THE UNIFORM LIMIT OF LIPSCHITZ FUNCTIONS ON A BANACH SPACE*

J. V. HEROD

(Received 27 September 1971)

Communicated by E. Strzelecki

With R the set of real numbers and S a Banach space, let $\mathscr{L}$ be the class of functions A from $R \times S$ to S which have following properties:

(1) if B is a bounded subset of S then the family $\{A(\cdot\,,P): P \text{ is in } B\}$ is equicontinuous; i.e., if t is a number and $\varepsilon > 0$ then there is a positive number δ such that if $|s-t| < \delta$ and P is in B then $|A(s,P) - A(t,P)| < \varepsilon$.

(2) A is Lipschitz continuous; i.e., there is a continuous, real valued function α such that if t is a number and P and Q are in S then $|A(t,P)-A(t,Q)| \leqq \alpha(t) \cdot |P-Q|$, and

(3) if t is a number then $A(t,\cdot)$ is dissipative; i.e., if $c > 0$, P and Q are in S, and t is a number then

$$|[P - cA(t,P)] - [Q - cA(t,Q)]| \geqq |P-Q|.$$

THEOREM 1. *Suppose that A_0 is a function from $R \times S$ to S. These are equivalent:*

I. *There is a sequence $\{A_p\}_{p=1}^{\infty}$ in $\mathscr{L}$ having the property that if $a < b$ and B is a bounded subset of S then $\{A_p\}_{p=1}^{\infty}$ converges uniformly to A_0 on $[a,b] \times B$.*

II. *The function A_0 has the following properties:*
   - (a) *if B is a bounded subset of S then the family $\{A_0(\cdot\,,P): P \text{ is in } B\}$ is equicontinuous in the sense of* (1) *above,*
   - (b) *if $a < b$ and B is a bounded subset of S then*
     - (i) *A_0 is bounded on $[a,b] \times B$, and*
     - (ii) *if $\varepsilon > 0$ then there is a positive number δ such that if $a \leqq t \leqq b$, P is in B, Q is in S, and $|P-Q| < \delta$ then $|A_0(t,P) - A_0(t,Q)| < \varepsilon$, and*
   - (c) *if t is a number then $A_0(t,\cdot)$ is dissipative.*

* The main portion of the research in this paper was completed while the author was supported by a Frederick Gardner Cottrell grant in aid from Research Corporation.

COROLLARY. (*See also* [4, *Theorem* 4.1]) *If case I of the above Theorem holds, then there is a sequence of functions* $\{M_n\}_{n=0}^{\infty}$ *such that if P is in S then* $M_n(x,y)P = P + \int_y^x A_n(s, M_n(s,y)P)ds$ *for* $x \geqq y$ *and* $n = 0,1,2,\cdots$. *Moreover,* $M_0 = \lim_p M_p$ *and the convergence is uniform on bounded subsets of* $R \times R \times S$.

For each non-negative integer n, the existence of the function M_n which solves the integral equation indicated in the Corollary has been established. See, for example, [3] for the Lipschitz case and, otherwise, [7, p. 277]. In case A_n is in $\mathscr{L}$ or has the properties of statement II in the above Theorem and in case $x > y$ and P is in S, then $M_n(x,y)P$ may be found to be

$$ {}_x\Pi^y[1 - ds \cdot A_n(s, \cdot)]P $$

which is approximated by

$$ [1-(s_1-s_0)A_n(s_0, \cdot)][1-(s_2-s_1)A_n(s_1, \cdot)]\cdots[1-(s_m-s_{m-1})A_n(s_m, \cdot)]P $$

with $x = s_0 \geqq \cdots \geqq s_m = y$. In more general situations ([7, Theorem 1.] or [2]), $M_n(x,y)P$ is given by

$$ {}_x\Pi^y[1 + ds \cdot A_n(s, \cdot)]^{-1}P. $$

If $\{A_p\}_{p=1}^{\infty}$ is a sequence as indicated in statement I of the Theorem, then the fact that A_0 has the properties indicated in II follows from the standard inequalities. On the other hand, beginning with statement II, the question is how to construct the sequence $\{A_p\}_{p=1}^{\infty}$ of Lipschitz continuous functions which will converge uniformly on bounded subsets to A_0. A prototype is found in [5, Theorem 1.].

With A_0 as in statement II of the above Theorem, define $V(x,y)P$ to be $\int_y^x A_0(s,P)ds$ for $x \geqq y$ and P in S. Then V has the following properties: [2, Theorem 6.1]

1A. *if P and Q are in S,* $c > 0$, *and* $x \geqq y$ *then*

$$ |[1 - cV(x,y)]P - [1 - cV(x,y)]Q| \geqq |P - Q|, $$

2A. *if* $x \geqq y \geqq z$ *and P is in S then*

$$ V(x,y)P + V(y,z)P = V(x,z)P, $$

3A. *if* $a < b$ *and B is a bounded subset of S then there is a number L such that if* $b \geqq x \geqq y \geqq a$ *and P is in B then*

$$ |V(x,y)P| \leqq L \cdot (x - y), \text{ and} $$

4A. *if* $a < b$, *B is a bounded subset of S, and* $\varepsilon > 0$, *then there is a positive number* δ *such that if P is in B, Q is in S such that* $|Q - P| < \delta$, *and*

$b \geqq x \geqq y \geqq a$, then

$$|V(x,y)P - V(x,y)Q| \leqq (x-y)\varepsilon.$$

The main Theorem of [2] states that there is a function M from $R \times R \times S$ to S related to V by the following formulas: if $x \geqq y$ and P is in S, then

i. $\quad M(x,y)P = P + \int_x^y V[M(\cdot, y)P]$,

ii. $\quad M(x,y)P = {}_x\Pi^y[1-V]^{-1}P$, *and*

iii. $\quad V(x,y)P = {}_x\Sigma^y[M-1]P$.

Moreover, M has the following properties:

1M. *if P and Q are in S and $x \geqq y$ then*

$$|M(x,y)P - M(x,y)Q| \leqq |P-Q|,$$

2M. *if $x \geqq y \geqq z$ and P is in S then $M(x,y)M(y,z)P = M(x,z)P$,*

3M. *if $a < b$ and B is a bounded subset of S then there is a number L such that if $b \geqq x \geqq y \geqq a$ and P is in B then*

$$|M(x,y)P - P| \leqq L \cdot (x-y), \text{ and}$$

4M. *if $a < b$, B is a bounded subset of S, and $\varepsilon > 0$, then there is a positive number δ and a positive number d such that if P is in B, Q is in S such that $|Q-P| < \delta$, and $b \geqq x \geqq y \geqq a$ such that $x - y < d$, then*

$$|[M(x,y)P - P] - [M(x,y)Q - Q]| \leqq (x-y)\varepsilon.$$

These results will be used to establish the Theorem.

INDICATION OF PROOF FOR II $\Rightarrow$ I. For each positive integer n, let $A_n(t,P) = n[M(t+1/n,t)P - P]$. If n is a positive integer and B is a bounded subset of S then the family $\{A_n(\cdot,P): P \text{ is in } B\}$ is equicontinuous for: if $s \leqq t$ and P is in B then

$$\begin{aligned}
&|n[M(t+1/n,t)P - P] - n[M(s+1/n,s)P - P]| \\
&\quad \leqq n|M(t+1/n,t)P - M(t+1/n,t)M(t,s)P| \\
&\qquad + n|M(t+1/n,s+1/n)M(s+1/n,s)P - M(s+1/n,s)P| \\
&\quad \leqq n|[M(t+1/n,t)P - P] - [M(t+1/n,t)-1]M(t,s)P| \\
&\qquad + n|P - M(t,s)P| + n|M(t+1/n,s)P - M(s+1/n,s)P|.
\end{aligned}$$

These inequalities, together with the properties 3M. and 4M., establish the equicontinuity of the family $\{A_n(\cdot,P): P \text{ is in } B\}$.

If n is a positive integer and t is a number then $A_n(t,\cdot)$ is Lipschitz for; if P and Q are in S then

$$|A_n(t,P) - A_n(t,Q)| = n|[M(t+1/n,t)P - M(t+1/n,t)Q] - [P-Q]|$$
$$\leqq 2n|P-Q|.$$

And, if n is a positive integer and t is a number, then $A_n(t,\cdot)$ is dissipative for if $c > 0$ and P and Q are in S then

$$|\{P - cn[M(t+1/n,t)P - P]\} - \{Q - cn[M(t+1/n,t)Q - Q]\}|$$
$$\geqq [1+cn] \quad |P-Q| - cn|P-Q| = |P-Q|.$$

Finally, if $a < b$ and B is a bounded subset of S then $\{A_p\}_{p=1}^{\infty}$ converges uniformly on $[a,b] \times B$ for: suppose that $\varepsilon > 0$. Let δ be as specified in 4A and L be as in 3A. Suppose that $x - y < \delta/L$ and that

$$b \geqq x = t_0 \geqq t_1 \geqq \cdots \geqq t_n = y \geqq a.$$

If p is an integer in $[1,n]$ and P is in B, then

$$\left|\prod_{i=p}^{n}[1 - V(t_{i-1},t_i)]^{-1}P - P\right| \leqq \sum_{i=p}^{n}\left|V(t_{i-1},t_i)P\right| < \delta.$$

Thus

$$\left|\prod_{p=1}^{n}[1 - V(t_{i-1},t_i)]^{-1}P - P - V(x,y)P\right|$$
$$= \left|\sum_{p=1}^{n} V(t_{p-1},t_p)\prod_{i=p}^{n}[1 - V(t_{i-1},t_i)]^{-1}P - V(t_{p-1},t_p)P\right| \leqq (x-y)\varepsilon.$$

Hence, if $b \geqq x \geqq y \geqq a$ and $x - y < \delta/L$ then

$$|M(x,y)P - P - V(x,y)P| \leqq (x-y)\varepsilon.$$

(See [2, Lemma 4.1].) Moreover, by II (a), there is a number d such that if $a \leqq s \leqq t \leqq b$, $t - s < d$, and P is in B then

$$|V(t,s)P - (t-s)A_0(s,P)| = \left|\int_s^t A_0(z,P)dz - (t-s)A_0(s,P)\right| < (t-s)\varepsilon.$$

Hence, if n is a positive integer such that $1/n <$ minimum $\{\delta/L, d\}$, t is in $[a,b]$, and P is in B then

$$|A_n(t,P) - A_0(t,P)| = |n[M(t+1/n,t)P - P] - A_0(t,P)|$$
$$\leqq n|M(t+1/n,t)P - P - V(t+1/n,t)P|$$
$$+ n|V(t+1/n,t)P - A_0(t,P)/n| < 2\varepsilon.$$

Hence, $\{A_p\}_{p=1}^{\infty}$ converges uniformly on $[a,b] \times B$ and has limit A_0.

Indication of proof for corollary. For each non-negative integer n, let $V_n(x,y)P = \int_y^x A_n(s,P)ds$ for $x \geqq y$ and P in S. If n is a non-negative integer,

https://doi.org/10.1017/S1446788700013240 Published online by Cambridge University Press

then V_n has properties 1A.-4A. and there is a function M_n related to V_n as indicated above and in the main Theorem of [2]. Now suppose that P is in S and $x = t_0 \geqq \cdots \geqq t_n = b$. Then

$$\left|\prod_{p=1}^{n}[1-V_n(t_{p-1},t_p)]^{-1}P - \prod_{p=1}^{n}[1-V_0(t_{p-1},t_p)]^{-1}P\right|$$

$$= \left|\sum_{p=1}^{n}\left\{\prod_{q=1}^{p}[1-V_n(t_{q-1},t_q)]^{-1}\prod_{q=p+1}^{n}[1-V_0(t_{q-1},t_q)]^{-1}P\right.\right.$$

$$\left.\left. - \prod_{q=1}^{p-1}[1-V_n(t_{q-1},t_q)]^{-1}\prod_{q=p}^{n}[1-V_0(t_{q-1},t_q)]^{-1}P\right\}\right|$$

$$\leqq \sum_{p=1}^{n}\left|\prod_{q=p+1}^{n}[1-V_0(t_{q-1},t_q)]^{-1}P\right.$$

$$\left. - [1-V_n(t_{p-1},t_p)]\prod_{q=p}^{n}[1-V_0(t_{q-1},t_q)]^{-1}P\right|$$

$$= \sum_{p=1}^{n}\left| - V_0(t_{p-1},t_p)\prod_{q=p}^{n}[1-V_0(t_{q-1},t_q)]^{-1}P\right.$$

$$\left. + V_n(t_{p-1},t_p)\prod_{q=p}^{n}[1-V_0(t_{q-1},t_q)]^{-1}P\right|.$$

Suppose that $a < b$ and B is a bounded subset of S. Let B' be the set of points Q in S for which there is a decreasing sequence $\{t_p\}_{p=0}^{n}$ in $[a,b]$ with $t_n = a$ and a member P in B such that

$$Q = \prod_{p=1}^{n}[1-V_0(t_{p-1},t_p)]^{-1}P.$$

The set B' is bounded. (See [2, Lemma 2.0].) Corresponding to B', let N be an integer such that if $n > N$, Q is in B', and $a \leqq t \leqq b$ then

$$|A_n(t,Q) - A_0(t,Q)| < \varepsilon. \text{ For } a \leqq y \leqq x \leqq b, \ |V_n(x,y)Q - V_0(x,y)Q| < (x-y)\varepsilon;$$

and, the above inequalities show that if P is in B and $a \leqq y \leqq x \leqq b$ then

$$|M_n(x,y)P - M_0(x,y)P| \leqq (x-y)\varepsilon.$$

THEOREM 2. *Suppose that A_0 is a function from $R \times S$ to S and that ρ is a continuous function from R to R. These are equivalent:*

1. *There is a sequence $\{A_p\}_{p=1}^{\infty}$ of functions from $R \times S$ to S with the following properties:*

https://doi.org/10.1017/S1446788700013240 Published online by Cambridge University Press

a. *if p is a positive integer and B is a bounded subset of S then the family* $\{A_p(\cdot, P)$: *P is in B*$\}$ *is equicontinuous,*

b. *there is a sequence* $\{\alpha_p\}_{p=1}^{\infty}$ *of continuous functions from R to R such that if p is a positive integer, t is a number, and P and Q are in S, then*

$$|A_p(t,P) - A_p(t,Q)| \leqq \alpha_p(t)|P-Q|,$$

c. *there is a sequence* $\{\beta_p\}_{p=1}^{\infty}$ *of continuous functions such that* $\lim_{p\to\infty}\beta_p = \rho$ *and, if t is a number, n is a positive integer, P and Q are in S, and* $c > 0$, *then*

$$|[P - cA_n(t,P)] - [Q - cA_n(t,Q)]| \geqq [1 - c\beta_n(t)]|P-Q|, \text{ and}$$

d. $\{A_p\}_{p=1}^{\infty}$ *converges uniformly on bounded subsets and has limit* A_0:

II. *The function* A_0 *has the following properties:*

a. *if B is a bounded subset of S then*
   1. $\{A_0(\cdot, P)$: *P is in B*$\}$ *is equicontinuous,*
   2. *if* $a < b$ *then*
      (i) A_0 *is bounded on* $[a,b]\times B$, *and*
      (ii) *if* $\varepsilon > 0$ *then there is a positive number* δ *such that if* $a \leqq u \leqq b$, *P is in B, and Q is in S such that* $|Q - P| < \delta$ *then*

$$|A_0(u,Q) - A_0(u,P)| < \varepsilon, \text{ and}$$

b. *if t is a number, P and Q are in S, and* $c > 0$, *then*

$$|[P - cA_0(t,P)] - [Q - cA_0(t,Q)]| \geqq [1 - c\rho(t)]|P-Q|.$$

REMARK. Suppose that P is in S and $\{G_p\}_{p=0}^{\infty}$ is a sequence of functions defined by $G_0(t) = P$ and

$$G_n(t) = P + \int_0^t A_0(s, G_{n-1}(s))ds$$

for $t \geqq 0$ and $n = 1, 2, \cdots$. One might conjecture that, if A_0 has the properties in statement II of Theorem 1, and, hence, is the uniform limit (on bounded subsets) of Lipschitz functions, then the sequence $\{G_p\}_{p=0}^{\infty}$ converges to the solution for $y(0) = P$ and $y'(t) = A_0(t, y(t))$. That this is not the case may be seen by examining an example by Müller [6] (or [1, p. 53]).

References

[1] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955.)

[2] J. V. Herod, 'A pairing of a class of evolution systems with a class of generators', *Trans. Amer. Math. Soc.* 157 (1971), 247–260.

[3] J. S. Mac Nerney, 'A nonlinear integral operation', *Illinois J. Math.* 8 (1964), 621–638.
[4] J. L. Mermin, 'An exponential limit formula for nonlinear semigroups', *Trans. Amer. Math. Soc.* 150 (1970), 469–476.
[5] I. Miyadera and S. Oharu, 'Approximation of semigroups of nonlinear operators', *Tohoku Math. J.* 22 (1970), 22–47.
[6] M. Müller, 'Über das Fundamentaltheorem in der Theorie der gewöhnliche Differentielgleichungen', *Math. Zeit.* 26 (1927), 619–645.
[7] G. F. Webb, 'Product integral representation of time dependent nonlinear evolution equations in Banach spaces,' *Pacific J. Math.* 32 (1970), 269–281.

Georgia Institute of Technology
Atlanta, Georgia 30332, USA

https://doi.org/10.1017/S1446788700013240 Published online by Cambridge University Press