CORRESPONDENCE.

To the Editors of the Journal of the Institute of Actuaries.

SIRS,—With reference to Mr. A. T. Traversi's letter (*J. I. A.*, vol. li, p. 304), I beg to make the following observations:

The distinction which Mr. Traversi makes between "rates" and "probabilities" has been recently the subject of new investigations which have shown how the former, which are not exactly the same as differential coefficients or central rates, can be based on the calculus of probabilities instead of being derived from an empirical conception of the so-called "exposed to risk." They can then be identified with Karup's "pure, independent or partial probabilities" in accordance with a definite hypothesis as to the distribution over the years of age of entrants and withdrawals in the experience under consideration and as to the instantaneous rate of mortality or as to the probabilities of death $q(x+t, x+1)$ for the fractional intervals $(x+t, x+1)$ on the hypothesis that no other cause of elimination takes place.

If $\lambda(x)$ denotes the number of individuals in the experience who have attained the exact age x, $I(x+t)$ those who enter on observation at age $x+t$, where t may have any value from 0 to 1, $U(x+t)$ those who pass out of observation by causes other than death, and $q(x+t, x+1)$ the above-mentioned probability of death between ages $x+t$ and $x+1$, so that $\Sigma_0^1 I(x+t) = I_x$, $\Sigma_0^1 U(x+t) = U_x$ and

https://doi.org/10.1017/S0020268100029358 Published online by Cambridge University Press

$q(x, x+1) = q_x$, then M_x, the probable number of deaths between ages x and $x+1$, is evidently given by the relation

$$M_x = \lambda(x)q_x + \Sigma_0^1 I(x+t)q(x+t, x+1) - \Sigma_0^1 U(x+t)q(x+t, x+1).$$

If by way of hypothesis $q(x+t, x+1)$ is taken as $=(1-t)q_x$, then

$$q_x = M_x \Big/ [\lambda(x) + \Sigma_0^1 I(x+t)(1-t) - \Sigma_0^1 U(x+t)(1-t)].$$

The denominator of this formula coincides exactly with the "exposed to risk" according to the exact-age method, and q_x—the "rate" of English actuarial literature—is seen to be, from the method by which it has been obtained, Karup's "pure probability."

If the hypothesis of uniform distribution is adopted, then the infinitesimals $I_x dt$ and $U_x dt$ are to be substituted for $I(x+t)$ and $U(x+t)$, and the evaluation of the integrals with respect to t between the limits 0 and 1 gives for the denominator the exposed to risk according to the mean age method.

In addition to showing clearly to what hypothesis the usual methods based on an empirical conception of the "exposed to risk" correspond, the foregoing formulas show also the evident coincidence of "rates" with Karup's "pure probabilities" based on the indicated hypothesis.

On the conception of pure probabilities and on their importance in actuarial mathematics an extensive literature exists. The subject has been exhaustively treated in a few memoirs submitted to the 7th International Actuarial Congress, vol ii, p. 327, *et seq.*

More recently the subject has been fully developed in Italy in connection with the statistical and actuarial application of independent probabilities, especially in the following papers :

F. CANTELLI.—*Genesi e construzione delle tavole di mutualità* in "Bollettino di notizie sul Credito e sulla Previdenza", no. 3, 1914 ;

I. MESSINA.—*Le probabilità parziali nella matematica attuariale,* nello stesso periodico no. 4 a 6 del 1915 ;

F. CANTELLI.—*Sull 'applicazione delle probabilità parziali alla statistica,* in "Giornale di Matematica finanziaria", n. 1, 2 e 3 del 1919.

With special reference to the construction of tables of mortality the writer has dealt with the subject in a note entitled *Costruzione e critica delle tavole di mortalità* in the "Giornale degli economisti e rivista di statistica" for December 1917.

If the circulation and study of the papers just mentioned had not been prevented by the war, Mr. Traversi would not perhaps have written his letter, since he would have been able to see clearly how the so-called "rates" of English actuarial literature can only correspond theoretically to independent probabilities in the sense in which Karup uses that term and must be regarded strictly as

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referring to independent* and compatible events so that it is quite legitimate to apply to them Karup's formula

$$p_{ng} = p_n \times p_g \qquad (1)$$

just as, in an experience subject to decrement by death and invalidity, the probability of not dying or being invalided is given by the product of the respective independent probabilities (*cf.* H. A. Van den Belt's memoir in the Transactions of the Seventh Congress).

It appears from the above-mentioned investigations that the formulas used by T. B. Sprague, when he considered the two decremental causes "death" and "marriage" in so far as the direct determination of pure probabilities is concerned, involved the employment of hypotheses which cannot be treated as mutually consistent, and this explains the incongruities produced by the simultaneous use of Sprague's formulas for the two decremental causes. Mr. Traversi has adopted these formulas, and yet claims to establish the inapplicability of (1) to the case dealt with by Prof. Cantelli. Such incongruities have been clearly indicated by this author in the first of the two works referred to above.

I have thought it useful to submit these observations to the *Journal* because, having myself devoted some attention to the subject, it seemed to me convenient not to allow the points raised by Mr. Traversi to pass without remark, as this would tend to re-establish conceptions which, as the above-mentioned have shown, can no longer be maintained consistently with the more recent developments of actuarial science, and because it seemed desirable to draw attention to published investigations which, owing to the war, may have been unnoticed outside Italy.

I am, &c.,

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P.S.—I am late in commenting on Mr. Traversi's letter, but I have only recently seen the numbers of the *Journal* for 1918 and 1919. I may take the opportunity of mentioning that the formula employed by Mr. Savory (*J.I.A.*, vol. li, p. 65) for calculating the excess of mortality due to the war, namely, $10000(q_{ng} - q_n)$, in which q_{ng} and q_n are central rates, is not exact but is sufficiently approximate in view of the numbers involved.—G. B.

* And not therefore mutually dependent as Mr. Traversi maintains.

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