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LOGICAL ASPECTS OF COMBINATORIAL DUALITY

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ABSTRACT. D. R. Woodall has introduced closely-related notions of Menger and König duals which can be applied to a broad range of combinatorial contexts. The present paper considers these two notions for finite ground sets in terms of syntactic duality principles. Specific graph-theoretic interpretations are cited.

0. **Introduction.** The notions of Menger and König duals were introduced by D. R. Woodall in [10] and [11] as points of departure for a number of well-known results including Menger's theorem from graph theory, the Ford–Fulkerson max-flow min-cut theorem for network flows, and the duality theorem of linear programming. In particular, Menger duality generalizes matroid duality, which itself generalizes the familiar cycle/cocycle duality of graph theory. We shall view these dualities in terms of syntactic duality principles based on quantifiers appropriate to the combinatorial contexts.

Suppose S is a given set of elements, with a subset of S being called *nontrivial* whenever it contains at least two elements. A nonempty family $\mathcal{F}$ of subsets (called *blocks* of $\mathcal{F}$) will be called a *set system* whenever both the following hold.

(0.1) No block of $\mathcal{F}$ properly contains another block of $\mathcal{F}$.

(0.2) All blocks of $\mathcal{F}$ are nontrivial.

The first axiom makes $\mathcal{F}$ a "clutter" (or "Sperner family"). The second is nonstandard, but serves to simplify the subject from our point of view and removes only very special or trivial cases.

A set system $\mathcal{F}'$ is the *Menger dual* of $\mathcal{F}$ whenever its members are precisely the minimal nontrivial subsets of S having at least one element in common with each block of $\mathcal{F}$. Similarly, a set system $\mathcal{F}'$ is the *König dual* of $\mathcal{F}$ whenever its members are precisely the maximal nontrivial subsets of S having at most one element in common with each block of $\mathcal{F}$.

As a simple, but not too combinatorial, example, let S be the point set of the Cartesian plane $\mathbf{R}\times\mathbf{R}$, $\mathcal{F}$ the family of all vertical lines (viewed as subsets of

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S), and $\mathscr{F}'$ the family of all functions having domain $\mathbf{R}$. Then $\mathscr{F}$ and $\mathscr{F}'$ are each both Menger and König duals of each other.

In the following sections we shall restrict our attention to finite sets S and develop a more significant graph-theoretic example which illustrates practical uses of our logical orientation.

1. **Menger duality.** We shall use lower-case variables $(x, y, \ldots)$ and constants $(a, b, \ldots)$ for elements of S and upper-case variables $(X, Y, \ldots)$ and constants $(A, B, \ldots)$ for blocks of set systems. In addition to the usual logical connectives $\vee$ for disjunction and $\neg$ for negation, and the universal $(\forall)$ and existential $(\exists)$ quantifiers, we shall freely use informal abbreviations such as $(\forall x \in B)$ and $(\exists X \in \mathscr{F})$. We shall let an element predicate (that is, a property of elements) such as $P(x)$ correspond to a subset P of S; thus $P(x)$ and $x \in P$ will be used interchangeably. For instance, $(\exists X \in \mathscr{F}')(\forall x \in X)P(x)$ will mean that each element of some block belongs to P.

THEOREM 1. *$\mathscr{F}'$ is the Menger dual of $\mathscr{F}$ if and only if the following equivalence holds for all predicates P:*

$$(1.1) \qquad (\exists X \in \mathscr{F}')(\forall x \in X)P(x) \quad \textit{iff} \quad (\forall X \in \mathscr{F})(\exists x \in X)P(x).$$

Proof. First assume $\mathscr{F}'$ is the Menger dual of $\mathscr{F}$. The left side of (1.1) implies the existence of $B \in \mathscr{F}'$ such that $B \subset P$. By assumption, B must meet each block of $\mathscr{F}$ at least once, and so P must also, as required by the right side. Conversely, since S is assumed to be finite, the right side of (1.1) together with the assumed Menger duality forces P to contain some block of $\mathscr{F}'$, as asserted by the left side.

Now suppose (1.1) and $B \in \mathscr{F}'$, towards showing that $\mathscr{F}'$ is the Menger dual of $\mathscr{F}$. So $(\exists X \in \mathscr{F}')(\forall x \in X)(x \in B)$, and so by (1.1) B meets each block of $\mathscr{F}$ at least once. To show B's minimality, suppose $B' \subset B$ such that B' meets each block of $\mathscr{F}$ at least once. So $(\forall X \in \mathscr{F})(\exists x \in X)(x \in B')$, and so by (1.1) there exists some block B'' of $\mathscr{F}'$ such that $B'' \subset B'$. Since $B'' \subset B' \subset B$, (0.1) implies $B' = B$. Thus $\mathscr{F}'$ is a subset of the Menger dual of $\mathscr{F}$. Suppose B is any block in the Menger dual of $\mathscr{F}$, so $(\forall X \in \mathscr{F})(\exists x \in X)(x \in B)$. By (1.1), there exists some block B' of $\mathscr{F}'$ (and so of the Menger dual) such that $B' \subset B$. By minimality, $B = B'$ and so B is in $\mathscr{F}'$; thus $\mathscr{F}'$ is the Menger dual.

Notice that we have proved something stronger than Theorem 1: The "only if" (=left-to-right) implication in (1.1) corresponds to the "at least one" portion of the definition of Menger duality; the "if" implication, to the "minimal" portion.

The importance of (1.1) as a duality principle is best shown by a graph-theoretic example. Consider a two-port connected multigraph, the "ports" being distinguished source and sink vertices. The edge set is taken as S, with the blocks of $\mathscr{F}$ being *tiesets*—minimal paths connecting the source to the sink—and the blocks of $\mathscr{F}'$ being *cutsets*—minimal sets of edges whose removal would disconnect the source from the sink. In accord with axiom (0.2), we forbid tiesets or cutsets having only one edge. The set systems $\mathscr{F}$ and $\mathscr{F}'$ are Menger duals of each other, and so (1.1) is applicable.

If a distinguished source-sink edge is added, tiesets and cutsets become (respectively) cycles and cocycles. This cycle/cocycle duality is studied in [4] using (1.1) which is shown to correspond to this duality in so far as, as is sometimes claimed, graph-theoretic duality is explained by matroid duality. This correspondence rests on a self-dual axiomatization of matroids quoted in [2, page 41]: axiom 2 there corresponds to our (0.1), while the two very different axioms 1 and 3 there correspond to the two converse implications which make up (1.1). (See [4] for details and [7] for a strengthened duality principle.) Duality principles such as (1.1) allow statements which have been suitably expressed in terms of tiesets to be equivalently stated in terms of cutsets. When recast as an exclusive disjunction, equivalence (1.1), now under the name of "Minty's theorem", occurs repeatedly in combinatorial mathematics and its applications; for instance, [9] describes its role in circuit theory, and [5] and [6] describe additional interpretations.

It is natural to think of $(\exists X \in \mathscr{F}')(\forall x \in X)P(x)$ as determining a quantifier on $P(x)$, asserting that $P(x)$ holds "for all x in some block of $\mathscr{F}'$." We shall abbreviate the left side of (1.1) as $(\supset Q'x)P(x)$, a generalized quantifier (in the sense of [3, pages 100–101]) corresponding to the family $\supset \mathscr{F}'$ of all supersets of blocks of $\mathscr{F}'$; $(\supset Qx)P(x)$ will correspond to $\supset \mathscr{F}$, defined similarly. Equivalence (1.1) can now be expressed as

$$(\supset Q'x)P(x) \quad \text{iff} \quad \neg(\supset Qx)\neg P(x),$$

and so can be viewed as asserting that the quantifier $\supset Q$ is the logical dual of $\supset Q'$. (This connection with Boolean duality is also mentioned, in different terms, in [1, page 303].)

2. **König duality.** We begin by reexamining the traditional universal and existential quantifiers when restricted to subsets of S. Each can be viewed as containing an unstated "at least": $(\forall x \in B)P(x)$ means P holds "for at least all $x \in B$" and possibly x outside of B as well; $(\exists x \in B)P(x)$ means P holds "for at least one $x \in B$". Replacing "at least" with "at most" produces a useful pair of *converse quantifiers*. We shall use $(\subseteq \forall x \in B)P(x)$ to mean that P holds "for at most all x in B"; that is, P holds "for no x outside of B"; that is, $P \subset B$. Similarly, $(\subseteq \exists x \in B)P(x)$ will mean that P holds "for at most one $x \in B$"; that is, that the cardinality of $P \cap B$ is at most one.

THEOREM 2. *$\mathscr{F}'$ is the König dual of $\mathscr{F}$ if and only if the following equivalence holds for all predicates P:*

(2.1) $$(\exists X \in \mathscr{F}')(\subseteq \forall x \in X)P(x) \quad \text{iff} \quad (\forall X \in \mathscr{F})(\subseteq \exists x \in X)P(x).$$

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We omit the proof, which is almost the same as for Theorem 1, except for noting that the "only if" implication in (2.1) corresponds to the "at most one" portion of the definition of König duality; the "if" implication, to the "maximal" portion.

König duality differs from Menger duality in a very major way. Taking the contrapositive of (1.1) produces the equivalent principle

$$(\exists X \in \mathscr{F})(\forall x \in X)P(x) \quad \text{iff} \quad (\forall X \in \mathscr{F}')(\exists x \in X)P(x).$$

Thus $\mathscr{F}'$ being the Menger dual of $\mathscr{F}$ is equivalent to each being the Menger dual of the other—i.e., ([10, Prop. 1], [11, Thm 7.1]) Menger duality is symmetric. But not so with König duality. The alternative formulation

$$(2.2) \qquad (\exists X \in \mathscr{F})(\subseteq\forall x \in X)P(x) \quad \text{iff} \quad (\forall X \in \mathscr{F}')(\subseteq\exists x \in X)P(x)$$

is not equivalent to (2.1) without an additional assumption. Equivalence (2.2) is not the contrapositive of (2.1) because negation does not behave as simply on $(\subseteq\forall x \in X)$ and $(\subseteq\exists x \in X)$ as it does on $(\forall x \in X)$ and $(\exists x \in X)$; in other words, these converse quantifiers are not logical duals of each other. The additional assumption needed is that $\mathscr{F}$ and $\mathscr{F}'$ are *conformal* (called "clique-complete" in [11]), where a set system is conformal whenever, for each subset P of S, the containment of each pair of elements of P in a common block implies that the entire set P is contained in a block.

We can illustrate the use of (2.1) to prove the following observation of [10] and [11]: If $\mathscr{F}'$ is the König dual of $\mathscr{F}$, then $\mathscr{F}'$ is conformal. To see this, suppose, for every $a, b \in P$, that $(\exists X \in \mathscr{F}')(\subseteq\forall x \in X)(x = a \vee x = b)$. Equivalence (2.1) then implies $(\forall X \in \mathscr{F})(\subseteq\exists x \in X)(x = a \vee x = b)$ for all $a, b \in P$. So $(\forall X \in \mathscr{F})(\subseteq\exists x \in X)P(x)$ and so by (2.1) again $(\exists X \in \mathscr{F}')(\subseteq\forall x \in X)P(x)$; that is, P is contained in a block of $\mathscr{F}'$.

THEOREM 3 [10, Prop. 2], [11, Thm 7.2]. *If $\mathscr{F}'$ is the König dual of $\mathscr{F}$ and $\mathscr{F}$ is conformal, then $\mathscr{F}$ is the König dual of $\mathscr{F}'$.*

Proof. Suppose $\mathscr{F}$ is conformal, (2.1) holds for all predicates P, and a particular P is given, towards proving (2.2). Let a and b be arbitrary elements of P (noting that (2.2) is trivial if P has fewer than two elements). Then (2.1) implies

$$(\exists X \in \mathscr{F}')(\subseteq\forall x \in X)(x = a \vee x = b) \quad \text{iff} \quad (\forall X \in \mathscr{F})(\subseteq\exists x \in X)(x = a \vee x = b).$$

Negating both sides and paying careful attention to the meaning of $(\subseteq\forall x \in X)$ and $(\subseteq\exists x \in X)$ shows

$$(\forall X \in \mathscr{F}')(\subseteq\exists x \in X)(x = a \vee x = b) \quad \text{iff} \quad (\exists X \in \mathscr{F})(\subseteq\forall x \in X)(x = a \vee x = b).$$

This formulation is easily shown to be equivalent to (2.2), using the conformality of $\mathscr{F}$ for the "if" implication of (2.2).

The key point in this argument is that, when P has cardinality two, each of $(\subseteq\forall x \in X)P(x)$ and $(\subseteq\exists x \in X)P(x)$ is the negation of the other. So while they are not logical duals, they are nicely related for such near-trivial P. This observation leads to the following.

COROLLARY. *Two conformal set systems are König duals of each other if and only if every pair of elements is in a common block of exactly one of the two set systems.*

While Woodall (with a more restricted objective) observes [11, page 259] that in practice, all the most interesting examples tend to be conformal, the families of tiesets and cutsets in our graph-theoretic example fail to be. But adding the assumption that every pair of edges is in either a common tieset or cutset, but not both, causes these families to become conformal and, indeed, each to be simultaneously the Menger and König dual of the other. Within the more general cycle/cocycle context mentioned in Section 1, this assumption corresponds to the multigraph being series-parallel and nonseparable; see [8] for details. In fact, these two important graph-theoretic concepts correspond to the "only if" and "if" implications of (2.1) respectively, thereby showing that series-parallel and nonseparable are converses of each other in this very dual-like fashion.

Since there are natural examples of joint Menger/König duality, it is natural to seek a simple duality principle corresponding to the conjunction of (1.1) and (2.1). One possibility would be to replace each "at most" in (2.1) with "exactly", producing, in effect

$$(2.3) \qquad P \in \mathcal{F}' \quad \text{iff} \quad (\forall X \in \mathcal{F})(\exists!\, x \in X)P(x).$$

But while (2.3) follows simply from (1.1) and (2.1) (and so does hold in all examples of joint Menger/König duality), and even though its "only if" implication implies the "only if" implications of (1.1) and (2.1), it is easy to find examples satisfying (2.3) without being Menger duals.

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