7. COMMISSION DE LA MECANIQUE CELESTE

Report of Meetings

PRESIDENT: Y. Hagihara.

VICE-PRESIDENT acting as Secretary: W. J. Eckert.

Meetings of the Commission were held as follows:

Wednesday 26 August 1964	$9^h\ 00^m$
Friday 28 August	$9^h\ 00^m$
Friday 28 August	$15^h\ 45^m$
Monday 31 August	$10^h\ 45^m$

The first session and the first part of the fourth were devoted to business, and the others to technical papers on the orbits of natural bodies, and on the orbits of artificial bodies. Dr J. Kovalevsky acted as interpreter at all sessions.

Business Meetings

The minutes of the first session are as follows:

The President announced the schedule for the meetings of the Commission and distributed the programme for the scientific sessions.

The President discussed the Draft Report and asked for corrections.

The President announced that the Executive Committee had proposed that the President, Vice-President, and Organizing Committee of Commission 7 continue for another term of three years. The proposal was approved by the Commission.

The President opened the discussion of new members of the Commission and of consulting members, and presented proposed lists for discussion. After the discussion it was decided that the President recommend to the Nominating Committee of the Union for membership in the Commission the following:

(*a*) Proposed by the Commission President: J. M. A. Danby (U.S.A.), B. Garfinkel (U.S.A.), I. Iszak (U.S.A.), P. Musen (U.S.A.), C. L. Siegel (W. Germany), V. Szebehely (U.S.A.), A. H. Cook (U.K.).

(*b*) Proposed by the National Committee of the Adhering Organizations: E. P. Askenov (U.S.S.R.), J. Barlier (France), V. A. Brumberg (U.S.S.R.), D. N. Katsis (Greece), S. F. Mello (Brazil), Yu. A. Rjabov (U.S.S.R.), L. Sehnal (Czechoslovakia), Y. Thiry (France).

(*c*) Consulting Members for the coming three years: A. P. Arnold (U.S.S.R.), S. P. Diliberto (U.S.A.), W. M. Kaula (U.S.A.), D. G. King-Hele (U.K.), J. Moser (U.S.A.), J. P. Vinti (U.S.A.).

The President proposed for discussion two proposals made by the General Secretary some time ago:

(1) To combine Commissions 4 and 7.

(2) To transfer from Commission 17 to Commission 7 the part dealing with the motion of the Moon.

The President had previously corresponded with the members of the Organizing Committee concerning these proposals and he presented a review of this correspondence. Discussion of

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the first proposal emphasized the different roles of Commissions 4 and 7 and the small overlap in their functions. The former deals with the solution of problems in a certain area while the latter deals with the dissemination of information from the same and other areas. Furthermore if Commission 4 loses its identity, its functions will require the creation of a Committee with the same membership as the present Commission. Discussion of the second proposal concerned the definition of 'motion of the Moon'. The following *resolutions* were adopted:

(1) Commission 7 does not recommend the combination of Commissions 4 and 7.

(2) Commission 7 recommends the transfer from Commission 17 to Commission 7 of matter concerning the orbital motion of the Moon.

The President called attention to the two previous IAU Symposia of interest to members of Commission 7:

No. 21: The System of Astronomical Constants (Paris)

No. 25: The Theory of Orbits in the Solar System and in Stellar Systems (Thessaloniki), and expressed the appreciation of many people to those responsible, especially Dr Clemence and Dr Contopoulos. He pointed out that with so many symposia being organized by COSPAR, IUTAM, and IAU there was danger of overlapping, though he had not noticed any in the past. It was observed that members of Commission 7 on the committees of COSPAR and IUTAM would provide liaison.

Dr Eckert presented the following report of the Committee on Astronomical Records in Machine-Readable Form:

(1) At the meeting in Berkeley (*Trans. IAU*, **11B**, p. 173) it was agreed that members of the Committee would write to observatories in their geographical areas for information concerning material suitable for exchange. The response from the observatories was disappointing, for example, listings were received from only half a dozen institutions in the entire western hemisphere.

(2) Many astronomers mention their machine-readable records in their research publications and make the data available on a personal basis but are not willing to do the necessary documentation etc. for a formal exchange project.

(3) A number of institutions such as the Naval Observatory, Cincinnati, and NASA and their counterparts in other countries have been actively engaged in exchanging and providing such material on request. Dr Strand informs me that the Naval Observatory is willing to exchange data in its area of interest.

(4) A somewhat more detailed report will be published in the *IAU Circulars* and additional listings of material are requested.

Dr Duncombe outlined the informal arrangements now in effect between the Almanac Offices for the exchange of Star Catalogues, Ephemerides etc. and stated that these arrangements would probably be formalized at this meeting of the assembly.

Dr Wilkins proposed that the work of the Committee be transferred to Commission 4. The motion was seconded by Dr Duncombe and approved by the Commission.

Dr Wilkins expressed the gratitude of members of the Commission to the President for the effort and care used in preparing such an excellent Draft Report.

Dr Heinrich stated that he would like to add to the description of his work in the Draft Report, the statement that his complete integration of the integral equation includes eccentricity and forms a solid basis for the lunar theory. Reference: *Izv. Mat. Inst. Sof.* **2**, no. 2, 1959; *Acta mat. Stockh.* **88**, 1952; *Bull. astr. Insts. Csl.*, 1962.

The business session at the beginning of the fourth meeting of the Commission included three items: Eulerian angles, areas of interest of Commissions 7 and 17, and Colloquia to be held at future meetings.

During the technical session on the previous day *Dr Belorizky* (*Tech. Pap.* 10) had introduced the following resolution:

'Le sens des angles d'Euler varie d'un traité de Mécanique Céleste à un autre. Je propose que la Commission 7 accepte une définition conforme à celle de Tisserand (Vol. II, § 169) et recommande cette définition à tous les Membres de l'Union Astronomique.'

The President had requested the following Committee to examine the proposal: Belorizky, Chebotarev, Clemence, Garfinkel, Herget, Mikhailov. The President reported that the Committee had met but was not prepared to recommend any action by the Commission.

The President reported that there was a conflict between recommendations by Commission 17 (The Motion and Figure of the Moon) and the resolution of Commission 7 recommending the transfer to Commission 7 of 'matter concerning the orbital motion of the Moon'. The discussion indicated that the members of the Commission feel that discussions of the *observed* orbital motion of the Moon should be in Commission 17 and discussions of the *theory* of the orbital motion in Commission 7.

The President reported that the Executive Committee recommends that at future meetings of the General Assembly each Commission should have one business session and one or more colloquia. He requested that suggestions for such colloquia in Commission 7 be sent to the President or Vice-President. Dr Herget proposed that one be organized on 'The use of electronic computers for analytical (not numerical) developments in celestial mechanics.'

This suggestion met with general approval and the President requested the Vice-President to organize such a colloquium.

Technical Sessions

The programme for the three technical sessions is as follows:

(1) W. J. Eckert: The solution of the main problem of the lunar theory.

(2) E. Rabe: The determination of the impressed terms of short periods for librating Trojans in the elliptic restricted problem.

(3) P. E. Kustaanheimo: Relativistic spinor regularization of the Kepler motion.

(4) B. Popovič: Une méthode générale pour les perturbations spéciales des petites planètes.

(5) K. Stumpff: A variant of Encke's method of calculating special perturbations.

(6) F. L. Whipple: Evidence for a comet belt beyond Neptune.

(7) S. Herrick: Differential representation in observational differential corrections of orbits.

(8) V. G. Szebehely: Orbits with consecutive collisions in the restricted problem of three bodies.

(9) A. H. Cook: Exact solutions for orbits in a potential field.

(10) D. Belorizky: Sur les angles d'Euler.

(11) D. Brouwer: The orbit of Pluto.

(12) P. J. Message: Summary report on progress in recent years in the theory of the motion of artificial satellites.

(13) A. H. Cook: A comparison of solutions for the parameters of the Earth's gravitational field.

Numbers 9 and 10 were added after the programme was circulated at the first meeting. Papers 1–5 were presented at the second session, papers 6–11 and 13 at the third and paper 12 at the final session.

The abstracts and discussion follow. The review of orbits of artificial satellites is reported more fully than the rest of the technical programme. We have placed at the end of the abstracts a communication by Dr R. P. Cesco intended for the Draft Report and received after the Report was completed.

1. *W. J. Eckert*. The solution of the main problem of the lunar theory by W. J. Eckert and H. F. Smith, Jr., outlined at Joint Discussion C in Berkeley (*Trans. IAU*, **11B**, p. 447) has been completed. The method of solution and the numerical analysis were described at IAU Symposium no. 25 in Thessaloniki. The second approximations yielded corrections as large as $1 \times 10^{-10} = 0''.00\,002$ in only two or three terms. The results agree remarkably well with Brown's. The largest correction in the periodic terms is $''.07$ for the $2F - 2l$ term and the corrections to the annual motions of the perigee and node are

$$\delta\,\pi = +\,''\!\cdot\!216 \qquad\qquad \delta\,\Omega = -\,''\!\cdot\!037.$$

These corrections are to his solution in rectangular co-ordinates. There is a small uncertainty in his transformation to polar co-ordinates used in the comparison with observation; this transformation is now being recomputed. The $2F - 2l$ term can and should be tested by careful analysis of the observations. Corrections given later by Brown for omitted terms in the perigee and node were $+\,''\!\cdot\!14$, $-\,''\!\cdot\!04$. Using the currently accepted value of J_2 from satellite observations and the new value for the solar action, the comparison of the observed and computed values of the perigee and node give:

$$\frac{C'}{M'D'^2} = \cdot 60 \qquad\qquad \frac{C' - B'}{C' - A'} = \cdot 69$$

The former, which implies greater density at the surface of the Moon than at the centre, comes almost directly from the motion of the node. A large value for this quantity was found by Spencer Jones in 1937.

DISCUSSION

Dr Cook remarked that since the availability of a reliable value of J_2 from artificial satellites it has been hoped that the physically implausible value of $C'/M'D'^2 > 0\cdot4$ might have been due to an error in Brown's theory. The work here presented seems to eliminate the possibility and leaves a difficult problem. Dr Hertz asked if the energy integral had been used as a check; the reply was in the negative. Dr Zadunaisky asked if the effectiveness of the method for other satellites had been examined; the reply was in the negative.

2. *E. Rabe*. In the theory previously outlined by the author (**1**), the motion of any librating Trojan in the plane, elliptic restricted problem is described on the basis of a reference orbit which is a combination of a suitable periodic solution of the corresponding restricted problem with the elliptic motion of Jupiter. This chosen intermediate orbit is not a rigorous solution of the differential equations, but the method devised here for the determination of the coefficients of the required additional terms reveals a relatively small difference between the reference orbit and that rigorous solution which is the actual equivalent of the basic periodic libration of the restricted problem. The determination and successive improvement of the impressed terms of increasing order takes the form of a convenient algorithm, thanks to the convergence of the basic series expansions for the librational and elliptic components of the theory.

REFERENCE

1. Rabe, E. 'Elements of a Theory of Librational Motions in the Elliptical Restricted Problem'. Proceedings of the Summer Seminar in Space Mathematics, held at Cornell University, July/August 1963 (in print).

DISCUSSION

Prof. Wilkens asked about the relation between Rabe's method and those published by him and others about 50 years ago. Rabe explained that his method is a combined numerical-analytical method which gives high precision for large and small amplitude of libration. Hertz asked about Dr Rabe's plans concerning the effect of Saturn. Rabe replied that this would be considered after complete treatment of actual Trojans in inclined orbits.

3. *P. E. Kustaanheimo.* The method of conformal mapping between a 4-parameter spinor space and the 3-parameter space of the classical physics used in (1) and (2) for the purpose of regularizing the classical Kepler motion is in this paper extended to a conformal mapping between an 8-parameter spinor space and the 4-parameter Minkowski space which permits a similar treatment of the relativistic Kepler motion. The paper will be distributed as a *Publ. astr. Obs. Helsinki.*

REFERENCES

1. Kustaanheimo, P. E. *Publ. astr. Obs. Helsinki*, no. 102, 1964.
2. Kustaanheimo, P. E., Stiefel *Publ. astr. Obs. Helsinki.* Preprint 1964.

4. *B. Popovič.* The method is developed in such a manner that it may easily be applied with an electronic calculator or a simpler machine. By the utilization of the 'fundamental equation' of Stumpff, the method is applicable to parabolas and hyperbolas as well as ellipses of all excentricities. It is not necessary to calculate $\bar{v}$, and $\bar{r}$ is needed only with little precision (for it enters only in the perturbative force). The formulae are given.

5. *K. Stumpff.* Encke's well-known differential equation for the vector of perturbation contains terms of different behaviour. After integration, the indirect term of the perturbation increases slowly with the fourth power of the intermediate time, the direct terms, however small, increase more rapidly with the second power. In many cases, for instance if orbits from the Earth into the gravitational sphere of the Moon are to be computed, this asymmetry is disadvantageous. Instead of the osculating conic section as an intermediary orbit, K. Stumpff proposes another one, which includes several terms of the perturbations, among them all up to the third order of the intermediate time. This intermediate orbit consists of several parts, which can be computed with the formulae of the Kepler motion only. The remaining perturbation is of the fourth order and formally of absolute symmetry in respect to all attracting masses of the system.

6. *F. L. Whipple.* According to certain theories of planetary formation a thin stable belt of comets should exist near the invariable plane beyond Neptune. Calculations indicate that such a comet belt can account for the residuals in latitude for Neptune slightly better than the previously assumed attraction of Pluto. Modern definitive orbits of Neptune, Uranus and Pluto should be able to settle the question whether Pluto has an appreciable mass or whether a comet belt exists.

REFERENCE

Proc. nat. Acad. Sci. U.S.A., **51**, 711, 1964.

7. *S. Herrick.* The paper discusses the practical advantages to be found in basing differential corrections upon such equations as

$$\begin{aligned} \rho - \mathbf{L} \cdot \boldsymbol{\rho}_c &= \mathbf{L} \cdot \Delta\boldsymbol{\rho} \\ - \mathbf{A} \cdot \boldsymbol{\rho}_c &= \mathbf{A} \cdot \Delta\boldsymbol{\rho} \\ - \mathbf{D} \cdot \boldsymbol{\rho}_c &= \mathbf{D} \cdot \Delta\boldsymbol{\rho} \\ \dot{\rho} - \mathbf{L} \cdot \dot{\boldsymbol{\rho}}_c &= \mathbf{L} \cdot \Delta\dot{\boldsymbol{\rho}} \end{aligned}$$

rather than upon the residuals $\Delta\alpha = \alpha - \alpha_c$, $\Delta\dot{\rho} = \dot{\rho} - \dot{\rho}_c$, etc. where ρ, $\dot{\rho}$, α, δ are observed values, where

$$\mathbf{L} = \begin{Bmatrix} \cos\delta \ \cos\alpha \\ \cos\delta \ \sin\alpha \\ \sin\delta \end{Bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} -\sin\alpha \\ +\cos\alpha \\ 0 \end{Bmatrix}, \quad \mathbf{D} = \begin{Bmatrix} -\sin\delta \ \cos\alpha \\ -\sin\delta \ \sin\alpha \\ +\cos\delta \end{Bmatrix},$$

where $\boldsymbol{\rho}_c$ and $\dot{\boldsymbol{\rho}}_c$ are computed values, and where $\Delta\boldsymbol{\rho}$ and $\Delta\dot{\boldsymbol{\rho}}$ represent equations in terms of corrections to any given set of elements. Extensions to certain dynamical equations, especially in the orbit methods of Lagrange, Gauss, and Gibbs, were referred to. The work was supported by the U.S.A.F. Office of Aerospace Research.

8. *V. G. Szebehely.* Orbits are generated which go through both singularities representing the primaries in the restricted problem. Birkhoff's (1) transformation from the conventional rotating rectangular co-ordinates to regularized co-ordinates is used along with the appropriate time transformation. The general perturbation aspects of this problem were discussed by Szebehely (2). Results of a special perturbation study have been given in the present paper.

REFERENCES

1. Birkhoff, G. D. *Rc. Circ. mat. Palermo*, **39**, 1, 1915.
2. Szebehely, V. *Astr. J.*, **69**, 309, 1964.

9. *A. H. Cook.* It is well known that the Hamilton-Jacobi equation may be separated in those co-ordinate systems for which the Laplace equation separates provided that in addition, the potential is of the form

$$V = \sum_i \frac{v_i(x_i)}{h_i^2}$$

(The h_i are the coefficients in the metric $ds^2 = \sum_i h_i^2\, dx_i^2$). If the potential does satisfy Laplace's equation, then the problem of orbits in free space can be solved by quadratures without recourse to a perturbation procedure. It is of course well known that orbits about a point mass ($V = -\mu/r$) and about an oblate spheroid (Vinti's potential) can be found by quadratures and I have found appropriate forms of the potential for all co-ordinate systems for which solutions are possible. My original purpose was to see if a solution were possible for ellipsoidal co-ordinates, corresponding to orbits about a triaxial ellipsoid, but it seems that in this case no potential satisfies the conditions. Equally, it is unlikely that there is a solution for paraboloidal co-ordinates. The results of this study will be published in detail elsewhere.

DISCUSSION

Vinti remarked that there are applications of these methods in many fields other than celestial mechanics.

10. *D. Belorizky.* Eulerian angles appear frequently in the literature and considerable inconvenience arises from the lack of uniformity in the designation of the signs of the angles. Dr Belorizky proposed that the convention of Tisserand be adopted. After brief discussion the President referred Dr Belorizky's resolution to a special committee to make recommendations at the business meeting on Monday (see minutes of that meeting).

11. *D. Brouwer.* The speaker called attention to the important results obtained recently by D. J. Cohen and E. C. Hubbard at the U.S. Naval Weapons Laboratory, Dahlgren, Va. (*NWL Report*, 1945). The orbits of the five outer planets were computed by special perturbations over

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120 000 years. There was revealed a remarkable libration of the close approaches of Pluto to Neptune such that the distance between the bodies is never less than 18 A.U. It is concluded that the orbit of Pluto is safe from very close approaches to Neptune and no particular instability results from the fact that the radius of perihelion of Pluto is less than the radius of the orbit of Neptune. The libration is characterized by an oscillation in the angle, $3l_P - 2(l_N + \omega_N - \tilde{\omega}_P) - 180°$. The period of the oscillation is about 19 670 years and the amplitude is about 76°.

12. *P. J. Message.* This report makes no attempt to be exhaustive, but only attempts to illustrate the important topics in which progress has been made, by reference to some of the published work which I have seen. One might denote by the name, 'main drag-free problem', the study of the motion of a point mass in the zonal harmonic field, that is, a field whose potential is

$$V = -\frac{\mu}{r}\left\{1 - \sum_{n=1}^{\infty} J_n \left(\frac{r_{eq}}{r}\right)^n P_n(\sin\beta)\right\}$$

(here r is radial distance, β is latitude, and r_{eq} is the value used for the radius of the Earth's Equator). If the centre of mass of the Earth is used as the origin of the reference system, then $J_1 = 0$, and it is of interest to ask how well the value of this coefficient is known, that is to say, with what precision do we know the position of the centre of mass of the Earth.

Most of the important treatments of the main drag-free problem were published more than 4 years ago, so they will be mentioned only in briefest outline. King-Hele's theory, using spherical polar co-ordinates, and a rotating inclined reference plane, has the argument of latitude in this plane, plus a right angle, as independent variable. The solution, which is in the form of a series in the coefficients J_2, J_4, requires higher terms after a moderately short time. Merson used 'smoothed' values of the elements of the osculating ellipse, and Zhongolovitch also used osculating elements. Musen uses a modification of Hansen's method. Brouwer used Delaunay's canonical parameters for the osculating ellipse, and separated the short-period features of the perturbations from those of long period by a transformation of von Zeipel's type, and used another such to solve the long-period problem. The expressions used are in closed form, and need no power series expansions in the eccentricity. Kozai used a similar procedure, using osculating elements, and has computed the short term perturbations as far as the terms in J_2^2. It is found that, while the node always regresses, the argument of perigee increases or decreases, according as the inclination is less or greater, respectively, than a critical value of about 63·4 degrees. For values of the inclination close to this, small divisors appear in the expressions, making a different treatment necessary. Garfinkel made a different choice of the potential for the intermediate or 'unperturbed' solution, still leading to a Hamilton-Jacobi equation solvable by separation of variables, but so that the intermediate solution gives the apse and node their entire secular motions. Vinti, making use of oblate spheroidal co-ordinates, chose an intermediate potential satisfying Laplace's equation, enabling all of the effect of the second harmonic to be included in the intermediate solution. This potential corresponds to

$$J_{2n} = (-J_2)^n, \quad \text{if} \quad J_1 = 0.$$

In the last 3 years, these theories have been further developed to take account of further kinds of perturbations, and special cases of the elements, such as small eccentricities, and small inclinations, and have been shown to be of great convenience in the prediction of positions and the discussion of observations. The apparent disagreements between the expressions, derived in the different theories, for those parts of the secular motions which have the factor J_2^2, have been shown to be due in each case to the differences in the definitions of the parameters used, and the expressions become reconciled when these differences are taken into account. The odd harmonics (with factors J_3, J_5, etc.) do not give rise to secular motions of the apse and node, but to oscillations of long period. For the eccentricity (e) and argument of perigee (ω), Izsak gives the equations

$$\dot{e} = C \cos \omega$$
$$e\dot{\omega} = \bar{e}\bar{\dot{\omega}} - C \sin \omega$$

where C is a linear combination of the coefficients J_3, J_5, etc. The identification of these oscillations in a number of satellites, and the estimation of their amplitude, enables these coefficients to be determined.

For each of the treatments of the motion mentioned above, the case in which the orbital inclination has a value close to the critical value requires a special method of solution of the equations. Garfinkel reduced the equations to those of a simple pendulum, and hence showed that the possible types of motion were, periodic motion, libration of the argument of perigee, and circulation of this argument through all values. Hori used von Zeipel's transformation for the long-period problem, expanding the determining function in powers of $\sqrt{J_2}$, instead of J_2 as in Brouwer's theory. Rectangular-type variables have been used by Hagihara (who used $\sqrt{G} \cos g$ and $\sqrt{G} \sin g$), Kozai, and Izsak (who uses variables equivalent to $e \cos \omega$ and $e \sin \dot{\omega}$, which are necessary for the discussion of motion with small eccentricity), and by plotting the curves of constant values of the Hamiltonian for the long-period problem ($\bar{F}$), the possible types of motion are exhibited. Izsak finds that there are two possible arrangements of the periodic solutions, according to the values of the constants of integration, and in his preamble he says that he finds a further small divisor difficulty in the discussion of the equation $\bar{F}$ = constant by means of expansion in powers of $\sqrt{J_2}$. It is clear that there is still more to be learnt about this problem. Aoki solved the equations in terms of Weierstrassian elliptic functions, and described all the distinct cases in detail. In a second paper he described the effects of odd harmonics. Kevorkian (1) used his 'two independent variable' method, deriving an expansion for the solution in the non-critical case using the orbital longitude φ, and $\tilde{\varphi} = \epsilon\varphi$ as the two arguments ($\epsilon = \frac{3}{2} J_2$), and also an expansion for the critical case, using φ and $\tilde{\varphi} = (\epsilon)^{3/2} \varphi$ as arguments. The two solutions are matched, to give a uniformly convergent expansion valid for both cases. (He has also applied this method to the motion of a lunar probe, developing different expansions for the part of the motion near the Earth and the part near the Moon.)

The other very important influence on the motion of a close artificial satellite is the atmospheric drag, that is the resistance of the air to the motion of the satellite. The effect is always dissipative, and reduces the energy of the motion, and the long-term effect is to reduce both the major semi-axis and the eccentricity. Brouwer and Hori approached the problem with the use of Delaunay's canonical parameters, giving expressions for the changes, due to the drag, in the quantities (L'', G'', H'', etc.) which are constant in the drag-free problem. Expansions in powers of the eccentricity are unavoidable, and so the results are useful only for small values of the eccentricity. King-Hele (2) and his collaborators have developed the theory of the relative changes of the major semi-axis (a) and the quantity $x = ae$, in terms of the scale height (H) of a model of the atmosphere in which the density diminishes exponentially with height, and successive papers include the adaptation of the treatment to an oblate model of the atmosphere, a treatment for large eccentricities using expansions in powers of $1/z = H/(ea)$, and the determination of the changes of H over recent years as revealed by observations. In a recent paper, still unpublished, they allow for the day-to-night variation in the density of the atmosphere. When smaller influences on the motion of a close satellite are considered, we must take into account the longitude-dependent terms, or tesseral harmonics, in the Earth's field, and write the potential in the generalized form

$$V = -\frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{r_{eq}}{r}\right)^n P_n^m (\sin \beta) \{C_{nm} \cos m\lambda + S_{nm} \sin m\lambda\}\right]$$

where λ is terrestrial longitude, and P_n^m the associated Legendre polynomial. Sometimes reference is made to the quantity

$$J_{nm} = \sqrt{(C_{nm}^2 + S_{nm}^2)}.$$

The coefficients with $n = 2$ are given by:

$$\begin{aligned} C_{20} &= -J_2 = \{\tfrac{1}{2}(A + B) - C\}/(Mr_{eq}^2), \\ C_{21} &= -E/(Mr_{eq}^2), \\ S_{21} &= D/(Mr_{eq}^2), \\ C_{22} &= (A - B)/(Mr_{eq}^2), \\ \text{and } S_{22} &= -F/(Mr_{eq}^2), \end{aligned}$$

where M is the mass of the Earth, A, B and C are the moments of inertia of the Earth about the axes of the frame of reference, and D, E, and F are the products of inertia. We notice that if the polar axis of our reference system is a principal axis of inertia of the Earth, then $D = E = 0$, and so C_{21}, S_{21}, and therefore J_{21}, vanish. It is of interest to ask how close to zero these coefficients actually are, that is, how well does the axis of our reference system coincide with this principal axis of the Earth.

Groves (3), among others, has given expressions for the perturbations of the elements due to the tesseral harmonics in the usual case, when these perturbations are all of short period. Special interest attaches, however, to those cases in which the orbital period of the satellite is close to a small integer ratio of the period of rotation of the Earth, since the resonance-type effects then arising may bring the effect of the tesseral harmonics into larger importance. Allan (4) and Blitzer *et al.* have each given elegant treatments of the oscillations in longitude for the satellite of period one siderial day, taking the J_{22} term to be alone significant, and they find that oscillations are possible about the equilibrium positions on the line of the minor axis of the Equator (which is under their supposition approximately elliptical), while the positions on the major axis are unstable. However, it is now clear that other harmonics are of about the same size as J_{22}, and the situation will therefore be more complicated. Cook (5) has treated other harmonics, and Morando has solved the equations for cases where some of the other harmonics dominate, using elliptic functions.

Other effects, which are small for most satellites to date, are the repulsion due to the radiation pressure of the Sun, and the attractions of the Moon and Sun. These have been studied by the traditional methods of celestial mechanics, though it was pointed out by Vinti that if a uniform field was taken as an approximation to the effect of the radiation pressure, and the harmonics in the Earth's field neglected, the problem of the satellite motion become separable.

A topic of considerable theoretical interest which has been the subject of study is that related to the periodicity properties of the motion. Callender has shown that all motions in the field used for the intermediate solution by Vinti are almost periodic. MacMillan (6) had shown long ago that there existed periodic orbits in the field in which J_2 and J_4 are the only non-zero coefficients, and the subject of periodic orbits has been the subject of researches recently. One can distinguish several families. There are periodic orbits in the Equator plane, which become circular when J_2 and J_4 become zero, and so are analogous to the periodic solutions of the *première sorte* in the general three-body problem, described by Poincaré. There are orbits which are periodic in a rotating frame, which have arbitrary orbital inclination, and are also of the *première sorte*. There are orbits which are also periodic in a rotating frame, which exist near the critical value of the inclination, and which become elliptical when J_2 and J_4 become zero, and so are analogous to Poincaré's *seconde sorte*, or *troisième sorte*. Barrar has announced the existence of orbits periodic in a rotating frame, also of the *seconde sorte*, which exist for arbitrary values of the inclination, but with specially chosen value of the argument of perigee.

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5. Cook, A. H. *Geophys. J.*, **4**, 53, 1961.
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DISCUSSION

Dr Kozai remarked (1) J_1 cannot produce any long-period perturbations in any element; (2) we cannot detect effects due to J_1 if $|J_1|$ is less than 10^{-6} although effects due to other odd harmonics can be detected if they are larger than 10^{-8}; (3) Mr W. Kaula and Mr I. Izsak once tried to derive values of J_{21} by observations. Dr O'Keefe discussed the physical reasons for considering that the $J_{1,\,m}$ and $J_{2,\,1}$ are zero. The first requires that the origin of co-ordinates coincide with the centre of mass of the Earth. This in turn coincides with the centre of figure of the free-air co-geoid within a few meters. An attempt was made in the original vanguard system to ensure the coincidence of the origin and the Earth's centre. Later work on the tracking system has been done by Mrs Irene Fisher and W. Kaula. If Kaula's latest corrections are applied it is believed most tracking stations are related to the centre of the Earth within about $\pm$ 20 meters. The $J_{2,\,1}$ measures the deviation between the axis of figure and the co-ordinate axes. The axis of rotation describes a small cone about the axis of figure and the relation of J_{21} to J_{20} would be of the order of a few parts in a million if the co-ordinate axis of the system of tracking station is made to coincide with the axis of the Earth. To ensure this it is necessary and sufficient that the Laplace equation

$$(\alpha_A - \alpha_G) = \cos\varphi\,(\lambda_A - \lambda_G)$$

is satisfied everywhere. This equation is imposed as a condition at many points in modern triangulation nets. The orientation of the net is probably correct to a few seconds; much more precision than needed for satellite calculations. Thus the $J_{2,\,1}$ vanishes with all needed precision; the $J_{l,\,m}$ may not. It is suggested that the interest of the IAU in these problems be brought to the attention of the Association Internationale de Géodésie, a component of the IGGU.

13. *A. H. Cook.* Within the last 2 years, four estimates (**1, 2, 3, 4**) have been made of even harmonics in the external gravitational field of the Earth, using the observed secular motions of the nodes and perigees of close artificial satellites. In earlier work the available data did not allow harmonics beyond J_6 to be estimated but now that there are accurate observations of a number of satellites with inclinations covering a wide range, it is possible to consider harmonics up to J_{14}. The results of these most recent studies have not, however, been too satisfactory, the estimates of $10^6\,J_2$ for example, lying between 1082·48 and 1082·86, while the estimates of the higher harmonics are no more consistent. Six sets of data have been used in these four studies and I have examined them statistically to see if they are consistent.

The groups of data with the number of satellites and range of inclinations are

Kozai, 1962	nodal	11	32°·9 to	66°·9
	perigee	6	32°·9	66°·7
King-Hele, 1963	nodal	7	28°·8	97°·4
Kozai, 1964	nodal	7	28°·8	95°·9
	perigee	7	28°·8	95°·9
King-Hele, 1964	nodal	7	28°·8	95°·9

Some satellites are common to two or more of these sets. My definition of consistency is that two sets of data are consistent if a group of harmonics determined from one set represents the other within the uncertainties of the observations. On applying this test I find that the sets of perigee data are highly inconsistent with the sets of nodal data, even when the sets refer to the same satellites, and I find that sets of nodal data are, in general, inconsistent. All the evidence indicates that this is because the harmonics beyond those that are fitted to the data cannot be assumed to be zero, as is supposed in making the estimates; for instance, the effect of the neglected higher harmonics is usually greater on the motion of perigee than on the node, leading to the inconsistency between node and perigee data.

Almost all the satellites that contribute to the above sets of data have semi-major axes that do not greatly exceed the radius of the Earth, and so the effects of harmonics of orders up to 20 or so are comparable. By choosing data from satellites with larger semi-major axes it should be possible to reduce the effects of the higher harmonics and to make better estimates of the lower order. The data are not very suitable at present, but by choosing four satellites with a wide range of inclination, I have been able to make a solution for J_2, J_4 and J_6 (5):

$$10^6 J_2\colon 1082{\cdot}65 \pm 0{\cdot}05;\ 10^6 J_4\colon -1{\cdot}61 \pm 0{\cdot}1;\ 10^6 J_6\colon +0{\cdot}73 \pm 0{\cdot}2.$$

The data are not adequate for the determination of higher harmonics but it can be said that the order of magnitude of harmonics up to and beyond J_{14} is $0{\cdot}5 \times 10^{-6}$.

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DISCUSSION

Y. Kozai announced that he had recently derived almost the same values for J_2, J_4 and J_6 as these reported by Dr Cook by using not only high altitude satellites but also low satellites:

$$10^6 J_2\colon 1082{\cdot}65 \pm 0{\cdot}01;\ 10^6 J_4\colon -1{\cdot}65 \pm 0{\cdot}02;\ 10^6 J_6\colon +0{\cdot}65 \pm 0{\cdot}03$$

Inconsistency in the perigee motion mentioned by Dr Cook is partly due to the solar radiation pressure, which Kozai cannot compute for the perigee with sufficient accuracy.

14. *R. P. Cesco.* Some recent work on the general solution of the three-body problem with arbitrary masses and possibility of binary collisions permits the assumption that even from the numerical point of view, Sundman's theory can be very useful.

Cesco, P. C. Riú and other students are now working with the method of summability of divergent series of Mittag-Leffler, which give the analytical continuation of the co-ordinates and the time through the mentioned strip of Sundman, combined, if necessary, with the useful Borel's method, not only to theoretical examples, but also to some astronomical cases.

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