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WHITEHEAD PRODUCTS IN THE COMPLEX STIEFEL MANIFOLDS

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1. Introduction

The complex Stiefel manifold $W_{n,k}$, where $n \geqq k \geqq 1$, is a space whose points are k-frames in C^n. By using the formula of McCarty [**4**], we will make the calculations of the Whitehead products in the groups $\pi_*(W_{n,k})$. The case of real and quaternionic will be treated by Nomura and Furukawa [**7**]. The product $[[\eta], j_1 \iota]$ appears as generator of the isotropy group of the identity map of Stiefel manifolds. In this note we use freely the results of the 2-components of the homotopy groups of real and complex Stiefel manifolds such as Paechter [**8**], Hoo-Mahowald [**1**], Nomura [**5**], Sigrist [**9**] and Nomura–Furukawa [**6**].

2. Preliminaries

We identify $W_{n,k}$ with $SU(n)/SU(n-k)$ in the usual way, and consider the boundary homomorphism $\Delta: \pi_{r+1}(W_{n,k}) \to \pi_r(SU(n-k))$ in the homotopy sequence of a fibration $SU(n) \to W_{n,k}$. A homomorphism P_k^c is defined by requiring that the diagram

$$\begin{array}{ccc} \pi_{r+1}(W_{n,k}) & \xrightarrow{P_k^c} & \pi_{r+2n-2k+1}(S^{2n-2k+1}) \\ & {\scriptstyle\Delta}\searrow \quad \nearrow{\scriptstyle J_c} & \\ & \pi_r(SU(n-k)) & \end{array}$$

commutes where J_c is given by the McCarty product [**4**], $J_c(\gamma) = \langle \gamma, \iota_{2n-2k+1} \rangle$ for $\gamma \in \pi_r(SU(n-k))$. Since the inclusion $j_{k-1}: S^{2n-2k+1} \to W_{n,k}$ is $SU(n-k)$-equivariant, we have

Lemma 1. *Let* $\alpha \in \pi_{r+1}(W_{n,k})$. *Then we have* $[\alpha, j_{k-1}\iota_{2n-2k+1}] = \pm j_{k-1} P_k^c(\alpha) = \pm j_{k-1} E J i \Delta(\alpha)$, *where* $i: SU(n-k) \to R_{2n-2k}$ *is the map into the group of rotations.*

Proof. By McCarty [4], we have

$$\begin{aligned}[\alpha, j_{k-1}\iota_{2n-2k+1}] &= \pm\langle \Delta\alpha, j_{k-1}\iota_{2n-2k+1}\rangle \\ &= \pm j_{k-1}\langle \Delta\alpha, \iota_{2n-2k+1}\rangle \\ &= \pm j_{k-1} P_k^c(\alpha) \\ &= \pm j_{k-1} J_c \Delta(\alpha) \\ &= \pm j_{k-1} EJi\Delta(\alpha).\end{aligned}$$

The following diagram commutes;

$$\begin{array}{ccccccc}
\pi_{r+1}(V_{2n,2k}) & \xrightarrow{i_l} & \pi_{r+1}(V_{2n+l,2k+l}) & & \xrightarrow{p_{2k+l-1}} & & \pi_{r+1}(V_{2n+l,2k+l-1}) \\
{\scriptstyle s_*}\uparrow & & \downarrow & \searrow{\scriptstyle P_{2k+l}} & & & \downarrow \quad \searrow{\scriptstyle P_{2k+l-1}} \\
\pi_{r+1}(W_{n,k}) & & & \pi_{r+2n-2k}^{2n-2k} & \xrightarrow{E^1} & & \pi_{r+2n-2k+1}^{2n-2k+1} \\
{\scriptstyle \Delta}\downarrow & & & \nearrow{\scriptstyle J} & & & \nearrow{\scriptstyle J} \\
\pi_r(SU(n-k)) & \xrightarrow{i} & \pi_r(R_{2n-2k}) & & \xrightarrow{h} & & \pi_r(R_{2n-2k+1})
\end{array} \tag{1.1}$$

where s, i_l and h are the inclusions and p_{2k+l-1} is the projection. Denote by s_k a cross section of a fibration $W_{n,k} \xrightarrow{q_1} S^{2n-1}$ if it exists and $[\gamma]$ an element of the group $\pi_*(W_{n,k})$ such that $q_{1*}[\gamma]=\gamma$ for an element $\gamma \in \pi_*(S^{2n-1})$. For the real Stiefel fibration $V_{2n,k} \xrightarrow{p_1} S^{2n-1}$, we use the same notations. For $m<2k$, there is the commutative diagram;

$$\begin{array}{ccc}
\pi_{r+1}(W_{n,k}) & \xrightarrow{s_*} & \pi_{r+1}(V_{2n,2k}) \\
{\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_m} \\
\pi_{r+1}(S^{2n-1}) & \xrightarrow[s_m]{} & \pi_{r+1}(V_{2n,m}).
\end{array} \tag{1.2$_m$}$$

Let $\Delta_{k-1}: \pi_{r+1}(W_{n,k-1}) \to \pi_r(S^{2n-2k+1})$ denote the boundary homomorphism in the homotopy sequence of a fibration $S^{2n-2k+1} \xrightarrow{j_{k-1}} W_{n,k} \to W_{n,k-1}$. For Δ_{k-1}, we have the following :

Lemma 2. (i) *Let $\alpha \in \pi_r(S^{2n-2})$. Then $\Delta_1(E\alpha)=0$ for n even, $\Delta_1(E\alpha)=\eta_{2n-3}\alpha$ for n odd.*

(ii) *Let n be even, then $\Delta_2 j_{1*}E\alpha=\eta_{2n-5}\alpha$ for $\alpha \in \pi_r^{2n-4}$, $\Delta_2 s_{2*}E\gamma = \pm b_n(\nu_{2n-5}+\alpha_1)\gamma$ for $\gamma \in \pi_r^{2n-2}$.*

(iii) *Let n be odd, then $\Delta_2 j_{1*}E\alpha=0$ for $\alpha \in \pi_r^{2n-4}$, $\Delta_2\{j_{1*}, \eta, \gamma\} \equiv -a_n(\nu+\alpha_1)\gamma \bmod a_n(2\nu+\alpha_1)\gamma$, where $a_n=(12,(n-3)/2)$, $b_n=(12, n/2)$.*

3. Whitehead products in $\pi_r(W_{n,2})$ for n odd

For the homotopy groups of $W_{n,k}$, see [6] and [9]. Let n be odd. The group $\pi_{2n-1}(W_{n,2})=Z$ is generated by $[2\iota]$ with $q_{1*}[2\iota]=2\iota$.

Theorem 3. *We have* $[[2\iota],j_1\iota_{2n-3}]=0$.

Proof. Let $k=2$, $l=1$ and $r=2n-2$ in (1.1). Since the group $\pi_{2n-1}(V_{2n+1,5})=Z_4$ is generated by $i_4\nu$, we have $p_4\pi_{2n-1}(V_{2n+1,5})=0$. This implies

$$\begin{aligned}[[2\iota],j_1\iota_{2n-3}]&=\pm j_1EJi\Delta[2\iota]\\&=\pm j_1P_4p_4i_1s_*[2\iota]\\&=0.\end{aligned}$$

The 2-component of the group $\pi_{2n+2}(W_{n,2})$ is Z_8 generated by $[\nu]$.

Theorem 4. *We have* $[[\nu],j_1\iota_{2n-3}]=\pm j_1EP_4[\nu]\neq 0$ *for* $n\equiv 1 \bmod 4$, *and* 0 *for* $n\equiv 3 \bmod 4$.

Proof. Let $k=2$, $l=4$ and $r=2n+1$ in (1.1). The group $\pi_{2n+2}(V_{2n,4})=Z_2+Z_8$ is generated by $i_3\nu^2$ and $[\nu]$, so we may set $s_*[\nu]\equiv[\nu] \bmod i_3\nu^2$ by $(1.2)_2$. By Lemma 1, we have

$$\begin{aligned}[[\nu],j_1\iota_{2n-3}]&=\pm j_1P_7p_7i_4s_*[\nu]\\&=\pm j_1EP_4[\nu]\\&=\pm j_1P_3[\nu].\end{aligned}$$

Consider the commutative diagram

$$\begin{array}{ccc}\pi^{2n-2}_{4n-2} & \xrightarrow{\ E\ } & \pi^{2n-1}_{4n-1}\\ \eta_{2n-3*}\big\downarrow & \searrow^{\eta^*_{4n-2}} & \big\downarrow\eta_{2n-2*}\\ \pi^{2n-3}_{4n-2} & \xrightarrow[\ E\ ]{} & \pi^{2n-2}_{4n-1}\end{array}$$

If $n\equiv 3 \bmod 4$, we have

$$E\eta_{2n-3*}P_4s_4\iota_{2n+1}=\eta^*_{4n-2}P_4s_4\iota_{2n+1}=P_4s_4\eta_{2n+1}=EP_5s_5\eta$$

for $P_4s_4\iota_{2n+1}\in\pi^{2n-2}_{4n-2}$. This shows that $\eta_{2n-3*}P_4s_4\iota_{2n+1}=P_5s_5\eta_{2n+1}$, since $E\colon\pi^{2n-3}_{4n-2}\to\pi^{2n-2}_{4n-1}$ is the monomorphism. Since $i_4[\nu]=i_2s_5\eta$ we have

$$P_3[\nu_{2n-1}]=P_5s_5\eta_{2n+1}\in \text{Image}\,\Delta_1.$$

Therefore $[[\nu], j_1 \iota_{2n-3}] = 0$.

If $n \equiv 1 \bmod 4$, we have $0 \neq [\nu, \iota_{2n-1}] = E^3 P_4[\nu]$ and $H_1 P_4[\nu] = \nu^2$ by [3]. This shows $EP_4[\nu] \notin \text{Image}\, E^2$. The unstable parts $Z_8 + Z_{24}$ of the group π_{4n-2}^{2n-2} which is generated by $P_4 s_4 \iota$ and $[\nu, \iota_{2n-2}]$ vanish by η_{4n-2}^*, since $H_5[\iota_{2n+3}, \iota_{2n+3}] = i_1 s_4 \eta$ implies $P_4 s_4 \eta = 0$. Therefore the unstable element $P_3[\nu]$ does not lie in $\eta_{2n-3*}\pi_{4n-2}^{2n-2} = \text{Image}\ \Delta_1$. This shows that

$$[[\nu], j_1 \iota_{2n-3}] = \pm j_1 P_3[\nu] \neq 0.$$

The 2-components of the group $\pi_{2n+8}(W_{n,2})$ is $Z_2 + Z_8$ generated by $[\nu]\nu^2$ and $[\eta\varepsilon]$.

Theorem 5. *We have* $[[\eta\varepsilon], j_1 \iota_{2n-3}] = \pm 4 j_1 P_5[\sigma]$ *for* $n \equiv 1 \bmod 4$ *and* $n \equiv 27 \bmod 64$, $[[\eta\varepsilon], j_1 \iota_{2n-3}] = 0$ *otherwise.*

Proof. Let $k = 2$, $l = 5$ and $r = 2n+7$ in (1.1). The group $\pi_{2n+8}(V_{2n,4}) = Z_2 + Z_4$ is generated by $[\nu]\nu^2$ and $[\eta\varepsilon]$, so we may set $s_*[\eta\varepsilon] = [\eta\varepsilon]$ in $(1.2)_2$. By the result of the groups $\pi_{2n+8}(V_{2n+5,k})$ for $k = 8$ and 9, we have the following:

$$[[\eta\varepsilon], j_1 \iota_{2n-3}] = \pm 4 j_1 P_5 s_5 \sigma \text{ for } n \equiv 3 \bmod 4,$$

$$[[\eta\varepsilon], j_1 \iota_{2n-3}] = \pm 4 j_1 P_5[\sigma] \text{ for } n \equiv 1 \bmod 4.$$

If $n \equiv 3 \bmod 4$, except $n \equiv 27 \bmod 64$, the relations

$$H_{12} P_1 \iota_{2n+9} = \lambda i_7 s_5 \sigma \ (\lambda = 0, 1, 2, 4)$$

lead to $[[\eta\varepsilon], j_1 \iota_{2n-3}] = 0$.

4. Whitehead products in $\pi_r(W_{n,3})$ for n even

Let n be even. The group $\pi_{2n-1}(W_{n,3}) = Z$ is generated by $[(12/b_n)\iota]$.

Theorem 6. *The nontrivial Whitehead product* $[[(12/b_n)\iota], j_2 \iota_{2n-5}]$ *is equal to* $\pm j_2 P_3[\eta_{2n-3}^2]$ *for* $n \equiv 2 \bmod 4$, $\pm j_2 P_4[\eta_{2n-2}]$ *for* $n \equiv 4 \bmod 8$, $\pm j_2 P_5 s_5 \iota_{2n-1}$ *for* $n \equiv 0$ mod 8, *respectively.*

Proof. Let $k = 3$, $l = 1$ and $r = 2n-2$ in (1.1). In $(1.2)_3$ the group $\pi_{2n-1}(V_{2n,6}) = Z_8 + Z$ is generated by $i_1[\eta]$ and $[2\iota]$ (or $s_6 \iota_{2n-1}$ for $n \equiv 0$ mod 4), so we may set

$$s_*[(12/b_n)\iota] \equiv (6/b_n)[2\iota] \bmod 4 i_1[\eta] \quad \text{for} \quad n \equiv 2 \bmod 4,$$

$$s_*[(12/b_n)\iota] \equiv (12/b_n) s_6 \iota_{2n-1} \bmod 4 i_1[\eta] \quad \text{for} \quad n \equiv 0 \bmod 4.$$

If $n \equiv 2 \bmod 4$, the relations $E^2 P_3[\eta_{2n-3}^2] = [\eta^2, \iota_{2n-3}] = E^6 P_7[\eta^2]$, $H_1 P_7[\eta^2] = \varepsilon_{2n-10}$ and $2 i_2[\eta] = -2 i_1[2\iota] = i_3[\eta^2]$ in $\pi_{2n-1}(V_{2n+1,6})$ imply that $P_3[\eta^2]$ is the unstable element $2P_4[\eta]=-2P_5[2\iota]$ of the group $\pi_{4n-7}^{2n-5}=G_{2n-2}+Z_8$, where Z_8 is generated by $P_4[\eta]$ (cf. [10]). Thus

$$[[(12/b_n)\iota],j_2\iota_{2n-5}]=\pm j_2P_6p_6i_{12}[2\iota]$$

$$=\pm j_2P_3[\eta^2]\bmod j_2 2P_3[\eta^2]=0,$$

since $2P_3[\eta^2]\in\operatorname{Im}\Delta_2$.

Now the unstable parts Z_2+Z_2 of the group π_{4n-7}^{2n-4} (cf. [10]) are generated by $P_2s_2\eta_{2n-3}$ and $[\eta^2,\iota_{2n-4}]$. So we have

$$\eta_{2n-5*}\pi_{4n-7}^{2n-4}\not\ni P_3[\eta^2]$$

because of $\eta_{2n-5*}P_2s_2\eta_{2n-3}=P_2[\eta]\eta^2=2P_3[\eta^2]$. Hence $P_3[\eta^2]\notin\operatorname{Image}\Delta_2$. Thus $j_2P_3[\eta^2]\neq 0$.

For the case $n\equiv 4\bmod 8$, the relations $E^3P_4[\eta_{2n-2}]=[\eta,\iota_{2n-2}]=E^7P_8[\eta]$ and $H_1P_8[\eta]=\eta\sigma$ show that the element $P_4[\eta]$ is the unstable element $2P_5s_5\iota_{2n-1}$ of the group $\pi_{4n-7}^{2n-5}=G_{2n-2}+Z_{16}$, where Z_{16} is generated by $P_5s_5\iota_{2n-1}$ (cf. [10]). So

$$\eta_{2n-5*}\pi_{4n-7}^{2n-4}\not\ni P_4[\eta]$$

since $\eta_{2n-5*}P_2s_2\eta_{2n-3}=P_2[\eta]\eta^2=8P_5s_5\iota_{2n-1}$.

An argument similar to the above one shows that $\pm b_n\nu_{2n-5*}\pi_{4n-7}^{2n-2}\not\ni P_4[\eta]$. Therefore we may conclude that

$$[[(12/b_n)\iota],j_2\iota_{2n-5}]=\pm j_2P_6p_6i_1 2s_6\iota$$

$$=\pm j_2P_6p_6i_2[\eta]$$

$$=\pm j_2P_4[\eta]$$

$$\neq 0\bmod 4j_2P_4[\eta]=0,$$

since $\eta_{2n-5}\pi_{4n-7}^{2n-4}\ni 4P_4[\eta]$.

If $n\equiv 0\bmod 8$, the relations

$$\eta_{2n-5*}P_2s_2\eta_{2n-3}=P_2[\eta]\eta^2=4P_4[\eta]=8P_5s_5\iota_{2n-1}$$

imply $P_5s_5\iota_{2n-1}\notin\eta_{2n-5*}\pi_{4n-7}^{2n-4}$. Assume that $P_5s_5\iota\in b_n\nu_{2n-5}\pi_{4n-7}^{2n-2}$. Then we have a contradiction

$$0\neq P_5s_5\eta=\eta_{4n-7}^{*}P_5s_5\iota\in\eta_{4n-7}^{*}b_n\nu_{2n-5}\pi_{4n-7}^{2n-2}=0.$$

Hence

$$[[(12/b_n)\iota], j_2\iota_{2n-5}] = \pm j_2 P_6 p_6 i_1 s_6 \iota$$
$$= \pm j_2 P_5 s_5 \iota$$
$$\neq 0 \bmod 4 j_2 P_4[\eta] = 0.$$

We know that the 2 components of the group $\pi_{2n}(W_{n,3})$ is Z_2+Z_8, where Z_2 is generated by $[\eta]$.

Theorem 7. $[[\eta], j_2\iota_{2n-5}] = 0$.

Proof. Let $k=3$, $l=1$ and $r=2n-1$ in (1.1). In the case $n\equiv 0 \bmod 4$, the group $\pi_{2n}(V_{2n,6}) = Z_2+Z_2+Z_2+Z_2$ is generated by $i_5 v^2$, $i_2[v], i_1[\eta]\eta$ and $s_6\eta_{2n-1}$, so we may set

$$s_*[\eta] \equiv s_6 \eta_{2n-1} \bmod i_5 v^2.$$

In the following diagram, E is monomorphism

$$\begin{array}{ccc} \pi^{2n-4}_{4n-6} & & \\ {\scriptstyle \eta_{2n-5*}}\downarrow & \searrow {\scriptstyle \eta^*_{4n-6}} & \\ \pi^{2n-5}_{4n-6} & \xrightarrow[E]{} & \pi^{2n-4}_{4n-5} \end{array}$$

so the relations

$$E\eta_{2n-5*} P_4 s_4 \iota_{2n-1} = \eta^*_{4n-6} P_4 s_4 \iota_{2n-1} = P_4 s_4 \eta_{2n-1} = E P_5 s_5 \eta_{2n-1}$$

imply $P_5 s_5 \eta_{2n-1} \in \eta_{2n-5*}\pi^{2n-4}_{4n-6}$. Thus

$$[[\eta], j_2\iota_{2n-5}] = \pm j_2 P_6 p_6 i_1 s_6 \eta$$
$$= \pm j_2 P_5 s_5 \eta$$
$$= 0 \bmod 0.$$

If $n \equiv 2 \bmod 4$ the group $\pi_{2n}(V_{2n,6}) = Z_4+Z_2+Z_2$ is generated by $i_2[v], i_1[\eta]\eta$ and $[\eta]$, so we may set

$$s_*[\eta] \equiv [\eta] \bmod i_5 v^2.$$

Because of $i_1[\eta] = H_6 P_1 \iota_{2n+1}$ in $\pi_{2n}(V_{2n+1,6})$, the relations $P_5[\eta_{2n-1}] = P_6 i_1[\eta] = 0$ hold. Therefore

$$[[\eta], j_2\iota_{2n-5}] = \pm j_2 P_5[\eta] = 0.$$

We know that the group $\pi_{2n+2}(W_{n,3})=Z_{4(b_n,2)}+Z_{16}$, where the first component is generated by $[(2/(b_n,2))v]$.

Theorem 8. *We have* $[[(2/(b_n,2))v],j_2\iota_{2n-5}]=0$, *where the indeterminacy is* $j_2[\sigma,\iota_{2n-5}]$ *when* $n\not\equiv 2 \bmod 8$.

Proof. Let $k=3$, $l=1$ and $r=2n+1$ in (1.1). In the case $n\equiv 2 \bmod 4$, the group $\pi_{2n+2}(V_{2n,6})=Z_4+Z_4+Z_4$ is generated by $i_4s_2\sigma, i_3[v^2]$ and $[2\iota]v$, so we may set

$$s_*[2v]\equiv[2\iota]v \bmod i_4s_2\sigma, 2i_3[v^2]$$

by $(1.2)_4$. Since $EP_2[v_{2n-4}^2]=[v^2,\iota_{2n-4}]=E^3P_4[v^2]$ and $H_1P_4[v^2]=v^3$ hold, $P_2[v^2]$ is the unstable element $P_5[2\iota]v$ of the group π_{4n-4}^{2n-5}. By

$$\begin{aligned}\pm b_n v_{2n-5*}\pi_{4n-4}^{2n-2} &\supset \pm b_n v_{2n-5*}E^3\pi_{4n-7}^{2n-5}\\ &= \pm b_n v_{4n-7}^*\pi_{4n-7}^{2n-5}\\ &\ni v_{4n-7}^*P_5[2\iota]\\ &= P_5[2\iota]v,\end{aligned}$$

we have $P_2[v_{2n-4}^2]=P_5[2\iota]v\in \operatorname{Image}\Delta_2$. Thus

$$\begin{aligned}[[2v],j_2\iota_{2n-5}] &\equiv \pm j_2P_6p_6i_1[2\iota]v \bmod j_2P_6p_6i_5s_2\sigma\\ &= \pm j_2P_6p_6i_4[v^2]\\ &= \pm j_2P_2[v^2]\\ &\equiv 0 \bmod j_2[\sigma,\iota_{2n-5}].\end{aligned}$$

Especially in the case $n\equiv 2 \bmod 8$, we have $[\sigma,\iota_{2n-5}]=0$.

Let $n\equiv 0 \bmod 4$. Since the 2 components of the group $\pi_{2n+2}(V_{2n,6})$ is $Z_4+Z_4+Z_8$ generated by $i_4s_2\sigma, i_3[v^2]$ and s_6v, we have

$$s_*[v]\equiv s_6v \bmod i_4s_2\sigma, 2i_3[v^2].$$

If $n\equiv 4 \bmod 8$ we have

$$\begin{aligned}[[v],j_2\iota_{2n-5}] &= \pm j_2P_5s_5v_{2n-1}\\ &\equiv 0 \bmod j_2[\sigma,\iota_{2n-5}],\end{aligned}$$

since $H_8P_1\iota_{2n+3}=i_3s_5\nu$ implies $P_5s_5\nu=P_8i_3s_5\nu=0$. If $n\equiv 0 \bmod 8$, $n\geqq 24$, the relations $i_8\sigma=i_4s_5\nu$ and $[\sigma,\iota_{2n-5}]\neq 0$ (cf. [3]) imply

$$0\neq P_5s_5\nu_{2n-1}=[\sigma,\iota_{2n-5}]\notin b_n\nu_{2n-5*}\pi_{4n-4}^{2n-2}.$$

Hence

$$[[\nu],j_2\iota_{2n-5}]=\pm j_2P_5s_5\nu_{2n-1}$$

$$\equiv 0 \bmod j_2[\sigma,\iota_{2n-5}].$$

Let $(l{:}m)=l/(l,m)$, where (l,m) is the greatest common measure of the integers l and m. We have $\pi_{2n+6}(W_{n,3})=Z_{16(4:b_n)}+Z_{2(4:b_n)}+Z_2$, where $Z_{16(4:b_n)}$ is generated by $[\sigma]$.

Theorem 9. *We have* $[[\sigma],j_2\iota_{2n-5}]=0$ *for* $n\equiv 0$, $6 \bmod 8$, $[[\sigma],j_2\iota_{2n-5}]=\pm j_2P_5s_5\sigma_{2n-1}\neq 0$ *for* $n\equiv 4 \bmod 8$ (*the indeterminacy is* $j_2[\zeta,\iota_{2n-5}]$ *when* $n\equiv 60 \bmod 64$), $[[\sigma],j_2\iota_{2n-5}]=\pm j_2P_5[\sigma_{2n-1}]\neq 0$ *for* $n\equiv 2 \bmod 8$.

Proof. Let $k=3$, $l=3$ and $r=2n+5$ in (1.1). For the case $n\equiv 0 \bmod 4$, the group $\pi_{2n+6}(V_{2n,6})=Z_4+Z_8+Z_{16}$ is generated by $i_1[\bar{\nu}]$, $i_1[\eta]\sigma$ and $s_6\sigma$, so we may set $s_*[\sigma]=s_6\sigma$.

If $n\equiv 0 \bmod 8$ we have

$$[[\sigma],j_2\iota_{2n-5}]=\pm j_2P_5s_5\sigma_{2n-1}$$

$$\equiv 0 \bmod j_2[\zeta,\iota_{2n-5}]=0,$$

since $H_{12}P_1\iota_{2n+7}=i_7s_5\sigma$ implies $P_5s_5\sigma_{2n-1}=0$ and $[\zeta,\iota_{2n-5}]=8P_5s_5\sigma=0$.

If $n\equiv 4 \bmod 8$, we have

$$[[\sigma],j_2\iota_{2n-5}]=\pm j_2P_5s_5\sigma_{2n-1} \bmod j_2[\zeta,\iota_{2n-5}]=0$$

by $[\zeta,\iota_{2n-5}]=8P_5s_5\sigma=0$ (except $[\zeta,\iota_{2n-5}]\neq 0$ when $n\equiv 60 \bmod 64$). Since $E^4P_5s_5\sigma_{2n-1}=[\sigma,\iota_{2n-1}]=E^7P_8s_8\sigma$ and $H_1P_8s_8\sigma=\sigma^2$ hold, $P_5s_5\sigma$ is the unstable element.

Assume that $P_5s_5\sigma_{2n-1}\in\pm b_n\nu_{2n-5*}\pi_{4n}^{2n-2}$, then we have

$$EP_5s_5\sigma_{2n-1}\in\pm b_n\nu_{2n-4*}E\pi_{4n}^{2n-2}=\pm b_n\nu_{2n-4*}\pi_{4n+1}^{2n-1}$$

by the isomorphism $\pi_{4n}^{2n-2}\xrightarrow{E}\pi_{4n+1}^{2n-1}$. By the map $E^3\colon \pi_{4n+1}^{2n-1}\to\pi_{4n+4}^{2n-1}$, we have

$$0\neq[\sigma,\iota_{2n-1}]=E^4P_5s_5\sigma_{2n-1}$$

$$\in\pm E^3(b_n\nu_{2n-4*}\pi_{4n+1}^{2n-1})$$

$$=\pm b_n\nu_{4n+1}^{*}\pi_{4n+1}^{2n-1}.$$

Here the unstable part Z_8 of the group π_{4n+1}^{2n-1} generated by $P_4[\eta]$ (cf. [10]) vanishes by $b_n\nu_{4n+1}^*$, since $\pm b_n\nu_{4n+1}^*P_4[\eta]=\pm P_4(b_n[\eta]\nu)$ and $[\eta]\nu\in\pi_{2n+6}(V_{2n+3,4})=Z_2+Z_2$. This is a contradiction.

Similarly assume $P_5s_5\sigma_{2n-1}\in\eta_{2n-5*}\pi_{4n}^{2n-4}$. This leads to a contradiction as follows;

$$0\neq E^7P_8[\sigma]\in E^4(\eta_{2n-5*}\pi_{4n}^{2n-4})=\eta_{4n+3}^*E^3\pi_{4n}^{2n-4},$$

where the group π_{4n+3}^{2n-1} is stable. These show $P_5s_5\sigma_{2n-1}\notin\text{Image}\,\Delta_2$, therefore $j_2P_5s_5\sigma_{2n-1}\neq 0$.

In the case $n\equiv 2 \bmod 4$, the group $\pi_{2n+6}(V_{2n,6})=Z_4+Z_4+Z_{32}$ is generated by $i_1[\bar{\nu}], i_1[\eta]\sigma-4[\sigma]$ and $[\sigma]$, so we have

$$s_*[\sigma]\equiv[\sigma] \bmod i_4s_2\zeta.$$

If $n\equiv 6 \bmod 8$, we have

$$[[\sigma], j_2\iota_{2n-5}]=\pm j_2P_5[\sigma_{2n-1}]=0$$

because the relation $H_{12}P_1\iota_{2n+7}=i_7[\sigma]+i_3s_9\nu$ implies

$$\begin{aligned} P_5[\sigma] &= -P_9s_9\nu_{2n+3} \\ &\in \pm b_n\nu_{4n-3}^*\pi_{4n-3}^{2n-5} \\ &\subset \pm b_n\nu_{2n-5*}\pi_{4n}^{2n-2}. \end{aligned}$$

If $n\equiv 2 \bmod 8$, we have

$$[[\sigma], j_2\iota_{2n-5}]=\pm j_2P_5[\sigma_{2n-1}].$$

Assume $P_5[\sigma_{2n-1}]\in\pm b_n\nu_{2n-5*}\pi_{4n}^{2n-2}$, then we have

$$\begin{aligned} EP_5[\sigma_{2n-1}] &\in \pm b_n\nu_{2n-4*}E\pi_{4n}^{2n-2} \\ &= \pm b_n\nu_{2n-4*}\pi_{4n+1}^{2n-1} \end{aligned}$$

by the isomorphism $E:\pi_{4n}^{2n-2}\to\pi_{4n+1}^{2n-1}$. Here the unstable part Z_{16} of the group π_{4n+1}^{2n-1} vanishes by ν_{4n+1}^*, since

$$\begin{aligned} \pm b_nE^3\nu_{2n-4*}P_5s_5\iota_{2n+3} &= \pm b_n\nu_{4n+1}^*P_5s_5\iota_{2n+3} \\ &= \pm b_nP_5s_5\nu \\ &= \pm b_nP_8i_3s_5\nu \\ &= \pm b_nP_8H_8P_1\iota_{2n+7} \\ &= 0. \end{aligned}$$

This contradicts $E^4P_5[\sigma_{2n-1}]=[\sigma,\iota_{2n-1}]\neq 0$.

Assume that $P_5[\sigma_{2n-1}]\in\eta_{2n-5*}\pi_{4n}^{2n-4}$. Since the relations

$$E^4\eta_{2n-5*}\pi_{4n}^{2n-4}=\eta_{4n+3}^*E^3\pi_{4n}^{2n-4}\subset\eta_{4n+3}^*\pi_{4n+3}^{2n-1}$$

hold and the unstable part Z_2 of the group π_{4n+3}^{2n-1} vanishes by η_{4n+3}^*, we have a contradiction to $E^7P_8[\sigma]=[\sigma,\iota_{2n-1}]\neq 0$. Therefore $P_5[\sigma_{2n-1}]\notin\operatorname{Image}\Delta_2$. Hence, $j_2P_5[\sigma_{2n-1}]\neq 0$.

5. Whitehead products in $\pi_r(W_{n,3})$ for n odd

Let n be odd. The group $\pi_{2n-1}(W_{n,3})=Z$ is generated by $[(24/a_n)\iota]$.

Theorem 10. $[[(24/a_n)\iota].j_2\iota_{2n-5}]=0$.

Proof. Let $l=1$, $k=3$ and $r=2n-2$ in (1.1). Since $\pi_{2n-1}(V_{2n+1,7})=0$ we obtain the result.

We know the 2 components of the group $\pi_{2n+2}(W_{n,3})=Z_{4(a_n,2)}+Z_{16}$, where the first component is generated by $[(2/(a_n,2))\nu]$.

Theorem 11. *We have* $[[(2/(a_n,2))\nu],j_2\iota_{2n-5}]=\pm j_2P_5[\nu_{2n-1}]\not\equiv 0\bmod j_2[\sigma,\iota_{2n-5}]$, $j_2P_2[\nu^2]$ *if* $n\equiv 3\bmod 4$, $\equiv 0\bmod j_2[\sigma,\iota_{2n-5}],j_2P_2[\nu_{2n-4}^2]$ *if* $n\equiv 1\bmod 4$.

Proof. Let $l=1$, $k=3$ and $r=2n+1$ in (1.1). In the case $n\equiv 1\bmod 4$ the group $\pi_{2n+2}(V_{2n,6})=Z_2+Z_2+Z_2+Z_2+Z_4$ is generated by $i_5\bar{\nu},i_5\varepsilon,i_4s_2\sigma,i_3[\nu^2]$ and $[2\iota]\nu$, so we may set

$$s_*[2\nu]\equiv[2\iota]\nu\bmod i_5\bar{\nu},i_5\varepsilon,i_4s_2\sigma,i_3[\nu^2]$$

in $(1.2)_2$. Thus

$$[[2\nu],j_2\iota_{2n-5}]=\pm j_2P_6p_6i_1[2\iota]\nu$$
$$\equiv 0\bmod j_2[\sigma,\iota_{2n-5}],j_2P_2[\nu_{2n-4}^2]$$

because of $i_1[2\iota]\nu\equiv 0\bmod i_6\bar{\nu},i_6\varepsilon,i_5s_2\sigma,i_4[\nu^2]$. Assume that $P_2[\nu_{2n-4}^2]\in a_n\nu_{2n-5*}\pi_{4n-4}^{2n-2}$. Then we have the relations

$$\eta_{4n-4}^*P_2[\nu^2]=P_2[\nu^2]\eta=P_6[\eta]\eta^2=[\varepsilon,\iota_{2n-5}]\neq 0.$$

This contradicts $\eta_{4n-4}^*a_n\nu_{2n-5*}\pi_{4n-4}^{2n-2}=0$. Therefore $0\neq P_2[\nu_{2n-4}^2]\notin\operatorname{Image}\Delta_2$, so $j_2P_2[\nu^2]\neq 0$. A similar argument shows that $j_2[\sigma,\iota_{2n-5}]\neq 0$ by $\eta_{4n-4}^*[\sigma,\iota_{2n-5}]=[\eta\sigma,\iota_{2n-5}]\neq 0$.

In the case $n\equiv 3\bmod 4$, the group $\pi_{2n+2}(V_{2n,6})=Z_2+Z_2+Z_2+Z_2+Z_8$ is generated by $i_5\bar{\nu},i_5\varepsilon,i_4s_2\sigma,i_3[\nu^2]$ and $[\nu]$, so we may set

$$s_*[\nu]\equiv[\nu]\bmod i_5\bar{\nu},i_5\varepsilon,i_4s_2\sigma,i_3[\nu^2]$$

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in $(1.2)_2$. Since the unstable parts Z_2+Z_2 of the group π_{4n-4}^{2n-2} generated by $P_2 s_2 \iota_{2n-1}$ and $[\eta, \iota_{2n-2}]$ (cf. [10]) vanish by the map $a_n \nu_{2n-5*}$, so the unstable elements $P_5[\nu_{2n-1}]$, $P_1\sigma_{2n-5}$ and $P_2[\nu_{2n-4}^2]$ do not lie in Image Δ_2. Thus

$$\begin{aligned}[[\nu], j_2\iota_{2n-5}] &= \pm j_2 P_6 p_6 i_1[\nu] \\ &= \pm j_2 P_5[\nu_{2n-1}] \\ &\not\equiv 0 \bmod j_2[\sigma, \iota_{2n-5}], j_2 P_2[\nu^2].\end{aligned}$$

We know the 2 components of the group $\pi_{2n+6}(W_{n,3}) = Z_{16(4:a_n)} + Z_{2(a_n,4)}$, where $Z_{16(4:a_n)}$ is generated by $[2\sigma]$.

Theorem 12. $[[2\sigma], j_2\iota_{2n-5}] = 0$.

Proof. Let $l=1$, $k=3$ and $r=2n+5$ in (1.1). Since $\pi_{2n+6}(V_{2n+1,7})=0$, the result follows.

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