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FOLDINGS AND MONOMORPHISMS

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In this paper we generalize the folding process initiated by Stallings for graphs to a class of generalized covering spaces. These spaces are called pinched coverings or pinched cores, depending on the particular situation. We then apply our generalized folding process to manipulate these spaces into actual coverings. By using elementary homotopy arguments, we can calculate the fundamental groups of these spaces. As a corollary to our main result we obtain a generalization of a result due to Gersten concerning monomorphisms between free products of groups.

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All base spaces, namely B and C, considered in this paper will be assumed to have base points, b and c respectively. In addition B and C are assumed to be path-connected, locally path-connected and locally simply-connected. If $E \xrightarrow{p} B$ is a covering (E is not assumed to be connected), then a point v in E is called a vertex if $p(v)=b$ and a component of E is called a country.

If we are given an equivalence relation defined on A, a subset of Y, we can extend this equivalence relation to Y by defining the equivalence class of a point in $Y-A$ to consist of just itself. We call this equivalence relation on Y the trivial extension to Y.

Definition. Let $E \xrightarrow{p} B$ be a covering. A *pinching* of E is the trival extension of an equivalence relation defined on the set of vertices of E. The quotient space of a pinching together with the induced map to B is called a *pinched covering.* If $X \xrightarrow{s} B$ is a pinched covering then a *country* (*vertex*) in X is just the image of a country (vertex) in E.

Definition. Let $X \xrightarrow{s} B$ be a pinched covering and let G be a connected union of countries in X. If x and y are two points in G we say that $x_g y$ (denoting the relation by the lower case form of the name of the set) if $s(x)=s(y)$ and there is a path j in G between x and y such that $s(j)$ is a null-homotopic loop. The quotient space obtained by taking the trivial extension of this equivalence relation is denoted as X_g. We use the symbol s_g to denote the induced map from X_g to B.

It is a fact that $X_g \xrightarrow{s_g} B$ is also a pinched covering. This will readily follow from (1) in Proposition 1. If X is a labelled graph then the quotient map taking X into X_g is quite similar to the folding operation introduced by Stallings [3].

Proposition 1. *Let X be connected. Let $X \xrightarrow{s} B$ be a pinched covering and let $X \xrightarrow{H} X_x$ be the quotient map. Then:*

(1) *$X_x \xrightarrow{s_x} B$ is a covering;*

(2) *If x and y are vertices in X and k is a path from $H(x)$ to $H(y)$, then there is a path j from x to y such that $H(j)$ is end point homotopic to k.*

Proof. If one defines X_x as a quotient space of the original covering space rather than of X and uses the fact that B is locally-path connected then (1) becomes a straightforward exercise in point set topology and is left to the reader.

To prove (2), let k be the given path. Clearly there exists a path j_1 from x to z such that $H(j_1)=k$: this path j_1 and the point z are not necessarily unique. Since $y_x z$ there is a path j_2 from z to y such that $s(j_2)$ is null homotopic; hence $H(j_2)$ is null homotopic by part 1. To finish the proof we let $j=j_1 * j_2$.

Definition. Let $E \xrightarrow{p} B$ and $F \xrightarrow{q} C$ be coverings. A pinching of $E \cup F$, the disjoint union of E and F, is the trivial extension of an equivalence relation defined on the set of vertices in $E \cup F$. The quotient space of a pinching together with the induced map to $B \vee C$, the wedge of B and C is called a *pinched core.* The image in the pinched core of a country in $E(F)$ is called a $B(C)$ *country* respectively, the image of a vertex is called a *vertex.*

Definition. Let $Z \xrightarrow{t} B \vee V$ be a pinched core. If every vertex in Z belongs to at most one B and one C country and every country is unpinched, i.e. an actual covering space, then this pinched core is just called a *core.*

It is a standard fact that every connected core in which a base point has been chosen can be completed to a covering in a unique way, see e.g. [1], such that the inclusion of the core into the covering space induces an isomorphism on the fundamental group. Thus a core has the property that t_* is a monomorphism, a property not enjoyed in general by pinched cores. The main result of this paper is to prove that when a connected pinched core satisfies certain properties, which we define as being of type M, then t_* is a monomorphism. These properties were formulated by Gersten [2]. He has claimed that these properties were sufficient when C is a finite wedge of circles.

Notation. If G is a union of countries in Z then $Z-G$ denotes the union of the remaining countries. Hopefully no confusion will arise in this abuse of notation as $Z-G$ is really the closure of the complement of G rather than just the complement.

Definition. Let $Z \xrightarrow{t} B \vee C$ be a pinched core and let G be a either a connected union of B countries or a connected union of C countries in Z. Define an equivalence relation on G denoted $_g$ by the following: $x_g y$ if $t(x)=t(y)$ and there is a path j between x and y which lies in G such that $t(j)$ is a null homotopic loop. The quotient space obtained by taking the trivial extension to Z is denoted as Z_g and the induced map to $B \vee C$ is denoted as t_g.

Clearly $Z_g \xrightarrow{t_g} B \vee C$ is also a pinched core, since we can obtain this space by first mapping the pinched covering space G into a covering space G_g and then reattach the vertices of $Z-G$ onto their images in G_g.

Definition. Let $Z \xrightarrow{t} B \vee C$ be a pinched core and let some vertex z be chosen as the basepoint of Z. The pinched core is said to be of *type M* if:

(1) Z is connected;

(2) There is a unique copy of C at z. We denote this country as C;

(3) $\pi_1(Z-C) \xrightarrow{\beta_* t_*} \pi_1(B)$ is a monomorphism, where β is the map from $B \vee C$ to B which collapses C to the base point and is the identity on B and t is t restricted to $Z-C$.

Proposition 2. *Let $Z \xrightarrow{t} B \vee C$ be of type M. Assume that there is another C country, C_1, at z and let $G = C \cup C_1$. Then:*

(1) C_1 is an unpinched universal covering space;

(2) $Z_g \xrightarrow{t_g} B \vee C$ is of type M;

(3) H_* is an isomorphism where H is the quotient map from Z to Z_g.

Proof. Condition (3) of the definition guarantees that all C countries in $Z-C$ are unpinched universal covering spaces of C.

To prove (2), we examine the commutative diagram of spaces and maps indicated by figure 1. The vertical maps are inclusions. The horizontal map is obtained by collapsing C_1 to a point; hence this map induces an isomorphism between the fundamental groups. The homomorphism induced by the composition of the two left-sided maps is a monomorphism by assumption. Therefore the homomorphism induced by the composition of the right-sided maps is also a monomorphism; hence Z_g is also of type M. We observe that the diagram would not commute if we had put in the spaces $B \vee C$ or inserted the map H between Z and Z_g.

To prove (3), we refer to the commutative diagram represented by figure 2. The upper left-hand picture represents Z, the upper right-hand picture represents Z', a second pinched core. Z' is obtained from Z in the following manner. All B and C vertices which are attached to C_1 at vertices other than z are disconnected and reattached to z. The bottom left-hand picture represents Z_g and the bottom right-hand picture represents Z'_g, which is homeomorphic to Z_g. We have depicted B as a space in which the base point has a neighborhood homeomorphic to $[0, 1)$, where the basepoint corresponds to 0. The dotted segment corresponds to $[0,.5]$. In figure 2 the basepoint of B_1 is assumed to be attached to v, a vertex of C_1 which is different from z. The dotted curve in the upper right-hand picture represents a path in C_1 from z to v and the loop in the bottom right-hand picture represents the image of this path under the covering projection from C_1 to C. The two horizontal maps are homotopy equivalences and H'_* is an isomorphism, thus H_* is also an isomorphism.

Proposition 3. *Let $Z \xrightarrow{t} B \vee C$ be of type M. Let G be a connected union of either **B** or C countries in $Z-C$. Then:*

(1) Z_g is of type M;

(2) H_* is an isomorphism.

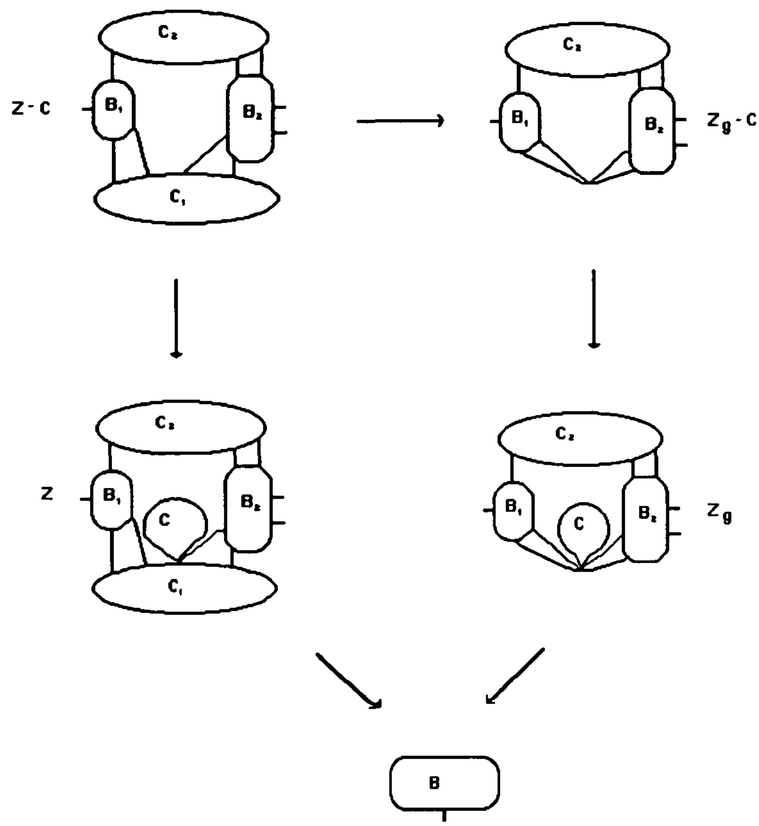

FIGURE 1

Proof. Since H maps C homeomorphically onto C and maps $Z-C$ onto Z_g-C it suffices to both prove (1) and (2) by showing that K_* is an isomorphism where K is the restriction of H to $Z-C$. K_* is one to one since $\beta \circ t$ factors through K and Z is of type M. We note that except for some possible pinching, the complement of G in Z is homeomorphic to the complement of G_g in Z_g. Thus any path in the complement of G_g has a well defined pre-image in the complement of G. We now appeal to Proposition 1. This guarantees that any path j in G_g can be "lifted" back to a path j' in G where the initial and terminal points of j' are prescribed subject to the fact that their images are the initial and terminal points of j and such that $K(j')$ is endpoint homotopic to j. Now suppose we are given a loop l in Z_g-C. We first pull back the subpaths of l which lie in Z_g-G_g, this prescribes endpoints for "lifting" back the remainder of l. If we now apply the above construction we obtain a loop l' in $Z-C$ such that $K(l')$ is homotopic to l. Therefore K_* is onto and thus is an isomorphism.

Definition. A pinched core is said to be *finite* if the number of B and C countries is finite.

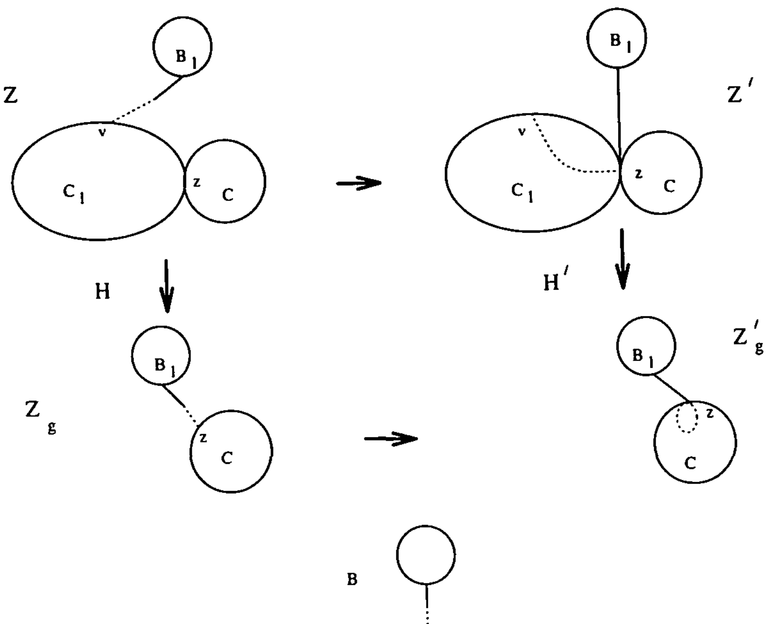

FIGURE 2

Theorem 4. *Let $Z \xrightarrow{t} B \vee C$ be a finite pinched core which is of type M. Then t_* is a monomorphism.*

Proof. If Z is a core we are finished, if not then Z has one of the following:
(1) Two B countries which meet in a vertex;
(2) Two C countries which meet in a vertex;
(3) A pinched B country, i.e, a B country which is not a covering space.
If (1) holds then we apply Proposition 3 where G is the union of these countries. This produces a new pinched core, which satisfies condition M and such that H_* is an isomorphism. Similarly if (2) holds then we apply either Proposition 2 or 3 depending on whether or not one of the C countries is C. In either case the number of countries has gone down as is clear from the remarks after the formal definition of Z_g. We repeat this process until (1) or (2) does not hold. By our finiteness condition this procedure stops after a finite number of steps.

If our new pinched core $Z' \xrightarrow{t'} B \vee C$ satisfies (3) then we let G be some pinched B country. Clearly the number of countries in Z'_g is the same as the number in Z'. We claim that the number of pinched countries has gone down by one. The only countries which are pinched in Z'_g but whose pre-images are not pinched in Z' must meet G; these by assumption are C countries. Since Z' is also of type M the only C countries are C and unpinched copies of the universal covering space of C. Since Z'_g is also of type M, any C country in Z'_g must be unpinched, therefore the number of pinched countries has gone down by one. It is possible, however, that Z'_g now satisfies (1) or (2). If this happens we apply our previous construction, if not we look for another pinched B country. It is clear that after a finite number of steps we arrive at a core which finishes the proof.

Definition. Let $M=\pi_1(B)$, $N=\pi_1(C)$ and let $F\xrightarrow{\Phi}M*N$ be a monomorphism. We say that (F,Φ) is of *finite type* if there is a finite core $Z\xrightarrow{t}B\vee C$, together with some base point of Z, such that image $(t_*)=$image (Φ).

By Kurosh's Theorem, any finitely generated subgroup of $M*N$ together with its inclusion is of finite type.

Corollary 5. *Let I_N denote the identify homomorphism on N and let ρ_M be the projection of $M*N$ onto M. If (F,Φ) is of finite type and $\rho_M\circ\Phi$ is a monomorphism then $\Phi*I_N$ is a monomorphism.*

Proof. Let $Z\xrightarrow{t}B\vee C$ be a core finite type corresponding to (Φ,F). At the basepoint of Z attach a copy of C and denote this pointed core as $Z'\xrightarrow{t'}B\vee C$. The homomorphism $\Phi*I_N$ is thus represented by t'_*. Since $Z'\xrightarrow{t'}B\vee C$ is a finite pinched core which is of type M then t'_* is a monomorphism by Theorem 4. □

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