2, for it is on S_1. The proof of Theorem 1 is completed. $\square$

PROOF OF THEOREM 2: Let M be an ovaloid, that is, a closed surface with positive Gauss curvature. If C is a line of curvature on M, then as in the proof of Theorem 1, we have $\tau = -\dot{\theta}$ and

$$(23)\quad T = -\int_0^L \dot{\theta} = \theta(0) - \theta(L).$$

Because the Gauss curvature of M is positive, two principal curvatures k_1 and k_2 of M are either all positive or all negative. Without loss of generality, we may choose the unit normal vector field $\mathbf{n}$ of M in the outward direction of M and so k_1 and k_2 are both negative, then by Euler's theorem, the normal curvature k_n in the direction

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of C is always negative, that is, $k_n < 0$. Moreover, by Meusnier's theorem, we have $\kappa \cos\theta = k_n$, where κ is the curvature of C and always positive, thus we know that $\cos\theta = k_n/\kappa < 0$, consequently, we obtain

$$\theta(s) \in \left(\frac{\pi}{2} + 2m\pi,\ \frac{3\pi}{2} + 2m\pi\right), \tag{24}$$

where m is an integer.

Since M is a closed, oriented surface, we have, as in the proof of Theorem 1, $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(L)$ and $\boldsymbol{\beta}(0) = \boldsymbol{\beta}(L)$, and also $\mathbf{n}(0) = \mathbf{n}(L)$. Then we may obtain

$$\theta(0) - \theta(L) = 0 \bmod (2\pi). \tag{25}$$

Thus, by the continuity of $\theta(s)$, we may finally obtain, combining (24) and (25),

$$\theta(0) - \theta(L) = 0,$$

which implies that $T = 0$ by (23). The proof of Theorem 2 is completed. $\square$

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