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CYCLIC TARSKI ALGEBRAS

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The variety of cyclic Boolean algebras is a particular subvariety of the variety of tense algebras. The objective of this paper is to study the variety $\mathcal{T}$ of $\{\to, g, h\}$-subreducts of cyclic Boolean algebras, which we call cyclic Tarski algebras. We prove that $\mathcal{T}$ is generated by its finite members and we characterise the locally finite subvarieties of $\mathcal{T}$. We prove that there are no splitting varieties in the lattice $\Lambda(\mathcal{T})$ of subvarieties of $\mathcal{T}$. Finally, we prove that the subquasivarieties and the subvarieties of a locally finite subvariety of $\mathcal{T}$ coincide.

1. Introduction and Preliminaries

The variety of *tense algebras* [8] is defined as the variety of algebras $\langle A; \wedge, \vee, \neg, g, h, 0, 1\rangle$, where $\langle A; \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and g and h are unary operators satisfying the following identities: $g1 = 1$, $h1 = 1$; $g(x \wedge y) = gx \wedge gy$, $h(x \wedge y) = hx \wedge hy$; $\neg x \vee g\neg h\neg x = 1$; $\neg x \vee h\neg g\neg x = 1$.

Tense algebras are the algebraic counterpart of tense logics.

In [2] a particular subvariety of the variety of tense algebras, called Boolean algebras with a distinguished automorphism (or cyclic Boolean algebras), was studied. By a *cyclic Boolean algebra* we understand a tense algebra $\langle A; \wedge, \vee, \neg, g, h, 0, 1\rangle$, such that g is an automorphism of A (and so $h = g^{-1}$).

The purpose of this paper is to investigate the variety of $\{\to, g, h\}$–subreducts of cyclic Boolean algebras. This variety models the implication operation of cyclic Boolean algebras in the same way as Tarski algebras (or implication algebras) model the implication operation of Boolean algebras.

Definition 1.1: The variety $\mathcal{T}$ of cyclic Tarski algebras consists of algebras $\mathbf{A} = \langle A; \to, g, h, 1\rangle$ of type $(2, 1, 1, 0)$ fulfilling the following conditions:

1. $1 \to x = x$,
2. $x \to x = 1$,
3. $x \to (y \to z) = (x \to y) \to (x \to z)$,

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4. $(x \to y) \to y = (y \to x) \to x$,
5. $g(x \to y) = gx \to gy$, $h(x \to y) = hx \to hy$,
6. $hgx = ghx = x$.

That is, $\mathbf{A} = \langle A; \to, g, h, 1\rangle \in \mathcal{T}$ if $\langle A; \to, 1\rangle$ is a Tarski algebra and g and h are operators fulfilling conditions (5) and (6). Observe that if $\mathbf{A} \in \mathcal{T}$, then g and h are automorphisms of the Tarski algebra $\langle A; \to, 1\rangle$, and $h = g^{-1}$.

We assume familiarity with the theory of Tarski algebras. In particular, we recall the following properties:

THEOREM 1.2. *In any Tarski algebra* $\mathbf{A}$,

1. $x \to (y \to x) = 1$,
2. $\big(x \to (y \to z)\big) \to \big((x \to y) \to (x \to z)\big) = 1$,
3. $x \to 1 = 1$,
4. *if* $x \to y = 1$ *and* $y \to x = 1$ *then* $x = y$,
5. $(x \to y) \to x = x$,
6. *For* $x, y \in A$, *the relation* $x \leqslant y$ *if* $x \to y = 1$, *is a partial order with last element* 1. *Relative to this partial order,* $\mathbf{A}$ *is a join-semilattice and the join of two elements* a *and* b *is given by* $a \vee b = (a \to b) \to b$. *Besides, for each* a *in* A, $[a) = \{x \in A : a \leqslant x\}$ *is a Boolean algebra in which, for* $b, c \geqslant a$, $b \wedge c = \big(b \to (c \to a)\big) \to a$ *gives the meet and* $b \to a$ *is the complement of* b ([5]).

A *deductive system* in a Tarski algebra $\mathbf{A}$ is a subset D of A that satisfies $1 \in D$, and $b \in D$ whenever a and $a \to b \in D$. A *gh-deductive system* in an algebra $\mathbf{A} \in \mathcal{T}$ is a deductive system closed by g and h. Congruences in a Tarski algebra are determined by deductive systems [5] and, similarly, congruences in an algebra $\mathbf{A} \in \mathcal{T}$ are determined by gh-deductive systems. The congruence relation associated to a gh-deductive system $D \subseteq A$ is $x \equiv y \pmod{D}$ if and only if $x \to y \in D$ and $y \to x \in D$.

For $H \subseteq A$, the *gh-deductive system generated by* H is the intersection of all gh-deductive systems of $\mathbf{A}$ containing H.

It is well known the following characterisation of a deductive system in a Tarski algebra [5].

LEMMA 1.3. *Let* $\mathbf{A}$ *be a Tarski algebra and* D *a subset of* A. D *is a deductive system of* $\mathbf{A}$ *if and only if the following conditions are satisfied:*

1. D *is non-empty,*
2. D *is an increasing subset of* A,
3. *if* $a, b \in D$ *and there exists* $a \wedge b$, *then* $a \wedge b \in D$.

As a consequence of Lemma 1.3 we obtain the following characterisation of the deductive system generated by a subset H. The proof is easy.

Lemma 1.4. *For a non-empty subset H of a Tarski algebra $\mathbf{A}$, the deductive system generated by H consists of all elements of the form $a_1 \wedge a_2 \wedge \ldots \wedge a_n$, $a_i \in A$, $n \in \mathbb{N}$, such that for each $i = 1, 2, \ldots, n$, $a_i \geqslant x_i$ with $x_i \in H$.*

A *Kripke frame*, or simply a *frame*, is a triple $\mathcal{F} = \langle W; R, R^{-1}\rangle$, where W is a set, R is a binary relation defined on W and R^{-1} is the inverse relation of R. For $w \in W$, let $R(w) = \{z \in W : wRz\}$ and $R^{-1}(w) = \{z \in W : zRw\}$. For a given frame $\mathcal{F} = \langle W; R, R^{-1}\rangle$, $\mathrm{Alg}(\mathcal{F}) = \langle 2^W; \rightarrow, g, h, W\rangle$, where g and h are defined by

$$g(X) = \{w \in W : R(w) \subseteq X\} \quad \text{and} \quad h(X) = \{w \in W : R^{-1}(w) \subseteq X\}$$

and the set-theoretical implication is $X \rightarrow Y = (W \backslash X) \cup Y$, for X, $Y \subseteq W$, is a cyclic Tarski algebra which is called the *canonical algebra* of $\mathcal{F}$. Observe that, in fact, $\mathrm{Alg}(\mathcal{F})$ is a cyclic Boolean algebra.

Let $\mathbf{A} = \langle A; \rightarrow, g, h, 1\rangle \in \mathcal{T}$ and let $\mathfrak{D}(\mathbf{A})$ denote the set of maximal deductive systems of $\mathbf{A}$. Then $\mathcal{F}(\mathbf{A}) = \langle \mathfrak{D}(\mathbf{A}); R, R^{-1}\rangle$, where $R \subseteq \mathfrak{D}(\mathbf{A}) \times \mathfrak{D}(\mathbf{A})$ is defined by URW if and only if $g^{-1}(U) \subseteq W$ (or, equivalently, by $UR^{-1}W$ if and only if $h^{-1}(U) \subseteq W$), is called the *canonical frame* for $\mathbf{A}$. Since g is an automorphism, it is clear that for each $U \in \mathfrak{D}(\mathbf{A})$, $R(U)$ has just one element, $h(U)$, and g and h induce bijections on $\mathfrak{D}(\mathbf{A})$. So it is immediate to check the following.

Theorem 1.5. *If $\mathbf{A} \in \mathcal{T}$, then $\mathrm{Alg}(\mathcal{F}(\mathbf{A})) \in \mathcal{T}$.*

Theorem 1.6. *If $\mathbf{A} \in \mathcal{T}$, $\mathbf{A}$ is isomorphic to a subalgebra of $\mathrm{Alg}(\mathcal{F}(\mathbf{A}))$.*

Proof: An embedding can be taken as $s : A \rightarrow 2^{\mathfrak{D}(\mathbf{A})}$ with $s(a) = \{U \in \mathfrak{D}(\mathbf{A}) : a \in U\}$. ∎

The *orbit* of an element $U \in \mathfrak{D}(\mathbf{A})$ is the set $O(U) = \{g^n(U), n \in \mathbb{Z}\}$.

It is clear that $\mathcal{F}(\mathbf{A})$ is the disjoint union of its connected components $\mathcal{F}_j$, $j \in J$, where $\mathcal{F}_j = \langle O(U_j); R'_j, R'^{-1}_j\rangle$ for some $U_j \in \mathfrak{D}(\mathbf{A})$ and $R'_j = R \cap (O(U_j) \times O(U_j))$. Hence it can be proved that $\mathrm{Alg}(\mathcal{F}(\mathbf{A})) \cong \prod_{j \in J} \mathrm{Alg}(\mathcal{F}_j)$.

2. k-cyclic Tarski algebras

Our next objective is to study, for each $k \in \mathbb{N}$, the subvariety of $\mathcal{T}$, which we call the variety of k-cyclic Tarski algebras, characterised by the equation $g^k x = x$. The case $k = 2$ was used in [1] to obtain a representation of cubic lattices.

The next Lemma provides a characterisation of maximal deductive systems of a finite Tarski algebra. The proof can be found in [3].

Lemma 2.1. *Let $\mathbf{A}$ be a non-trivial finite Tarski algebra with n antiatoms. If a is an antiatom of $\mathbf{A}$, then $A \setminus (a]$ is a maximal deductive system of $\mathbf{A}$. Moreover, every maximal deductive system of $\mathbf{A}$ is of the form $A \setminus (a]$, for some antiatom a of $\mathbf{A}$. In particular, $\mathbf{A}$ has exactly n maximal dedutive systems.*

As a consequence of Lemma 2.1 we obtain the following characterisation of maximal gh-deductive systems of a finite k-cyclic Tarski algebra. Observe that in a k-cyclic Tarski algebra, $h = g^{k-1}$.

Lemma 2.2. *Let $\mathbf{A}$ be a finite k-cyclic Tarski algebra. If a is an antiatom of $\mathbf{A}$, then $\bigcap_{i=1}^{k} A \setminus (g^i a]$ is a maximal gh-deductive system of $\mathbf{A}$. Moreover, every maximal gh-deductive system of $\mathbf{A}$ is of the form $\bigcap_{i=1}^{k} A \setminus (g^i a]$, for some antiatom a of $\mathbf{A}$.*

Proof: It is clear that $\bigcap_{i=1}^{k} A \setminus (g^i a]$ is a deductive system. Let $x \in \bigcap_{i=1}^{k} A \setminus (g^i a]$ and suppose that $gx \notin \bigcap_{i=1}^{k} A \setminus (g^i a]$. Then for some $i = 1, \dots, k$, $gx \leqslant g^i a$. Thus $hgx \leqslant hg^i a$, and as $hgx = x$ and $hg^i a = g^{k-1} g^i a = g^{k+i-1} a$, we have $x \leqslant g^{k+i-1} a$. Since $\mathbf{A}$ is a k-cyclic algebra, $g^{k+i-1} a = g^l a$ for some l, $1 \leqslant l \leqslant k$, a contradiction.

If D is a gh-deductive system such that $\bigcap_{i=1}^{k} A \setminus (g^i a] \subsetneqq D$, there exists $x \in D$ such that $x \notin \bigcap_{i=1}^{k} A \setminus (g^i a]$, that is, $x \leqslant g^i a$ for some i. Since $x \in D$, $g^i a \in D$, and thus, D contains the antiatoms $a, ga, \dots, g^{k-1} a$. Let us see that D contains all the antiatoms of $\mathbf{A}$. Indeed, if b is an antiatom and $b \neq g^i a$ for every i, then $b \in \bigcap_{i=1}^{k} A \setminus (g^i a]$ and, consequently, $b \in D$. Now, D is closed under existing infima, and in any finite Tarski algebra, every element is an infimum of antiatoms, so $D = A$. Hence $\bigcap_{i=1}^{k} A \setminus (g^i a]$ is a maximal gh-deductive system.

Let us see that every maximal gh-deductive system M is of this form. As M is maximal, M is proper, so there exists an antiatom a of $\mathbf{A}$ not belonging to M. Since M is increasing, it follows that $M \subseteq A \setminus (a]$. Besides, as M is closed under g, $M \subseteq A \setminus (g^i a]$ for $i = 1, \dots, k$. So $M \subseteq \bigcap_{i=1}^{k} A \setminus (g^i a]$ and by maximality, $M = \bigcap_{i=1}^{k} A \setminus (g^i a]$. $\square$

The intersection of all maximal deductive systems of a Tarski algebra is $\{1\}$. So we have:

Corollary 2.3. *Let $\mathbf{A}$ be a finite k-cyclic Tarski algebra. The intersection of all maximal gh-deductive systems of $\mathbf{A}$ is $\{1\}$.*

Since every finite algebra in $\mathcal{T}$ is k-cyclic for some k, we obtain the following corollary.

Corollary 2.4. *The intersection of all maximal gh-deductive systems of a finite algebra $\mathbf{A} \in \mathcal{T}$ is $\{1\}$.*

Lemma 2.5. *Every finite subdirectly irreducible algebra of $\mathcal{T}$ is simple.*

Proof: Let $\mathbf{A} \in \mathcal{T}$, $\mathbf{A}$ finite and $\{D_i\}_{1 \leqslant i \leqslant n}$ the collection of maximal gh-deductive systems of $\mathbf{A}$. From Corollary 2.4, the canonical map $i : \mathbf{A} \to \prod_{j=1}^{n} \mathbf{A}/D_i$ is injective. Since $\mathbf{A}$ is subdirectly irreducible, $\mathbf{A}$ is isomorphic to $\mathbf{A}/D_i$ for some i. Hence $\mathbf{A}$ is simple. $\square$

LEMMA 2.6. *Every subdirectly irreducible k-cyclic Tarski algebra is finite and, consequently, simple.*

PROOF: Let $\mathbf{A}$ be a subdirectly irreducible k-cyclic Tarski algebra and $\mathcal{F}(\mathbf{A}) = \bigcup_{j \in J} \mathcal{F}_j$, the disjoint union of its connected components. We have that $g^k(U) = U$ for every maximal deductive system of $\mathbf{A}$ and, consequently, $\mathcal{F}_j$ is a finite frame for every j. Consider the embedding

$$s : \mathbf{A} \to \mathrm{Alg}\big(\mathcal{F}(\mathbf{A})\big) \cong \prod_{j \in J} \mathrm{Alg}(\mathcal{F}j).$$

For each $i \in J$, consider $\pi_i : \prod_{j \in J} \mathrm{Alg}(\mathcal{F}_j) \to \mathrm{Alg}(\mathcal{F}_i)$ the i-th projection and let $s_i = \pi_i \circ s$. We have that $\mathrm{Ker}(s_i)$ is a proper gh-deductive system of $\mathbf{A}$. If $\mathbf{A}$ is simple, $\mathrm{Ker}(s_i) = \{1\}$, and so s_i is an embedding of $\mathbf{A}$ into $\mathrm{Alg}(\mathcal{F}_i)$. If $\mathbf{A}$ is non-simple subdirectly irreducible with monolith M and we assume that for every i, $\mathrm{Ker}(s_i) \neq \{1\}$, then $M \subseteq \cap \mathrm{Ker}(s_i) = \cap\{U : U \in \mathfrak{D}(\mathbf{A})\} = \{1\}$, a contradiction. So there exists i such that s_i is an embedding. Since every component $\mathcal{F}_i$ is finite, $\mathrm{Alg}(\mathcal{F}_i)$ is finite and, consequently, so is $\mathbf{A}$. ∎

Finally, we give a characterisation of subdirectly irreducible (simple) k-cyclic Tarski algebras. For a k-cyclic Tarski algebra $\mathbf{A}$, we say that d is the *period* of $\mathbf{A}$ if d is the least natural such that $g^d a = a$ for all a in A. It is clear that d is a divisor of k.

For each $k \in \mathbb{N}$, let $W_k = \{w_1, w_2, \ldots, w_k\}$ and $\mathcal{F}_k = \langle W_k; R_k, R_k^{-1}\rangle$ where $w_i R_k w_{i+1}$ for $1 \leqslant i \leqslant k-1$ and $w_k R_k w_1$.

$$\begin{tikzcd} \underset{w_1}{\bullet} \arrow[r] & \underset{w_2}{\bullet} \arrow[r] & \cdots\cdots \arrow[r] & \underset{w_k}{\bullet} \arrow[lll, bend right] \end{tikzcd} \qquad (*)$$

Let $\mathbf{B}_k = \mathrm{Alg}(\mathcal{F}_k)$. Observe that $g\big(\{w_i\}\big) = \{w_{i-1}\}$ for $2 \leqslant i \leqslant k$ and $g\big(\{w_1\}\big) = \{w_k\}$, and then, g acts transitively on the set of antiatoms of $\mathbf{B}_k$, and consequently, g is an automorphism of $\mathbf{B}_k$. It is clear that $g^k x = x$ for every $x \in 2^{W_k}$ and that $\mathbf{B}_k$ is a k-cyclic Tarski algebra.

THEOREM 2.7. *Let $\mathbf{A}$ be a k-cyclic Tarski algebra of period d. Then $\mathbf{A}$ is subdirectly irreducible if and only if $\mathbf{A}$ is isomorphic to a nontrivial increasing subalgebra of $\mathbf{B}_d$.*

PROOF: Let $\mathbf{A}$ be a subdirectly irreducible k-cyclic Tarski algebra of period d. By Lemma 2.6, $\mathbf{A}$ is simple and finite, and so $\mathcal{F}(\mathbf{A})$ is connected being that every connected component yields a congruence of $\mathbf{A}$. In addition, the maximal deductive systems of $\mathbf{A}$ are $\{A \setminus (g^i a], 0 \leqslant i < d\}$ for an antiatom a of $\mathbf{A}$, that is, $\mathcal{F}(\mathbf{A})$ has the form $(*)$, and hence $\mathrm{Alg}\big(\mathcal{F}(\mathbf{A})\big)$ is isomorphic to $\mathbf{B}_d$.

Let $s : \mathbf{A} \to \mathrm{Alg}\big(\mathcal{F}(\mathbf{A})\big)$ be the embedding of Theorem 1.6. If $a_1, a_2, \ldots, a_d$ are the antiatoms of $\mathbf{A}$ and $i \neq j$, then $a_i \notin (a_j]$, that is, $a_i \in A \setminus (a_j]$ for every $j \neq i$. Thus

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$s(a_i) = \{A \setminus (a_j] : j \neq i\}$. So $s(a_i)$ is an antiatom of $\operatorname{Alg}(\mathcal{F}(\mathbf{A}))$ and, consequently, s induces a bijection from the set of antiatoms of $\mathbf{A}$ on the set of antiatoms of $\operatorname{Alg}(\mathcal{F}(\mathbf{A}))$. Since every element of a finite Tarski algebra is an infimum of antiatoms, it follows that $s : \mathbf{A} \to [\operatorname{Min}(s(\mathbf{A})))$ is an isomorphism, where with $\operatorname{Min}(s(\mathbf{A}))$ we denote the set of minimal elements of $s(\mathbf{A})$. Hence $\mathbf{A}$ is isomorphic to an increasing subset of $\mathbf{B}_d$, closed under g and h, that is, an increasing subalgebra of $\mathbf{B}_d$.

The converse is trivial being that every nontrivial subalgebra of a simple algebra is simple. ∎

Besides giving a characterisation of subdirectly irreducible k-cyclic Tarski algebras, Theorem 2.7 allows us to prove that $\mathbf{B}_k$ is a generator of the variety of k-cyclic Tarski algebras.

Corollary 2.8. *The variety of k-cyclic Tarski algebras is generated by* $\mathbf{B}_k$.

3. The variety $\mathcal{T}$

Let $2^{\mathbb{Z}}$ be the field of subsets of $\mathbb{Z}$ with the set-theoretical operation of implication. Let g be the automorphism of $2^{\mathbb{Z}}$ induced by the mapping $n \mapsto n+1$, $n \in \mathbb{Z}$ and $h = g^{-1}$. It is clear that $\mathbf{2}^{\mathbf{Z}} = \langle 2^{\mathbf{Z}}; \to, g, h, \mathbb{Z}\rangle \in \mathcal{T}$. Observe that $\mathbf{2}^{\mathbf{Z}} = \operatorname{Alg}(\langle \mathbb{Z}; R, R^{-1}\rangle)$, where $zR(z-1)$, for $z \in \mathbb{Z}$.

A subset of $\mathbb{Z}$ is called *k-periodic*, k a positive integer, if it coincides with the set obtained by adding k to each of its elements. It is clear that the set of k-periodic subsets of $\mathbb{Z}$ is a subalgebra of $\mathbf{2}^{\mathbf{Z}}$. We say that a subset of $\mathbb{Z}$ is *periodic* if it is k-periodic for some k.

Lemma 3.1. $\mathbf{B}_k$ *is isomorphic to a subalgebra of* $\mathbf{2}^{\mathbf{Z}}$.

Proof: If we consider the congruence modulo k in $\mathbb{Z}$ and we use the notation $[x]_k$ for the equivalence class of x, then $\mathbb{Z} \setminus [0]_k$, $\mathbb{Z} \setminus [1]_k$, …, $\mathbb{Z} \setminus [k-1]_k$ are the antiatoms of the subalgebra of k-periodic subsets of $\mathbb{Z}$. It is clear that g acts transitively on these antiatoms, that is, this subalgebra is isomorphic to the algebra $\mathbf{B}_k$. ∎

The proof of the following Theorem is similar to that of [**2**, Theorem 2.3].

Theorem 3.2. *The variety* $\mathcal{T}$ *is generated by* $\mathbf{2}^{\mathbf{Z}}$. *Moreover,* $\mathcal{T} = ISP(\mathbf{2}^{\mathbf{Z}})$.

Proof: If $\mathbf{A} \in \mathcal{T}$, $\mathbf{A} \hookrightarrow \operatorname{Alg}(\mathcal{F}(\mathbf{A})) \cong \prod_{j\in J} \operatorname{Alg}(\mathcal{F}_j)$, where $\{\mathcal{F}_j\}_{j\in J}$ is the family of distinct connected components of $\mathcal{F}(\mathbf{A})$. If $\mathcal{F}_j$ is finite, then $\operatorname{Alg}(\mathcal{F}_j) \cong \mathbf{B}_n$ for some n, and if $\mathcal{F}_j$ is infinite then $\operatorname{Alg}(\mathcal{F}_j) \cong \mathbf{2}^{\mathbf{Z}}$. Consequently $\operatorname{Alg}(\mathcal{F}(\mathbf{A}))$ is isomorphic to a product $\prod_{j\in J} \mathbf{A}_j$, where $\mathbf{A}_j \cong \mathbf{B}_n$ or $\mathbf{A}_j \cong \mathbf{2}^{\mathbf{Z}}$. In both cases there exists a monomorphism $\mathbf{A}_j \hookrightarrow \mathbf{2}^{\mathbf{Z}}$. Thus $\mathbf{A} \hookrightarrow \prod_{j\in J} \mathbf{A}_j \hookrightarrow (\mathbf{2}^{\mathbf{Z}})^J$, and so $\mathbf{A} \in ISP(\mathbf{2}^{\mathbf{Z}})$. ∎

In [**2**, Theorem 2.6] it is proved that the variety of cyclic Boolean algebras is generated by any infinite set of algebras $\mathbf{B}_k$. This could be done by establishing an embedding

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from $\mathbf{2}^{\mathbf{Z}}$ into a homomorphic image of a product of an infinite family of $\mathbf{B}_k'$s. Since a Boolean homomorphism is, in particular, a Tarski homomorphism, the proof of that theorem also holds for the variety $\mathcal{T}$. Consequently we have:

THEOREM 3.3. *If J is an infinite subset of $\mathbb{N}$, then $\mathcal{T} = V(\{\mathbf{B}_j : j \in J\})$.*

COROLLARY 3.4. *$\mathcal{T}$ is generated by its finite members.*

THEOREM 3.5. *If $\mathbf{A}$ is a subdirectly irreducible algebra of $\mathcal{T}$, then $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{2}^{\mathbf{Z}}$.*

PROOF: Let $\mathbf{A}$ be a subdirectly irreducible algebra of $\mathcal{T}$ and consider the embedding $s : \mathbf{A} \to \mathrm{Alg}(\mathcal{F}(\mathbf{A}))$ of Theorem 1.6. Arguing as in Lemma 2.6, there is a connected component $\mathcal{F}_j$ such that $\mathbf{A}$ can be embedded into $\mathrm{Alg}(\mathcal{F}_j)$. But $\mathrm{Alg}(\mathcal{F}_j)$ is isomorphic to either $\mathbf{2}^{\mathbf{Z}}$ or $\mathbf{B}_n$, for some n. Since $\mathbf{B}_n$ is isomorphic to a subalgebra of $\mathbf{2}^{\mathbf{Z}}$, we have that there exists a monomorphism $\mathrm{Alg}(\mathcal{F}_j) \hookrightarrow \mathbf{2}^{\mathbf{Z}}$. Thus there exists an embedding from $\mathbf{A}$ into $\mathbf{2}^{\mathbf{Z}}$. $\square$

Our next objective is to prove that the locally finite subvarieties of $\mathcal{T}$ are the finitely generated ones.

In [6], Bezhanishvili calls a class K of algebras *uniformly locally finite* if for all $k \in \mathbb{N}$ there exists $m(k) \in \mathbb{N}$ such that the cardinality of every subalgebra k-generated of an algebra of K is less than or equal to $m(k)$. In that work it is proved that:

THEOREM 3.6. *If K has finite signature, then K is locally finite if and only if K is generated by a uniformly locally finite class.*

If $\mathbf{A}$ is a k-cyclic finite algebra and d is the period of $\mathbf{A}$, then $\mathbf{A}$ is isomorphic to an increasing subalgebra of $\mathbf{B}_d$. In particular, it contains the antiatoms of $\mathbf{B}_d$. Let $Ant(\mathbf{B}_d)$ be the subalgebra generated by the antiatoms of $\mathbf{B}_d$, that is, $Ant(\mathbf{B}_d)$ consists of the antiatoms of $\mathbf{B}_d$ and 1. Observe also that $Ant(\mathbf{B}_d)$ is generated by any element different from 1. Thus, if K is an infinite subset of $\mathbb{N}$, the family $\{Ant(\mathbf{B}_d) : d \in K\}$ is not a uniformly locally finite class.

These remarks together with Theorem 3.6 allow us to characterise the locally finite subvarieties of $\mathcal{T}$.

THEOREM 3.7. *Let V be a subvariety of $\mathcal{T}$. V is locally finite if and only if V is finitely generated. Consequently, V is locally finite if and only if $V \subseteq V(\mathbf{B}_k)$ for some k.*

PROOF: Let V be a non-finitely generated subvariety of $\mathcal{T}$ and let $X = \{\mathbf{A}_i, i \in I\}$ be a set of non-isomorphic subdirectly irreducible algebras that generate V. Since V is not finitely generated, X is not a finite set of finite algebras.

Suppose that X contains an infinite algebra $\mathbf{A}_i = \mathbf{A}$. Since $\mathbf{A}$ is subdirectly irreducible we may identify $\mathbf{A}$ with a subalgebra of $\mathbf{2}^{\mathbf{Z}}$. If A contains an element x which is not periodic, then the subalgebra $\langle x \rangle$ generated by x is not finite. So V is not finitely generated. Suppose that every element of A is periodic. Then $\mathbf{A} = \bigcup_{j \in J} \mathbf{S}_j$, where $\mathbf{S}_j$ is a

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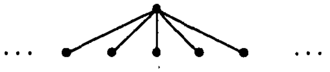
finite algebra of period d_j, and consequently, $Ant(\mathbf{B}_{d_j}) \subseteq S_j$. Since $\mathbf{A}$ is infinite, the set $K = \{d_j : \mathbf{S}_j \text{ is a subalgebra of } \mathbf{A}\}$ is infinite and, by the previous remark, the family $\{Ant(\mathbf{B}_{d_j}) : d_j \in K\}$ generates a variety which is not locally finite. Since $Ant(\mathbf{B}_{d_j}) \subseteq A$ for every $d_j \in K$, it follows that $\{Ant(\mathbf{B}_{d_j}) : d_j \in K\} \subseteq V(\mathbf{A})$, so $V(\mathbf{A})$ is not locally finite, and consequently, V is not locally finite.

If X does not contain an infinite algebra, then X contains infinitely many non-isomorphic finite algebras $\mathbf{A}_i$. If d_i is the period of $\mathbf{A}_i$, then $Ant(\mathbf{B}_{d_i}) \subseteq A_i$. The set $K = \{d_i : Ant(\mathbf{B}_{d_i}) \subseteq V\}$ is infinite and $Ant(\mathbf{B}_{d_i}) \in V$ for every $d_i \in K$. So, as in the previous case, V is not locally finite.

For the last part, observe that if V is finitely generated, V is generated by a finite set of finite algebras. But then there exists $k \in \mathbb{N}$ such that all these algebras are k-cyclic. So $V \subseteq V(\mathbf{B}_k)$. $\square$

Observe that there exist proper subvarieties of $\mathcal{T}$ that are not finitely generated. For instance, consider the subalgebra $\mathbf{A}$ of $\mathbf{2}^{\mathbf{Z}}$ whose elements are 1 and the antiatoms of $\mathbf{2}^{\mathbf{Z}}$.



It is clear that $\mathbf{A} \models \{x \vee g^n x = 1 : n \in \mathbb{N}\}$ and that $\mathbf{2}^{\mathbf{Z}} \notin V(\mathbf{A})$, so $V(\mathbf{A})$ is a proper subvariety of $\mathcal{T}$.

Now we are going to prove that there are no splitting subvarieties for the lattice $\Lambda(\mathcal{T})$ of subvarieties of $\mathcal{T}$. Let us see first the following lemma that characterises the gh-deductive system generated by a set H.

LEMMA 3.8. *Let $\mathbf{A} \in \mathcal{T}$. The gh-deductive system generated by a non-empty subset H of $\mathbf{A}$ consists of the elements of the form $a_1 \wedge a_2 \wedge \ldots \wedge a_n$, $a_i \in A$, such that for each $i = 1, 2, \ldots, n$, $a_i \geqslant g^{n_i} x_i$ with $x_i \in H$, $n_i \in \mathbb{Z}$ and $n \in \mathbb{N}$.*

PROOF: Let D be the set of elements of A of the form $a_1 \wedge a_2 \wedge \ldots \wedge a_n$, $a_i \in A$ such that for each $i = 1, 2, \ldots, n$, $a_i \geqslant g^{n_i} x_i$ with $x_i \in H$, $n_i \in \mathbb{Z}$, $n \in \mathbb{N}$.

By Lemma 1.4, D is a deductive system. In order to see that D is closed under g and h, it is enough to observe that if there exists $a_1 \wedge a_2 \ldots \wedge a_n \in D$ then $ga_1 \wedge ga_2 \ldots \wedge ga_n$ exists and $g(a_1 \wedge a_2 \ldots \wedge a_n) = ga_1 \wedge ga_2 \ldots \wedge ga_n$. Similarly for h. $\square$

Recall that an element a of a complete lattice $\mathbf{L}$ is said to be *strictly join prime* if $a \leqslant \bigvee_{i \in I} a_i$ implies that there exists $i \in I$ such that $a \leqslant a_i$. In [9] it was proved the following characterisation of splitting elements of a complete lattice $\mathbf{L}$.

THEOREM 3.9. *Let $\mathbf{L}$ be a complete lattice. Then $a \in L$ is a splitting element if and only if a is strictly join prime.*

For any fixed variety V, let us denote $\Lambda(V)$ the complete lattice of all subvarieties of V and call splitting elements of $\Lambda(V)$ *splitting varieties*. The following theorem was also established in [9].

THEOREM 3.10. *If a variety V is congruence-distributive and generated by its finite algebras, then every splitting variety in $\Lambda(V)$ is generated by a finite subdirectly irreducible algebra of V.*

THEOREM 3.11. *There are no splitting varieties in $\Lambda(\mathcal{T})$.*

PROOF: Let us prove that there are no strictly join prime varieties $V(\mathbf{A})$, where $\mathbf{A}$ is a finite subdirectly irreducible algebra in $\mathcal{T}$.

If $\mathbf{A}$ is a finite simple algebra of period $n > 1$, $V(\mathbf{A})$ is not strictly join prime. Indeed, since $\mathcal{T}$ is generated by any infinite family of algebras $\mathbf{B}_m$, we have that $V(\{\mathbf{B}_m : n \text{ does not divide a } m\}) = \mathcal{T}$. So $V(\mathbf{A}) \subseteq V(\{\mathbf{B}_m : n \text{ does not divide } m\})$, but $V(\mathbf{A}) \not\subseteq V(\mathbf{B}_m)$ for any m being that n does not divide m.

Let us see that $V(\mathbf{B}_1)$ is not strictly join prime either. Consider for each $n > 1$ the algebra $\mathbf{A}_n$ obtained from $\mathbf{B}_n$ by removing its first element and let $\mathbf{A} = \prod_{n>1} \mathbf{A}_n$. Let $a = (10, 100, 1000, \ldots) \in A$ and D the gh-deductive system generated by $a \to ga$ and $ga \to a$.

$a \to ga = (01, 011, 0111, \ldots)$, and in a similar way, $ga \to a = (10, 101, 1011, \ldots)$.

Let us see that $a \notin D$. Indeed, if we suppose that $a \in D$, then $a = \bigwedge_{i=i}^{k} a_i$, where either $a_i \geqslant g^{k_i}(a \to ga)$ or $a_i \geqslant g^{k_i}(ga \to a)$, $k_i \in \mathbb{Z}$. The element $\bigwedge_{i=i}^{k} a_i$ has in each component at most k zeros, whereas a has $n-1$ zeros in the component corresponding to $\mathbf{A}_n$, a contradiction.

Consequently, in the quotient $\mathbf{A}/D$, $\bar{a} \neq \bar{1}$ and $\bar{a} = \overline{ga}$. So $\{\bar{a}, \bar{1}\}$ is a subalgebra of $\mathbf{A}/D$ isomorphic to $\mathbf{B}_1$, that is,

$$V(\mathbf{B}_1) \subseteq SHP(\{\mathbf{A}_n : n > 1\}) \subseteq \bigvee_{n>1} V(\mathbf{A}_n).$$

But since the simple algebras of $V(\mathbf{A}_n)$ are in $HS(\mathbf{A}_n)$, it is clear that $V(\mathbf{B}_1) \not\subseteq V(\mathbf{A}_n)$ for any $n > 1$. ∎

The variety of cyclic Boolean algebras is generated by the algebra of finite and cofinite subsets of $\mathbb{Z}$ (see [2]). It is natural to expect that the algebra of cofinite subsets of $\mathbb{Z}$ generates $\mathcal{T}$. This is the next result.

THEOREM 3.12. *The algebra of cofinite subsets of $\mathbb{Z}$ generates $\mathcal{T}$.*

PROOF: Consider the algebra $\mathbf{A} = \prod_{j \in \mathbb{N}} \mathbf{A}_j$, where $\mathbf{A}_j$ is the algebra of cofinite subsets of $\mathbb{Z}$. Let us see that for every $n \in \mathbb{N}$, there is a gh-deductive system D of $\mathbf{A}$ such that $\mathbf{B}_n$ is isomorphic to a subalgebra of $\mathbf{A}/D$. In particular, this implies that for every $n \in \mathbb{N}$, $\mathbf{B}_n$ belongs to the variety generated by the algebra of cofinite subsets of $\mathbb{Z}$ and, consequently, this variety is $\mathcal{T}$.

Consider the element $a = (\mathbb{Z} \setminus \{0\}, \mathbb{Z} \setminus \{-n, 0, n\}, \mathbb{Z} \setminus \{-2n, -n, 0, n, 2n\}, \ldots)$ and let D be the gh-deductive system generated by $a \to g^n a$ and $g^n a \to a$. We have $a \to g^n a =$

$(\mathbb{Z}\setminus\{n\},\mathbb{Z}\setminus\{2n\},\mathbb{Z}\setminus\{3n\},\ldots)$ and $g^n a \to a = (\mathbb{Z}\setminus\{0\},\mathbb{Z}\setminus\{-n\},\mathbb{Z}\setminus\{-2n\},\ldots)$.

By Lemma 3.8, an element x of A belongs to D if and only if x is an infimum of elements $\{a_l\}$, where each a_l satisfies $a_l \geqslant g^{k_l}(a \to g^n a)$ or $a_l \geqslant g^{k_l}(g^n a \to a)$, $k_l \in \mathbb{Z}$. Arguing as in the proof of Theorem 3.11, it can be seen that a is not an element of D. So $\overline{a} \neq \overline{1}$. Besides, $g^i\overline{a} \neq \overline{a}$ for $1 \leqslant i \leqslant n-1$, since $a \to g^i a = g^i a$, $a \notin D$ and so $g^i a \notin D$.

Consider the subalgebra $\mathbf{S}$ of $\mathbf{A}/D$ generated by $\overline{a}$ and $\bigwedge_{i=1}^{n} g^i\overline{a}$. This infimum is an element of A/D since infima exist in $\mathbf{A}$. We have $g^i\overline{a} \vee g^j\overline{a} = \overline{1}$ and $g^j\overline{a} \to g^i\overline{a} = g^i\overline{a}$ for all $i \neq j$, $1 \leqslant i,j \leqslant n$ and $g^n\overline{a} = \overline{a}$. So $\mathbf{S}$ is isomorphic to $\mathbf{B}_n$. ∎

4. Quasivarieties of k-Cyclic Tarski algebras

Our next task is to prove that the subquasivarieties and the subvarieties of a locally finite subvariety of $\mathcal{T}$ coincide. Observe that V is a locally finite subvariety of $\mathcal{T}$ if and only if $V \subseteq V(\mathbf{B}_k)$ for some k.

Lemma 4.1. *Let $\mathbf{A}$ be a finite k-cyclic Tarski algebra and D a maximal gh-deductive system of $\mathbf{A}$. Then the quotient $\mathbf{A}/D$ is isomorphic to a subalgebra of $\mathbf{A}$.*

Proof: Let $D = \bigcap_{i=0}^{k-1} A \setminus (g^i a] = A \setminus \bigcup_{i=0}^{k-1} (g^i a]$, where a is an antiatom of $\mathbf{A}$ and let S be the gh-deductive system generated by a. Let us see that $\mathbf{S}$ is a subalgebra of $\mathbf{A}$ such that $S \cap D = \{1\}$. Indeed, S is an increasing subset of A closed under g and h, and consequently, $\mathbf{S}$ is a subalgebra of $\mathbf{A}$. If $z \in S \cap D$, $z \neq 1$, then $z = \bigwedge_{i \in I} g^i a$, for $I \subseteq \{1,\ldots,k\}$, and thus $z \leqslant g^i a$ for all $i \in I$. As a consequence, $g^i a \in D$, a contradiction.

Now we are going to prove that $\mathbf{A}/D \simeq \mathbf{S}$. First observe that if $x, y \in S$ and $\overline{x} = \overline{y}$ in $\mathbf{A}/D$, then $x \to y$, $y \to x \in D \cap S = \{1\}$, so $x = y$.

If $x \notin S$ and $x \notin D$, let $m = \wedge\{y \in S : x \leqslant y\}$. From $x \notin D$ there exists an antiatom b of $\mathbf{A}$ not belonging to D such that $x \leqslant b = g^s a$ for some s. Since $g^s a \in S$, it follows that $m \neq 1$.

Let us see that $\overline{x} = \overline{m}$. It is clear that $x \leqslant m$, and then, $x \to m = 1 \in D$. If we suppose that $m \to x \notin D$, then $m \to x \leqslant g^i a$ for some i. Then $m \leqslant g^i a \to m \leqslant (m \to x) \to m = m$, that is, $m = g^i a \to m$. Hence $g^i a = (g^i a \to m) \to g^i a = m \to g^i a$, that is, $g^i a = m \to g^i a$. The proof finishes if we prove that $m \to g^i a = 1$, since in that case we would have $g^i a = 1$, a contradiction.

Let us prove then that $m \to g^i a = 1$. Observe that, since $m \in S$ and $m \neq 1$, $m \notin D$. So there exists an antiatom of $\mathbf{A}$ of the form $g^j a$ such that $m \leqslant g^j a$. Since $x \leqslant m \leqslant g^j a$ and $x \leqslant m \to x \leqslant g^i a$, x is a lower bound of $\{g^i a, g^j a\}$ and consequently, the element $g^i a \wedge g^j a$ exists and belongs to S. By the definition of m, we have that $m \leqslant g^i a \wedge g^j a$. In particular $m \leqslant g^i a$, that is, $m \to g^i a = 1$. ∎

Corollary 4.2. *Every simple homomorphic image of a finite algebra $\mathbf{A}$ is a retract of $\mathbf{A}$.*

If K is a class of algebras, we denote by $Q(K)$ the quasivariety generated by K. Every variety is a quasivariety.

A *critical algebra* is a finite algebra not belonging to the quasivariety generated by its proper subalgebras.

THEOREM 4.3. (See [7]) *Every non-trivial locally finite quasivariety is generated by its critical algebras.*

THEOREM 4.4. *The set of critical algebras of $V(\mathbf{B}_k)$ coincides with its set of simple algebras.*

PROOF: Observe first that every simple finite algebra is critical. Suppose now that $\mathbf{A}$ is a finite algebra that is not simple. Then the set $\{D_i\}_{j=1}^{n}$ of non-trivial gh-deductive systems of $\mathbf{A}$ is not empty. Let $i : \mathbf{A} \to \prod_{j=1}^{n} \mathbf{A}/D_i$ be the subdirect representation of $\mathbf{A}$. By Lemma 4.1, for each j, there exists a proper subalgebra $\mathbf{S}_j$ of $\mathbf{A}$ such that $\mathbf{S}_j \cong \mathbf{A}/D_j$. Thus $\mathbf{A} \in ISP(\{\mathbf{S}_j\}_{j=1}^{n})$ and consequently, $\mathbf{A}$ is not critical. $\square$

REMARK If $\mathbf{A} \in \mathcal{T}$ is a finite simple algebra, then $Q(\mathbf{A}) = V(\mathbf{A})$. Indeed, as $\mathbf{A}$ is finite, the subdirectly irreducible algebras of $V(\mathbf{A})$ are in $HS(\mathbf{A})$. Since $\mathbf{A}$ is simple, every subalgebra of $\mathbf{A}$ is simple and thus $HS(\mathbf{A}) = S(\mathbf{A})$. If $\mathbf{B} \in V(\mathbf{A})$ and K is the set of subdirectly irreducible algebras of $V(\mathbf{A})$, then

$$\mathbf{B} \in SP(K) \subseteq SPS(\mathbf{A}) \subseteq Q(\mathbf{A}).$$

COROLLARY 4.5. *For a finite k-cyclic Tarski algebra $\mathbf{A}$,*

$$Q(\mathbf{A}) = V(\mathbf{A}) = V(\mathbf{A}_1, \dots, \mathbf{A}_n),$$

where $\mathbf{A}_1, \dots, \mathbf{A}_n$ are the simple homomorphic images of $\mathbf{A}$.

PROOF: Let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be the simple homomorphic images of $\mathbf{A}$ and $D_1, D_2, \dots, D_n$ the maximal gh-deductive systems of $\mathbf{A}$ such that $\mathbf{A}_i = \mathbf{A}/D_i$. Then $Q(\mathbf{A}_i) = V(\mathbf{A}_i)$ and so,

$$Q(\mathbf{A}_1, \dots, \mathbf{A}_n) = \bigvee_{i=1}^{n} Q(\mathbf{A}_i) = \bigvee_{i=1}^{n} V(\mathbf{A}_i) = V(\mathbf{A}_1, \dots, \mathbf{A}_n) = V(\mathbf{A}).$$

In order to see that $Q(\mathbf{A}) = Q(\mathbf{A}_1, \dots, \mathbf{A}_n)$, consider $i : \mathbf{A} \hookrightarrow \prod_{i=1}^{n} \mathbf{A}/D_i = \prod_{i=1}^{n} \mathbf{A}_i$. Then $\mathbf{A} \in SP(\mathbf{A}_1, \dots, \mathbf{A}_n) \subseteq Q(\mathbf{A}_1, \dots, \mathbf{A}_n)$, that is, $Q(\mathbf{A}) \subseteq Q(\mathbf{A}_1, \dots, \mathbf{A}_n)$. On the other hand, by Lemma 4.1, $\mathbf{A}_i \in S(\mathbf{A}) \subseteq Q(\mathbf{A})$, so $Q(\mathbf{A}_1, \dots, \mathbf{A}_n) \subseteq Q(\mathbf{A})$. $\square$

From Theorem 4.4 and Corollary 4.5 we have the following result:

THEOREM 4.6. *The subvarieties and the subquasivarieties of a locally finite subvariety of $\mathcal{T}$ coincide.*

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REFERENCES

[1] M. Abad and J.P. Díaz Varela, 'Representation of cubic lattices by symmetric implication algebras' (to appear).

[2] M. Abad, J.P. Díaz Varela and M. Zander, 'Boolean algebras with a distinguished automorphism', *Rep. Math. Logic* **37** (2003), 101–112.

[3] M. Abad, J.P. Díaz Varela and M. Zander, 'Varieties and quasivarieties of monadic Tarski algebras', *Sci. Math. Jpn.* **56** (2002), 599–612.

[4] J.C. Abbott, 'Semi-boolean algebras', *Mat. Vesnik* **19** (1967), 177–198.

[5] J.C. Abbott, 'Implicational algebras', *Bull. Math. Soc. Sci. Math R. S. Roumaine* **11** (1967), 3-23.

[6] G. Bezhanishvili, 'Locally finite varieties', *Algebra Universalis* **46** (2001), 531–548.

[7] J. Gispert and A. Torrens, 'Locally finite quasivarieties of MV-algebras', (preprint).

[8] T. Kowalski, 'Varieties of tense algebras', *Rep. Math. Logic* **32** (1998), 53–95.

[9] R. McKenzie, 'Equational bases and nonmodular lattice varieties', *Trans. Amer. Math. Soc.* **174** (1972), 1–43.

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