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BIRTH AND DEATH PROCESSES IN RANDOMLY FLUCTUATING ENVIRONMENTS

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Abstract. A birth and death process in a time-dependent random environment is introduced. We will discuss the recurrence and transience properties for the process.

§1. Introduction and results

Let E be a finite subset of a hyperplane in $\mathbb{R}^{2d}$ whose element has positive components. Each element in E is denoted by $a = (\lambda_+^1(a), \lambda_-^1(a), \dots, \lambda_+^d(a), \lambda_-^d(a)) \in E$.

Suppose we are given a stationary, continuous time, irreducible Markov chain a_t on E. On a probability space $(\Omega, \mathfrak{F}, P)$, construct a family of independent copies of a_t, indexed by $x \in \mathbb{Z}^d$, the d-dimensional space lattice. Call these processes $a_t(x)$. Set $\mathbf{a}_t = (a_t(x))_{x\in\mathbb{Z}^d}$.

For a given realization $\{\mathbf{a}_t, t \geq 0\}$, define a jump process X_t as the non-stationary Markov process with state space $\mathbb{Z}^d$ and transition probability determined by

$$(1) \qquad \begin{cases} P\{X_{t+h} = x \pm e_i \mid X_t = x\} = h\lambda_\pm^i(a_t(x)) + o(h) \\ P\{X_{t+h}=x \mid X_t=x\}=1-h\sum_{i=1}^{d}\{\lambda_+^i(a_t(x))+\lambda_-^i(a_t(x))\}+o(h) \end{cases}$$

where $e_i = (0, \dots, \overset{i}{\check{1}}, \dots, 0) \in \mathbb{Z}^d$. This process X_t is called a birth and death process in the randomly fluctuating environment $\{\mathbf{a}_t, t \geq 0\}$. In this paper we shall discuss the recurrence properties of the process X_t. The paired process $(X_t, \mathbf{a}_t)$ will be used to investigate the problem.

It is to be mentioned that another type of processes in randomly fluctuating environments have been introduced by Madras [4]. In the paper the recurrence and transience problem in one dimension was treated.

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Before proceeding to the existence of the above two processes, we shall introduce some notations and definitions.

Let $p_t(a,b)$ be the transition probability of the Markov process a_t. Since the process a_t is irreducible on the finite state space, it has the unique invariant probability measure μ. It is well known that there exists a positive constant α such that

$$|p_t(a,b) - \mu(b)| \leq e^{-\alpha t} \tag{2}$$

for $t > 0$ and $a, b \in E$. Define $\boldsymbol{\mu} = \mu \overset{\mathbb{Z}^d}{\otimes}$ as the direct product of μ on $\mathbb{Z}^d$. Let $E^{\mathbb{Z}^d}$ be the space of configurations of jump rates on $\mathbb{Z}^d$ and let it be endowed with the usual direct topology. Then $\mathbf{a}_t$ is regarded as a Markovian system as in e.g. Liggett [3] and $\boldsymbol{\mu}$ is its unique invariant probability measure. Let G be the infinitesimal generator of $\mathbf{a}_t$. $P_{\mathbf{a}}$ denotes the probability law of $\mathbf{a}_t$ starting at $\mathbf{a}_0 = \mathbf{a}$. Set

$$P_{\boldsymbol{\mu}}(B_1 \times B_2) = \int_{B_1} P_{\mathbf{a}}(B_2)\boldsymbol{\mu}(d\mathbf{a}).$$

Define a difference operator L on $\mathbb{Z}^d \times E^{\mathbb{Z}^d}$ by

$$\begin{aligned} Lf(x,\mathbf{a}) = \sum_{i=1}^{d} \big\{ &\lambda_+^i(a(x))(f(x+e_i,\mathbf{a}) - f(x,\mathbf{a})) \\ &+\lambda_-^i(a(x))(f(x-e_i,\mathbf{a}) - f(x,\mathbf{a})) \big\}. \end{aligned} \tag{3}$$

We now explain briefly about the existence of the above processes X_t and $(X_t, \mathbf{a}_t)$. Let $\eta_t(x)$, $(t,x) \in [0,+\infty) \times \mathbb{Z}^d$ be a function with values in E which is right continuous and has left limits in t. Set $\boldsymbol{\eta}_t = \{\eta_t(x)\}_{x\in\mathbb{Z}^d}$ and $\theta.$ denotes the shift of η i.e. $\theta_s \eta_t(x) = \eta_{t+s}(x)$. Define

$$\begin{aligned} L_t^{\boldsymbol{\eta}\cdot} f(x) = \sum_{i=1}^{d} \big\{ &\lambda_+^i(\eta_t(x))(f(x+e_i) - f(x)) \\ &+\lambda_-^i(\eta_t(x))(f(x-e_i) - f(x)) \big\}. \end{aligned}$$

Then as in Ethier and Kurtz [1], page 163–164, we can easily construct a (nonstationary) Markov process on $\mathbb{Z}^d$ associated with the operator $L_t^{\boldsymbol{\eta}\cdot}$. Denote the process by $X_t^{\boldsymbol{\eta}\cdot}$ and its transition probability by $p^{\boldsymbol{\eta}\cdot}(s,x;t,y)$. Then $X_t^{\mathbf{a}\cdot}$ is nothing but the process introduced in (1).

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Define a family of operators $\{T_t, t \geq 0\}$ on $\mathbb{C}_0 = \mathbb{C}_0(\mathbb{Z}^d \times E^{\mathbb{Z}^d})$, the space of all continuous functions on $\mathbb{Z}^d \times E^{\mathbb{Z}^d}$ vanishing at the infinity, by

$$T_t f(x, \mathbf{a}) = E_{\mathbf{a}} \left[\sum_{y \in \mathbb{Z}^d} p^{\mathbf{a}_\cdot}(0, x; t, y) f(y, \mathbf{a}_t) \right]. \tag{4}$$

Then $\{T_t\}_{t \geq 0}$ is a Feller semigroup on $\mathbb{C}_0(\mathbb{Z}^d \times E^{\mathbb{Z}^d})$. Indeed, we have

$$\begin{aligned} &T_{t+s} f(x, \mathbf{a}) \\ &= E_{\mathbf{a}} \left[\sum_{y \in \mathbb{Z}^d} p^{\mathbf{a}_\cdot}(0, x; t+s, y) f(y, \mathbf{a}_{t+s}) \right] \\ &= E_{\mathbf{a}} \left[\sum_{y, z \in \mathbb{Z}^d} p^{\mathbf{a}_\cdot}(0, x; s, z) p^{\mathbf{a}_\cdot}(s, z; t+s, y) f(y, \mathbf{a}_{t+s}) \right] \\ &= E_{\mathbf{a}} \left[\sum_{y, z \in \mathbb{Z}^d} p^{\mathbf{a}_\cdot}(0, x; s, z) p^{\theta_s \mathbf{a}_\cdot}(0, z; t, y) f(y, \theta_s \mathbf{a}_t) \right] \\ &= E_{\mathbf{a}} \left[\sum_{z \in \mathbb{Z}^d} p^{\mathbf{a}_\cdot}(0, x; s, z) E_{\mathbf{a}} \left[\sum_{y \in \mathbb{Z}^d} p^{\theta_s \mathbf{a}_\cdot}(0, z; t, y) f(y, \theta_s \mathbf{a}_t) \;\middle|\; \sigma(\mathbf{a}_r; r \leq s) \right] \right] \\ &= E_{\mathbf{a}} \left[\sum_{z \in \mathbb{Z}^d} p^{\mathbf{a}_\cdot}(0, x; s, z) T_t f(z, \mathbf{a}_s) \right] \\ &= T_s \cdot T_t f(x, \mathbf{a}). \end{aligned} \tag{5}$$

It is also easy to see that $T_t f \in \mathbb{C}_0$ if $f \in \mathbb{C}_0$. Furthermore it can be shown that the infinitesimal generator of $\{T_t\}$ is $\Delta = L + G$. This semigroup is associated with the bichain $(X_t^{\mathbf{a}_\cdot}, \mathbf{a}_t)$.

Set $\lambda_{\pm}^{i,0} = \int_E \lambda_{\pm}^i(a)\mu(da)$. Let X_t^i be the i-th component of X_t. Denote by $P_x^{\mathbf{a}_\cdot}(\,\cdot\,)$ the probability law of the birth and death process $X_t^{\mathbf{a}_\cdot}$ starting at x, in the randomly fluctuating environment $\{\mathbf{a}_t, t \geq 0\}$.

We are now in a position to state our first result.

THEOREM 1. *Suppose* $\lambda_{+}^{i_0,0} > \lambda_{-}^{i_0,0}$ *for some* $i_0 \in \{1, \ldots, d\}$. *Then there exists a positive constant* α_0^d *depending only on* $\lambda_{\pm}^i$, $i = 1, \ldots, d$, d

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and E such that if $\alpha > \alpha_0^d$,

$$P_0^{\mathbf{a}\cdot}\left[\lim_{t\to+\infty} X_t^{i_0} = +\infty\right] = 1 \quad a.s. \quad P_{\boldsymbol{\mu}}$$

The environment treated in Theorem 1 is biased in some sense to one direction. Thus we shall next consider the case where the environment is statistically balanced i.e. $\lambda_+^{i,0} = \lambda_-^{i,0}$. This case is more delicate. Therefore we need to put stronger assumptions on the environment;

There exist positive constants $p_i, q_i (p_i \neq q_i)$, $i = 1, \dots, d$ such that

(I) $$E = \left\{(\lambda_+^1, \lambda_-^1, \dots, \lambda_+^d, \lambda_-^d)\ ;\ (\lambda_+^i, \lambda_-^i) = (p_i, q_i) \text{ or } (q_i, p_i)\right\}$$

and the transition probability of the Markov chain a_t on E is given by

(II) $$p_t(a_1, a_2) = \begin{cases} \dfrac{1}{2^d}(1 - e^{-\alpha t}), & a_1 \neq a_2 \\ \dfrac{1}{2^d}(1 + (2^d - 1)e^{-\alpha t}), & a_1 = a_2. \end{cases}$$

In this case, $\mu(a) = \frac{1}{2^d}$ for $a \in E$. Under the assumptions (I) and (II),

$$\lambda_+^{i,0} = \lambda_-^{i,0} = \frac{p_i + q_i}{2}, \quad i = 1, \dots, d.$$
$$(\equiv \lambda^{i,0})$$

We have

THEOREM 2. *Let $d = 1$. Suppose the conditions* (I) *and* (II) *hold. Then there exists a positive constant β_0^1 depending only on p_1, q_1 such that if $\alpha > \beta_0^1$,*

$$P_0^{\mathbf{a}\cdot}[\ X_t = x \quad \textit{for some} \quad t > 0\ ] = 1 \quad a.s. \quad P_{\boldsymbol{\mu}}$$

for any $x \in \mathbb{Z}^1$.

THEOREM 3. *Let $d \geq 3$. Suppose the conditions* (I) *and* (II) *hold. Then there exists a positive constant β_0^d depending only on p_i and q_i, $i = 1, \dots, d$ such that if $\alpha > \beta_0^d$,*

$$P_0^{\mathbf{a}\cdot}\left[\lim_{t\to+\infty} |X_t| = +\infty\right] = 1 \quad a.s. \quad P_{\boldsymbol{\mu}}$$

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In order to handle the two dimensional case, a further restriction on the environment is required.

$$\text{(I)}' \qquad p_1 = p_2 (\equiv p) \quad \text{and} \quad q_1 = q_2 (\equiv q).$$

THEOREM 4. *Let $d = 2$. Suppose the conditions* (I), ((I)′) *and* (II) *hold. Then there exists a positive constant β_0^2 depending only on p and q such that if $\alpha > \beta_0^2$,*

$$P_0^{\mathbf{a}.} [X_t = x \quad \textit{for some} \quad t > 0] = 1 \quad a.s. \quad P_{\boldsymbol{\mu}}$$

for any $x \in \mathbb{Z}^2$.

It is to be remarked that all the constants α_0^d, β_0^d can be explicitly identified.

Note that it suffices to prove all the theorems under the law of the bichain $(X_t^{\mathbf{a}.}, \mathbf{a}_t)$ instead of the law of the original process.

The proofs of our theorems are based on Lyapunov's method. We shall construct Lyapunov functions by a perturbation of the ones for simple random walks. The point is to apply a sort of renormalization technique through the generator G.

§2. Proof of theorems

Let $P_{(x,\mathbf{a})}$ be the probability law of $(X_t, \mathbf{a}_t)$ starting at $(x, \mathbf{a})$.

We start with;

[1] *Proof of Theorem* 1. Assume $i_0 = 1$ and set $x = (x_1, \dots, x_d)$. In order to prove Theorem 1, we need the following two lemmas.

LEMMA 1. *We have*

$$P_{(x,\mathbf{a})} \left[X_t^1 = 0 \quad \textit{for some} \quad t > 0 \right] = 1$$

for any $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ with $x_1 < 0$ and any $\mathbf{a} \in E^{\mathbb{Z}^d}$.

LEMMA 2. *Let k be a positive number. Then there exist positive constants C_i, $i = 1, 2$ such that*

$$P_{(x,\mathbf{a})} \left[X_t^1 \le C_1 \quad \textit{for some} \quad t > 0 \right] \le \frac{C_2}{x_1^k}$$

for any $x = (x_1, \dots, x_d)$ with $x_1 > C_1$ and any $\mathbf{a} \in E^{\mathbb{Z}^d}$.

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Proof of Lemma 1. For $x = (x_1, \dots, x_d)$, set $V_0(x, \mathbf{a}) = x_1 + \varphi_0(x, \mathbf{a})$, where φ_0 will be determined later on. Assuming φ_0 belongs to the domain of G, compute

$$\begin{aligned}\Delta V_0(x, \mathbf{a}) &= LV_0(x, \mathbf{a}) + GV_0(x, \mathbf{a}) \\ &= \lambda_+^{1,0} - \lambda_-^{1,0} + \bar\lambda_+^1(a(x)) - \bar\lambda_-^1(a(x)) + L\varphi_0(x, \mathbf{a}) + G\varphi_0(x, \mathbf{a}),\end{aligned}$$

where $\bar\lambda_\pm^1(a) = \lambda_\pm^1(a) - \lambda_\pm^{1,0}$.
Since $\displaystyle\int_E \bar\lambda_\pm^1(a)\mu(da) = 0$,

$$\varphi_\pm(x, \mathbf{a}) = -\int_0^\infty E_{a(x)}\left[\bar\lambda_\pm^1(a_t(x))\right] dt$$

is well defined. Let $\varphi_0(x, \mathbf{a}) = \varphi_+(x, \mathbf{a}) - \varphi_-(x, \mathbf{a})$. Obviously

$$G\varphi_0(x, \mathbf{a}) + \bar\lambda_+^1(a(x)) - \bar\lambda_-^1(a(x)) = 0.$$

From (2),

$$|\varphi_\pm(x, \mathbf{a})| \le \frac{M_0}{\alpha}$$

where $M_0 = \max\left\{\sum_{a\in E}\left|\bar\lambda_+^i(a)\right|, \sum_{a\in E}\left|\bar\lambda_-^i(a)\right| \;;\; i = 1, \dots, d\right\}$.
Hence

$$|L\varphi_0(x, \mathbf{a})| \le \frac{8M_0M_1d}{\alpha},$$

with $M_1 = \max\left\{\lambda_\pm^1(a), \dots, \lambda_\pm^d(a) \;;\; a \in E\right\}$.
Thus if $\alpha > \beta_0^d = \dfrac{8M_0M_1d}{\lambda_+^{1,0} - \lambda_-^{1,0}}$, then $\Delta V_0(x, \mathbf{a}) \ge 0$ for any $(x, \mathbf{a}) \in \mathbb{Z}^d \times E^{\mathbb{Z}^d}$.

Define stopping times by

$$\sigma^0 = \inf\{t > 0 \;;\; X_t^1 \ge 0\}$$

and

$$\tau^{-N} = \inf\{t > 0 \;;\; X_t^1 \le -N\}, \quad N \ge 1.$$

By Dynkin's formula, we get

$$E_{(x,\mathbf{a})}\left[V_0(X_{\tau^{-N}\wedge\sigma^0}, \mathbf{a}_{\tau^{-N}\wedge\sigma^0})\right] \ge V_0(x, \mathbf{a})$$

for any $x = (x_1, \dots, x_d)$ with $x_1 \in (-N, 0)$ and any $\mathbf{a} \in E^{\mathbb{Z}^d}$.

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Therefore

$$(-N + \frac{2M_0}{\alpha})P_{(x,\mathbf{a})}(\tau^{-N} < \sigma^0) + \frac{2M_0}{\alpha}P_{(x,\mathbf{a})}(\tau^{-N} > \sigma^0) \geq V_0(x, \mathbf{a})$$

for any $(x, \mathbf{a})$ with $x_1 \in (-N, -1]$. Letting $N \uparrow +\infty$, we obtain

$$\lim_{N \to +\infty} P_{(x,\mathbf{a})}(\tau^{-N} < \sigma^0) = 0,$$

which implies $P_{(x,\mathbf{a})}(X_t^1 = 0 \quad \text{for some} \quad t > 0) = 1$ for $x_1 < 0$.

Proof of Lemma 2. For a given positive constant k, set $g_1(x) = |x_1|^{-k}$ and

$$V_1(x) = g_1(x) + g_{1,+}^1(x)\varphi_+(x, \mathbf{a}) + g_{1,-}^1(x)\varphi_-(x, \mathbf{a}),$$

where $g_{1,+}^1(x) = g_1(x+e_1) - g_1(x)$, $g_{1,-}^1(x) = g_1(x-e_1) - g_1(x)$ and $\varphi_\pm(x, \mathbf{a})$ are the ones in the proof of Lemma 1.
Then it is easy to see;

$$\mathcal{E}_0 \equiv \inf_{\substack{(x,\mathbf{a}) \\ x_1 = C_1}} V_1(x, \mathbf{a}) > 0 \quad \text{and} \quad \Delta V_1(x, \mathbf{a}) \leq 0, \quad \text{if} \quad x_1 \geq C_1 \tag{6}$$

for some positive integer C_1 and

$$\lim_{x_1 \to +\infty} V_1(x, \mathbf{a}) = 0.$$

Let

$$\sigma^N = \inf\{t > 0 \,;\; X_t^1 \geq N\}, \quad N > C_1,$$
$$\tau^{C_1} = \inf\{t > 0 \,;\; X_t^1 \leq C_1\}.$$

Apply Dynkin's formula to obtain

$$E_{(x,\mathbf{a})}\left[V_1(X_{\sigma^N \wedge \tau^{C_1}}, \mathbf{a}_{\sigma^N \wedge \tau^{C_1}})\right] \leq V_1(x, \mathbf{a}) \tag{7}$$

for any $(x, \mathbf{a})$ with $x_1 \in (C_1, N)$.
Letting $N \uparrow \infty$ in (7), we get

$$P_{(x,\mathbf{a})}(\tau^{C_1} < +\infty) \leq \frac{V_1(x, \mathbf{a})}{\mathcal{E}_0} \quad \text{for} \quad x_1 > C_1.$$

Since $V_1(x, \mathbf{a}) = O(|x_1|^{-k})$, the proof of Lemma 2 is complete.

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In order to prove Theorem 1, we first note that there exist a point $(x_1^0, \mathbf{a}_1^0) \in \mathbb{Z}^d \times E^{\mathbb{Z}^d}$ and an integer m_1 such that

$$P_{(x_1^0,\mathbf{a}_1^0)}(X_t^1 = m_1 \quad \text{i.o. as} \quad t \uparrow +\infty) > 0$$

if the assertion in Theorem 1 is false.

We can assume $m_1 = C_1$ without loss of generality. Choose an integer $\rho_0 > C_1$ such that $\frac{C_2}{\rho_0^k} < 1$.

Define a sequence of stopping times as follows;

$$\begin{aligned}
\tau_1^{C_1} &= \inf\{t \geq 0\ ;\ X_t^1 \leq C_1\} \\
\sigma_1^{\rho_0} &= \inf\{t \geq \tau_1^{C_1}\ ;\ X_t^1 \geq \rho_0\} \\
&\cdots\cdots \\
\tau_n^{C_1} &= \inf\{t \geq \sigma_{n-1}^{\rho_0}\ ;\ X_t^1 \leq C_1\} \\
\sigma_n^{\rho_0} &= \inf\{t \geq \tau_n^{C_1}\ ;\ X_t^1 \geq \rho_0\}.
\end{aligned}$$

Note that $\sigma_n^{\rho_0}$ is finite with probability one for all n from Lemma 1. We want to estimate

$$P_{(x_1^0,\mathbf{a}_1^0)}(\tau_n^{C_1} < +\infty) = P_{(x_1^0,\mathbf{a}_1^0)}[\tau_1^{C_1} < \sigma_1^{\rho_0} < \cdots < \sigma_{n-1}^{\rho_0} < \tau_n^{C_1}].$$

By repeated use of the strong Markov property, the right-hand side is equal to

$$\begin{aligned}
&E_{(x_1^0,\mathbf{a}_1^0)}\left[\tau_1^{C_1} < +\infty, E_{(X_{\tau_1^{C_1}}, \mathbf{a}_{\tau_1^{C_1}})}\left[\tau_2^{C_1} < +\infty, E_{(X_{\tau_2^{C_1}}, \mathbf{a}_{\tau_2^{C_1}})}\left[\quad \cdots\right.\right.\right. \\
&\qquad \cdots \left.\left.\left.\left[\tau_{n-1}^{C_1} < +\infty, E_{(X_{\tau_{n-1}^{C_1}}, \mathbf{a}_{\tau_{n-1}^{C_1}})}\left[P_{(X_{\sigma_{n-1}^{\rho_0}}, \mathbf{a}_{\sigma_{n-1}^{\rho_0}})}\left[\tau_n^{C_1} < +\infty\right]\cdots\right]\right.\right.\right].
\end{aligned}$$

By Lemma 2, we get

$$P_{(x_1^0,\mathbf{a}_1^0)}(\tau_n^{C_1} < +\infty) \leq \left(\frac{C_2}{\rho_0^k}\right)^{n-1} P_{(x_1^0,\mathbf{a}_1^0)}\left[\tau_1^{C_1} < +\infty\right].$$

Hence $P_{(x_1^0,\mathbf{a}_1^0)}(X_t^1 = m_1 \quad \text{i.o. as} \quad t \uparrow +\infty) = 0$. Thus the assertion of Theorem 1 is valid. □

[2] *Proof of Theorem* 2. The basic idea is the same as in the proof of Theorem 1. To prove Theorem 2, we shall make use of a function $V_2(x, \mathbf{a})$,

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$(x,\mathbf{a}) \in \mathbb{Z}^1 \times E^{\mathbb{Z}^1}$ such that

(8) $$\lim_{|x|\to+\infty} \inf_{\mathbf{a}} V_2(x,\mathbf{a}) = +\infty$$

(9) $$\Delta V_2(x,\mathbf{a}) \leq 0 \quad \text{for any} \quad |x| \geq C_3 \quad \text{and} \quad \mathbf{a} \in E^{\mathbb{Z}^1}$$

with some positive constant C_3.

Set $g_2(x) = \log|x|$ and $g_2^1(x) = g_2(x+1) - g_2(x-1)$. A function V_2 as above will be constructed in a form of

(10) $$V_2(x,\mathbf{a}) = g_2(x) + g_2^1(x) \sum_{n=0}^{\infty} \varphi_n(x,\mathbf{a}).$$

For this V_2, we first perform a heuristic computation.

$$\Delta V_2(x,\mathbf{a}) = Lg_2(x,\mathbf{a}) + \sum_{n=0}^{\infty} L(g_2^1\varphi_n)(x,\mathbf{a}) + \sum_{n=0}^{\infty} g_2^1(x) G\varphi_n(x,\mathbf{a}).$$

First of all,

$$Lg_2(x,\mathbf{a}) = -\frac{\lambda^{1,0}}{x^2} + \bar{\lambda}_+^1(a(x)) g_2^1(x) + O(x^{-3})$$

and

$$\begin{aligned} L(g_2^1\varphi_n)(x,\mathbf{a}) = & \ \lambda_+^1(a(x))(g_2^1(x+1)\varphi_n(x+1,\mathbf{a}) - g_2^1(x)\varphi_n(x,\mathbf{a})) \\ & + \lambda_-^1(a(x))(g_2^1(x-1)\varphi_n(x-1,\mathbf{a}) - g_2^1(x)\varphi_n(x,\mathbf{a})) \\ = & \ L\varphi_n(x,\mathbf{a}) + P_n(x,\mathbf{a}) \end{aligned}$$

where

$$\begin{aligned} P_n(x,\mathbf{a}) = & \ \lambda_+^1(a(x))\varphi_n(x+1,\mathbf{a})(g_2^1(x+1) - g_2^1(x)) \\ & + \lambda_-^1(a(x))\varphi_n(x-1,\mathbf{a})(g_2^1(x-1) - g_2^1(x)). \end{aligned}$$

It is easily seen that

$$|P_n(x,\mathbf{a})| \leq 2\max(p_1,q_1) \sup_{(x,\mathbf{a})} |\varphi_n(x,\mathbf{a})| \cdot \left(\frac{2}{x^2} + O(x^{-3})\right).$$

Thus it follows formally that

$$\begin{aligned} \Delta V_2(x,\mathbf{a}) = & -\frac{\lambda^{1,0}}{x^2} + O(x^{-3}) + g_2^1(x)\left\{\bar{\lambda}_+^1(a(x)) + G\varphi_0(x,\mathbf{a})\right\} \\ & + g_2^1(x) \sum_{n=0}^{\infty} \{L\varphi_n(x,\mathbf{a}) + G\varphi_{n+1}(x,\mathbf{a})\} + \sum_{n=0}^{\infty} P_n(x,\mathbf{a}). \end{aligned}$$

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In order for V_2 to satisfy the conditions (8) and (9), we shall construct an infinite sequence of $\{\varphi_n\}$ such that

$$(11)\qquad \begin{cases} \bar{\lambda}_+^1(a(x)) + G\varphi_0(x, \mathbf{a}) = 0 \\ L\varphi_n(x, \mathbf{a}) + G\varphi_{n+1}(x, \mathbf{a}) = 0, \quad n \geq 0 \end{cases}$$

and $\{\varphi_n\}$ converges to zero sufficiently fast. Thus the following lemma is a key to the proof of Theorem 2.

Lemma 3. *For the equations* (11), *there exists a system of solutions* $\{\varphi_n,\ n \geq 0\}$ *such that*

$$(12)\qquad |\varphi_n(x, \mathbf{a})| \leq \frac{(p_1 + q_1)^{n+1} \cdot 5^n}{\alpha^{n+1}}, \quad n \geq 0.$$

Before proceeding to the proof of Lemma, we give

Sublemma 1. *Let m be a positive integer and $x_1, \ldots, x_m$ be m-distinct points in $\mathbb{Z}^1$. Set*

$$f(\mathbf{a}) = \prod_{i=1}^{m} \bar{\lambda}_{j_i}^1(a(x_i)), \quad j_i = + \text{ or } -.$$

Then $E_{\mathbf{a}}[f(\mathbf{a}_t)] = e^{-m\alpha t} f(\mathbf{a})$.

Proof of Lemma 3. Since $\int_E \bar{\lambda}_+^1(a)\mu(da) = 0$, the first equation in (11) has a solution φ_0 given by

$$\begin{aligned}\varphi_0(x, \mathbf{a}) &= -\int_0^\infty E_{a(x)}\left[\bar{\lambda}_+^1(a_t(x))\right] dt = -\frac{1}{\alpha}\bar{\lambda}_+^1(a(x)) \\ &= -\frac{1}{2\alpha}\left\{\bar{\lambda}_+^1(a(x)) - \bar{\lambda}_-^1(a(x))\right\}.\end{aligned}$$

Based on the following claims, we can construct $\{\varphi_n\}_{n\geq 1}$ by induction.

Claim 1. *Let $\varphi(x, \mathbf{a})$ be a polynomial of $\bar{\lambda}_+^1(a(x + \cdot))$, $\bar{\lambda}_-^1(a(x + \cdot))$. Suppose there exist two polynomials $\varphi_+(x, \mathbf{a})$ and $\varphi_-(x, \mathbf{a})$ of $\bar{\lambda}_+^1(a(x + \cdot))$ and $\bar{\lambda}_-^1(a(x + \cdot))$ such that*

(i) $\varphi(x, \mathbf{a}) = \varphi_+(x, \mathbf{a}) - \varphi_-(x, \mathbf{a})$

(ii) *by replacement between $\bar{\lambda}_+(a(x + p))$ and $\bar{\lambda}_-(a(x - p))$ $(p \in \mathbb{Z})$, we get $\varphi_-(x, \mathbf{a})$ from $\varphi_+(x, \mathbf{a})$.*

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Then

$$\int_{E^{\mathbb{Z}^1}} L\varphi(x,\mathbf{a})\boldsymbol{\mu}(d\mathbf{a}) = 0$$

and hence

$$\tilde{\varphi}(x,\mathbf{a}) = -\int_0^\infty E_{\mathbf{a}}\left[L\varphi(x,\mathbf{a}_t)\right] dt$$

is well defined. $\tilde{\varphi}(x,\mathbf{a})$ *is also a polynomial of* $\bar{\lambda}^1_+(a(x+\cdot))$ *and* $\bar{\lambda}^1_-(a(x+\cdot))$ *and is decomposed into a difference of two polynomials as in (i) and (ii).*

CLAIM 2. *For a polynomial* ψ *of* $\bar{\lambda}^1_\pm(a(x+\cdot))$ *as*

$$\psi(x,\mathbf{a}) = \sum c_{r_1,\dots,r_t\,;\,s_1,\dots,s_u} \prod_{i=1}^{t} \bar{\lambda}^1_+(a(x+r_i)) \prod_{j=1}^{u} \bar{\lambda}^1_-(a(x+s_j))$$

where integers $r_1,\dots,r_t$, $s_1,\dots,s_u$ *are distinct each other and summation is taken over distinct sets* $\{r_1,\dots,r_t,s_1,\dots,s_u\}$, *define its norm by*

$$\|\psi\| = \sum |c_{r_1,\dots,r_t\,;\,s_1,\dots,s_u}| \cdot \frac{|p_1-q_1|^{t+u}}{2^{t+u}}.$$

Then for φ *and* $\tilde{\varphi}$ *in Claim 1, we have*

$$\|\tilde{\varphi}\| \le \frac{5}{\alpha}(p_1+q_1)\|\varphi\|. \tag{13}$$

Proof of Claim 1. First note that the assumptions (i) and (ii) together with the definition of L imply that $L\varphi_+$ and $L\varphi_-$ are interchangeable as in (ii). Indeed the following replacement enables $L\varphi_+$ and $L\varphi_-$ to interchange each other;

$$\bar{\lambda}^1_+(a(x)) \prod_{j=1}^{l_1} \bar{\lambda}^1_+(a(x+r_j+1)) \prod_{k=1}^{l_2} \bar{\lambda}^1_-(a(x+s_k+1))$$
$$\longleftrightarrow \bar{\lambda}^1_-(a(x)) \prod_{j=1}^{l_1} \bar{\lambda}^1_-(a(x-r_j-1)) \prod_{k=1}^{l_2} \bar{\lambda}^1_+(a(x-s_k-1)).$$

Consequently, taking the configuration of environment into consideration, we see that if any constant terms appear in $L\varphi_+(x,\mathbf{a})$ and $L\varphi_-(x,\mathbf{a})$, they are the same. Thus it follows that

$$\int_{E^{\mathbb{Z}^1}} L\varphi(x,\mathbf{a})\boldsymbol{\mu}(d\mathbf{a}) = \int_{E^{\mathbb{Z}^1}} \left\{L\varphi_+(x,\mathbf{a}) - L\varphi_-(x,\mathbf{a})\right\} \boldsymbol{\mu}(d\mathbf{a}) = 0$$

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Hence from Sublemma 1,

$$\tilde{\varphi}(x, \mathbf{a}) = -\int_0^\infty E_{\mathbf{a}}\left[L\varphi(x, \mathbf{a}_t)\right] dt$$

is well defined. Now set

$$\tilde{\varphi}_\pm(x, \mathbf{a}) = -\int_0^\infty E_{\mathbf{a}}\left[L\varphi_\pm(x, \mathbf{a}_t) - \int_{E^{\mathbb{Z}^1}} L\varphi_\pm(x, \mathbf{a})\boldsymbol{\mu}(d\mathbf{a})\right] dt.$$

Obviously $\tilde{\varphi}(x, \mathbf{a}) = \tilde{\varphi}_+(x, \mathbf{a}) - \tilde{\varphi}_-(x, \mathbf{a})$. Since $L\varphi_+$ and $L\varphi_-$ are interchangeable, so are $\tilde{\varphi}_+$ and $\tilde{\varphi}_-$ by virtue of Sublemma 1.

Proof of Claim 2. Note that

$$\begin{aligned} L\varphi(x, \mathbf{a}) =& \bar{\lambda}^1_+(a(x))\varphi(x+1, \mathbf{a}) + \bar{\lambda}^1_-(a(x))\varphi(x-1, \mathbf{a}) \\ &+ \lambda^{1,0}\varphi(x+1, \mathbf{a}) + \lambda^{1,0}\varphi(x-1, \mathbf{a}) - 2\lambda^{1,0}\varphi(x, \mathbf{a}). \end{aligned}$$

Combining this with Sublemma 1, it is easy to see that (13) holds.

We now complete the proof of Lemma 3.

The existence of $\{\varphi_n\}$ follows from repeated use of Claim 1 started with $\varphi_0(x, \mathbf{a}) = -\frac{1}{2\alpha}\left\{\bar{\lambda}^1_+(a(x)) - \bar{\lambda}^1_-(a(x))\right\}$. As for the upper bounds, we apply (13) with $\|\varphi_0\| \le \frac{p_1+q_1}{\alpha}$.

Back to the proof of Theorem 2, we first assume $\frac{5(p_1+q_1)}{\alpha} < 1$. Then $\sum_{n=0}^{\infty}\varphi_n(x, \mathbf{a})$ is convergent uniformly in $(x, \mathbf{a})$.

Since $|P_n(x, \mathbf{a})| \le 4\max(p_1, q_1) \cdot \sup|\varphi_n(x, \mathbf{a})| \cdot (x^{-2} + O(x^{-3}))$, the conditions (8), (9) are fulfilled provided that $\alpha > 13(p_1 + q_1)$. Under this condition, $\sum_{n=0}^{\infty}\varphi_n(x, \mathbf{a})$ belongs to the domain of G (See Liggett [3] for its definition.), since $\varphi_n(x, \mathbf{a})$ depends only on at most $(2n+1)$ distinct points in $\mathbb{Z}^1$. This makes the computation for ΔV_2 rigorous.

To complete the proof of Theorem 2, we introduce

$$\begin{aligned} \tau^{C_3} &= \inf\{t > 0\,;\, |X_t| \le C_3\} \\ \sigma^N &= \inf\{t > 0\,;\, |X_t| \ge N\}, \quad N > C_3. \end{aligned}$$

Apply Dynkin's formula to get

$$E_{(x,\mathbf{a})}V_2(X_{\tau_{C_3}\wedge\sigma_N}, \mathbf{a}_{\tau_{C_3}\wedge\sigma_N}) \le V_2(x, \mathbf{a})$$

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for $C_3 < |x| < N$. Therefore for $C_3 < |x| < N$,

$$\Big\{\inf_{\substack{|x|\geq N\\ \mathbf{a}\in E^{\mathbb{Z}^1}}} V_2(x,\mathbf{a})\Big\}\cdot P_{(x,\mathbf{a})}(\tau_{C_3} > \sigma_N) + \Big\{\inf_{\substack{|x|\leq C_3\\ \mathbf{a}\in E^{\mathbb{Z}^1}}} V_2(x,\mathbf{a})\Big\}\cdot P_{(x,\mathbf{a})}(\tau_{C_3} < \sigma_N) \leq V_2(x,\mathbf{a}).$$

Notice that $P_{(x,\mathbf{a})}(\tau_{C_3} > \sigma_N) + P_{(x,\mathbf{a})}(\tau_{C_3} < \sigma_N) = 1$. Letting $N \uparrow \infty$, we obtain

$$P_{(x,\mathbf{a})}(\tau_{C_3} < +\infty) = 1.$$

Thus Theorem 2 is valid. ∎

[3] *Proof of Theorem* 3. We shall find a function $V_3(x,\mathbf{a})$, $(x,\mathbf{a}) \in \mathbb{Z}^d \times E^{\mathbb{Z}^d}$ such that

$$V_3(x,\mathbf{a}) \geq 0 \quad \text{and} \quad \Delta V_3(x,\mathbf{a}) \leq 0 \tag{14}$$

for any $|x| \geq C_4$ and $\mathbf{a} \in E^{\mathbb{Z}^d}$ with a positive constant C_4 and

$$\lim_{|x|\to+\infty} V_3(x,\mathbf{a}) = 0. \tag{15}$$

For a given constant $k \in (0, d-2)$, set $g_3(x) = r^{-k}$, $r^2 = \frac{x_1^2}{\lambda^{1,0}} + \frac{x_2^2}{\lambda^{2,0}} + \cdots + \frac{x_d^2}{\lambda^{d,0}}$, $x = (x_1, \ldots, x_d)$ and $g_3^i(x) = g_3(x+e_i) - g_3(x-e_i)$. A Lyapunov function V_3 as above will be constructed in the following form,

$$V_3(x,\mathbf{a}) = g_3(x) + \sum_{i=1}^{d} g_3^i(x) \sum_{n=0}^{\infty} \varphi_n^{d,i}(x,\mathbf{a}). \tag{16}$$

By a heuristic computation,

$$\Delta V_3(x,\mathbf{a}) = Lg_3(x,\mathbf{a}) + \sum_{i=1}^{d} g_3^i(x) \sum_{n=0}^{\infty} G\varphi_n^{d,i}(x,\mathbf{a}) + \sum_{i=1}^{d}\sum_{n=0}^{\infty} L(g_3^i\varphi_n^{d,i})(x,\mathbf{a}).$$

On the other hand,

$$Lg_3(x,\mathbf{a}) = -\frac{k(d-k-2)}{r^{k+2}} + O(r^{-k-3}) + \sum_{i=1}^{d} \bar{\lambda}_+^i(a(x))g_3^i(x)$$

with $\bar{\lambda}_+^i(a) = \lambda_+^i(a) - \lambda^{i,0}$ and

$$L(g_3^i\varphi_n^{d,i})(x,\mathbf{a}) = L\varphi_n^{d,i}(x,\mathbf{a}) + Q_n^i(x,\mathbf{a})$$

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where

$$Q_n^i(x,\mathbf{a}) = \sum_{j=1}^{d} \Big\{ \lambda_+^j(a(x))(g_3^i(x+e_j) - g_3^i(x))\varphi_n^{d,i}(x+e_j,\mathbf{a}) + \lambda_-^j(a(x))(g_3^i(x-e_j) - g_3^i(x))\varphi_n^{d,i}(x-e_j,\mathbf{a}) \Big\}.$$

Note that

$$|Q_n^i(x,\mathbf{a})| \le \sup_{x,\mathbf{a},i} \left|\varphi_n^{d,i}(x,\mathbf{a})\right| \cdot \left(\frac{C_5}{r^{k+2}} + O(r^{-k-3}) \right)$$

with a positive constant C_5.
Thus we have

$$\begin{aligned} \Delta V_3(x,\mathbf{a}) = & -\frac{k(d-k-2)}{r^{k+2}} + O(r^{-k-3}) \\ & + \sum_{i=1}^{d} g_3^i(x) \Big\{ \bar{\lambda}_+^i(a(x)) + G\varphi_0^{d,i}(x,\mathbf{a}) \Big\} \\ & + \sum_{i=1}^{d} g_3^i(x) \sum_{n=0}^{\infty} \Big\{ L\varphi_n^{d,i}(x,\mathbf{a}) + G\varphi_{n+1}^{d,i}(x,\mathbf{a}) \Big\} \\ & + \sum_{i=1}^{d} \sum_{n=0}^{\infty} Q_n^i(x,\mathbf{a}) \end{aligned}$$

As in the proof of Theorem 2, the following lemma is a key to the proof of Theorem 3.

LEMMA 4. *The following d-systems of equations;*

$$\begin{cases} \bar{\lambda}_+^i(a(x)) + G\varphi_0^{d,i}(x,\mathbf{a}) = 0 \\ L\varphi_n^{d,i}(x,\mathbf{a}) + G\varphi_{n+1}^{d,i}(x,\mathbf{a}) = 0, \quad n \ge 0 \quad \text{and} \quad i = 1,\dots,d \end{cases}$$

have solutions $\left\{\varphi_n^{d,i}\right\}_{n=0,1,\dots}^{i=1,\dots,d}$ *such that*

$$\left|\varphi_n^{d,i}(x,\mathbf{a})\right| \le \frac{(p_i+q_i)\left\{\max\{p_j+q_j \; ; \; j=1,\dots,d\}\right\}^n (4d+1)^n}{\alpha^{n+1}}.$$

For the proof of Lemma 3, we need the next claims.

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CLAIM 3. *Let $\varphi(x, \mathbf{a})$ be a polynomial of $\bar{\lambda}^i_\pm(a(x+\cdot))$, $i = 1, \dots, d$. Suppose there exist two polynomials $\varphi_+(x, \mathbf{a})$, $\varphi_-(x, \mathbf{a})$ of $\lambda^i_\pm(a(x+\cdot))$, $i = 1, \dots, d$ such that*

(i) $\varphi(x, \mathbf{a}) = \varphi_+(x, \mathbf{a}) - \varphi_-(x, \mathbf{a})$

(ii) *by replacement between $\bar{\lambda}^i_+(a(x+p))$ and $\bar{\lambda}^i_-(a(x-p))$ $(p \in \mathbb{Z}^d$, $i = 1, \dots, d)$ we get $\varphi_-(x, \mathbf{a})$ from $\varphi_+(x, \mathbf{a})$.*

Then

$$\int_{E^{\mathbb{Z}^d}} L\varphi(x, \mathbf{a}) \boldsymbol{\mu}(d\mathbf{a}) = 0$$

and hence

$$\tilde{\varphi}(x, \mathbf{a}) = -\int_0^\infty E_{\mathbf{a}}[L\varphi(x, \mathbf{a}_t)] dt$$

is well defined. $\tilde{\varphi}(x, \mathbf{a})$ is also a polynomial of $\lambda^i_\pm(a(x+\cdot))$ and is decomposed into a difference of two polynomials as in (i) *and* (ii).

CLAIM 4. *For a polynomial ψ of $\bar{\lambda}^i_\pm(a(x+\cdot))$, $i = 1, \dots, d$ as*

$$\psi(x, \mathbf{a}) = \sum c_{r^1_1, \dots, r^1_{t_1}, s^1_1, \dots, s^1_{u_1}; r^2_1, \dots, r^2_{t_2}, s^2_1, \dots, s^2_{u_2}; \cdots; r^d_1, \dots, r^d_{t_d}, s^d_1, \dots, s^d_{u_d}} \times \prod_{j=1}^d \left(\prod_{i=1}^{t_j} \bar{\lambda}^j_+ \left(a\left(x + r^j_i\right)\right) \prod_{k=1}^{u_j} \bar{\lambda}^j_- \left(a\left(x + s^j_k\right)\right)\right)$$

where for each j, integers $r^j_1, \dots, r^j_{t_j}$, $s^j_1, \dots, s^j_{u_j}$ are distinct and summation is taken over distinct direct products $\{r^1_1, \dots, r^1_{t_1}, s^1_1, \dots, s^1_{u_1}\} \times \{r^2_1, \dots, r^2_{t_2}, s^2_1, \dots, s^2_{u_2}\} \times \cdots \times \{r^d_1, \dots, r^d_{t_d}, s^d_1, \dots, s^d_{u_d}\}$, define

$$\|\psi\| = \sum |c_{r^1_1, \dots, r^1_{t_1}, s^1_1, \dots, s^1_{u_1}; \cdots; r^d_1, \dots, r^d_{t_d}, s^d_1, \dots, s^d_{u_d}}| \times \prod_{j=1}^d \frac{|p_j - q_j|^{t_j + u_j}}{2^{t_j + u_j}}.$$

Then for φ and $\tilde{\varphi}$ in Claim 3,

$$\|\tilde{\varphi}\| \le \frac{(4d+1) \max\{p_j + q_j \ ; \ j = 1, \dots, d\}}{\alpha} \|\varphi\|.$$

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Since the proofs of Claim 3, 4 and Lemma 4 are similar to those of Claim 1, 2 and Lemma 3, they are omitted.

The rest of the proof of Theorem 3 is also worked out as in the proof of Theorem 1. Therefore we omit it, too.

[4] *Proof of Theorem* 4. As in the proof of Theorem 2, the crucial step is to construct a Lyapunov function $V_4(x, \mathbf{a})$, $(x, \mathbf{a}) \in \mathbb{Z}^2 \times E^{\mathbb{Z}^2}$ such that

$$\Delta V_4(x, \mathbf{a}) \leq 0 \quad \text{for} \quad |x| \geq C_6 \quad \text{and} \quad \mathbf{a} \in E^{\mathbb{Z}^2} \tag{17}$$

with a positive constant C_6 and

$$\lim_{|x| \to \infty} \inf_{a \in E^{\mathbb{Z}^2}} V_4(x, \mathbf{a}) = +\infty. \tag{18}$$

However we need a more elaborate argument for this case.

Putting $g_4(x) = \log^2 |x| = \log(\log |x|)$, $|x| = \sqrt{x_1^2 + x_2^2}$, a function V_4 as above will be constructed in the following form;

$$\begin{aligned} V_4(x, \mathbf{a}) = g_4(x) &+ g_4^1(x) \sum_{n=0}^{\infty} \psi_n^1(x, \mathbf{a}) + g_4^2(x) \sum_{n=0}^{\infty} \psi_n^2(x, \mathbf{a}) \\ &+ g_4^{1,1}(x) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \psi_{n,m}^1(x, \mathbf{a}) \\ &+ g_4^{1,2}(x) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \psi_{n,m}^2(x, \mathbf{a}) \end{aligned}$$

where

$$\begin{aligned} g_4^1(x) &= g_4(x + e_1) - g_4(x - e_1), \quad & g_4^2(x) &= g_4(x + e_2) - g_4(x - e_2), \\ g_4^{1,1}(x) &= \frac{2(x_2^2 - x_1^2)}{|x|^4 \log |x|}, & g_4^{1,2}(x) &= -\frac{4x_1 x_2}{|x|^4 \log |x|}. \end{aligned}$$

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A heuristic computation shows,

$$\begin{aligned}\Delta V_4(x,\mathbf{a}) &= Lg_4(x,\mathbf{a}) + \sum_{n=0}^{\infty} g_4^1(x) G\psi_n^1(x,\mathbf{a}) + \sum_{n=0}^{\infty} g_4^2(x) G\psi_n^2(x,\mathbf{a}) \\ &\quad + \sum_{n=0}^{\infty} L(g_4^1\psi_n^1)(x,\mathbf{a}) + \sum_{n=0}^{\infty} L(g_4^2\psi_n^2)(x,\mathbf{a}) \\ &\quad + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_4^{1,1}(x) G\psi_{n,m}^1(x,\mathbf{a}) + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_4^{1,2}(x) G\psi_{n,m}^2(x,\mathbf{a}) \\ &\quad + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L(g_4^{1,1}\psi_{n,m}^1)(x,\mathbf{a}) + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L(g_4^{1,2}\psi_{n,m}^2)(x,\mathbf{a}).\end{aligned}$$

It is easy to see,

$$Lg_4(x,\mathbf{a}) = L_0 g_4(x,\mathbf{a}) + L_1 g_4(x,\mathbf{a})$$

where

$$\begin{aligned}L_0 g_4(x) &= \lambda^0\{(g_4(x+e_1) - g_4(x)) + (g_4(x-e_1) - g_4(x)) \\ &\qquad + (g_4(x+e_2) - g_4(x)) + (g_4(x-e_2) - g_4(x))\} \\ &= -\frac{\lambda^0}{|x|^2(\log|x|)^2} + O\left(\frac{1}{|x|^3\log|x|}\right)\end{aligned}$$

and

$$L_1 g_4(x,\mathbf{a}) = \bar\lambda_+^1(a(x)) g_4^1(x) + \bar\lambda_+^2(a(x)) g_4^2(x)$$

with $\lambda^0 = \lambda^{1,0} = \lambda^{2,0}$ and $\bar\lambda_+^i(a) = \lambda_+^i(a) - \lambda^0$ $(i = 1, 2)$.
Furthermore,

$$\begin{aligned}L(g_4^1\psi_n^1)(x,\mathbf{a}) &= g_4^1(x) L\psi_n^1(x,\mathbf{a}) + g_4^{1,1}(x) T_1\psi_n^1(x,\mathbf{a}) \\ &\quad + g_4^{1,2}(x) T_2\psi_n^1(x,\mathbf{a}) + E_1^n(x,\mathbf{a}), \\ L(g_4^2\psi_n^2)(x,\mathbf{a}) &= g_4^2(x) L\psi_n^2(x,\mathbf{a}) + g_4^{1,2}(x) T_1\psi_n^2(x,\mathbf{a}) \\ &\quad - g_4^{1,1}(x) T_2\psi_n^2(x,\mathbf{a}) + E_2^n(x,\mathbf{a}), \\ L(g_4^{1,1}\psi_{n,m}^1)(x,\mathbf{a}) &= g_4^{1,1}(x) L\psi_{n,m}^1(x,\mathbf{a}) + E_3^{n,m}(x,\mathbf{a}), \\ L(g_4^{1,2}\psi_{n,m}^2)(x,\mathbf{a}) &= g_4^{1,2}(x) L\psi_{n,m}^2(x,\mathbf{a}) + E_4^{n,m}(x,\mathbf{a})\end{aligned}$$

where

$$\begin{aligned}T_1\psi(x,\mathbf{a}) &= \lambda_+^1(a(x))\psi(x+e_1,\mathbf{a}) - \lambda_-^1(a(x))\psi(x-e_1,\mathbf{a}), \\ T_2\psi(x,\mathbf{a}) &= \lambda_+^2(a(x))\psi(x+e_2,\mathbf{a}) - \lambda_-^2(a(x))\psi(x-e_2,\mathbf{a}),\end{aligned}$$

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and the error terms satisfy,

$$|E_1^n(x,\mathbf{a})| \le C_7 \frac{\sup\limits_{x,\mathbf{a}} |\psi_n^1|}{|x|^2(\log|x|)^2},$$
$$|E_2^n(x,\mathbf{a})| \le C_7 \frac{\sup\limits_{x,\mathbf{a}} |\psi_n^2|}{|x|^2(\log|x|)^2},$$
$$|E_3^{n,m}(x,\mathbf{a})| \le C_7 \frac{\sup\limits_{x,\mathbf{a}} |\psi_{n,m}^1|}{|x|^3\log|x|},$$
$$|E_4^{n,m}(x,\mathbf{a})| \le C_7 \frac{\sup\limits_{x,\mathbf{a}} |\psi_{n,m}^2|}{|x|^3\log|x|},$$

for $\mathbf{a} \in E^{\mathbb{Z}^2}$ and $|x| \ge 2$ with a positive constant C_7 independent of n, m. Thus we have shown,

$$\begin{aligned}
\Delta V_4(x,\mathbf{a}) &= L_0 g_4(x) + g_4^1(x)(\bar{\lambda}_+^1(a(x)) + G\psi_0^1(x,\mathbf{a})) \\
&+ g_4^2(x)(\bar{\lambda}_+^2(a(x)) + G\psi_0^2(x,\mathbf{a})) + g_4^1(x)\sum_{n=0}^{\infty}(L\psi_n^1(x,\mathbf{a}) + G\psi_{n+1}^1(x,\mathbf{a})) \\
&+ g_4^2(x)\sum_{n=0}^{\infty}(L\psi_n^2(x,\mathbf{a}) + G\psi_{n+1}^2(x,\mathbf{a})) \\
&+ g_4^{1,1}(x)\sum_{n=0}^{\infty}\Big\{(T_1\psi_n^1(x,\mathbf{a}) - T_2\psi_n^2(x,\mathbf{a}) + G\psi_{n,0}^1(x,\mathbf{a})) \\
&\qquad + \sum_{m=0}^{\infty}(L\psi_{n,m}^1(x,\mathbf{a}) + G\psi_{n,m+1}^1(x,\mathbf{a}))\Big\} \\
&+ g_4^{1,2}(x)\sum_{n=0}^{\infty}\Big\{(T_2\psi_n^1(x,\mathbf{a}) + T_1\psi_n^2(x,\mathbf{a}) + G\psi_{n,0}^2(x,\mathbf{a})) \\
&\qquad + \sum_{m=0}^{\infty}(L\psi_{n,m}^2(x,\mathbf{a}) + G\psi_{n,m+1}^2(x,\mathbf{a}))\Big\} \\
&+ \sum_{n=0}^{\infty}\Big\{E_1^n(x,\mathbf{a}) + E_2^n(x,\mathbf{a}) + \sum_{m=0}^{\infty}E_3^{n,m}(x,\mathbf{a}) + \sum_{m=0}^{\infty}E_4^{n,m}(x,\mathbf{a})\Big\}.
\end{aligned}$$

To obtain a desired Lyapunov function in (17), (18), we first give,

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LEMMA 5. *There exist two infinite sequences of solutions* $\{\psi_n^i\}_{n\geq 0}^{i=1,2}$ *for the following equations ; for* $i = 1, 2$,

$$\begin{cases} \bar{\lambda}_+^i(a(x)) + G\psi_0^i(x, \mathbf{a}) = 0 \\ L\psi_n^i(x, \mathbf{a}) + G\psi_{n+1}^i(x, \mathbf{a}) = 0, \quad n \geq 0, \end{cases}$$

which have the following upper bounds,

$$|\psi_n^i(x, \mathbf{a})| \leq \frac{9^n(p+q)^{n+1}}{\alpha^{n+1}}, \quad n \geq 0.$$

This lemma is a special case of Lemma 4.

For the solutions $\{\psi_n^i\}$ obtained in Lemma 5, we have

LEMMA 6. *For each* $n \geq 0$, *there exists an infinite sequence of solutions* $\{\psi_{n,m}^1\}_{m\geq 0}$ *for the equations*;

$$\begin{cases} T_1\psi_n^1(x, \mathbf{a}) - T_2\psi_n^2(x, \mathbf{a}) + G\psi_{n,0}^1(x, \mathbf{a}) = 0 \\ L\psi_{n,m}^1(x, \mathbf{a}) + G\psi_{n,m+1}^1(x, \mathbf{a}) = 0, \quad m \geq 0, \end{cases}$$

which have the following upper bounds,

$$|\psi_{n,m}^1(x, \mathbf{a})| \leq \frac{4 \cdot 9^{m+n}(p+q)^{m+n+2}}{\alpha^{n+m+2}}.$$

LEMMA 7. *For each* $n \geq 0$, *there exists an infinite sequence of solutions* $\{\psi_{n,m}^2\}_{m\geq 0}$ *for the equations*;

$$\begin{cases} T_2\psi_n^1(x, \mathbf{a}) + T_1\psi_n^2(x, \mathbf{a}) + G\psi_{n,0}^2(x, \mathbf{a}) = 0 \\ L\psi_{n,m}^2(x, \mathbf{a}) + G\psi_{n,m+1}^2(x, \mathbf{a}) = 0, \quad m \geq 0, \end{cases}$$

which have the following upper bounds,

$$|\psi_{n,m}^2(x, \mathbf{a})| \leq \frac{4 \cdot 9^{m+n}(p+q)^{m+n+2}}{\alpha^{n+m+2}}.$$

For the proof of the above three lemmas, the following is also necessary.

SUBLEMMA 2. *For the Markov chain* a_t *on* E, *we have*

$$E_{a_0}\left[\bar{\lambda}_{j_1}^1(a_t)\bar{\lambda}_{j_2}^2(a_t)\right] = e^{-\alpha t}\bar{\lambda}_{j_1}^1(a_0)\bar{\lambda}_{j_2}^2(a_0)$$

with $j_1, j_2 = +$ *or* $-$.

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Proof of Lemma 6. From the definitions of T_i and $\psi_n^i(x, \mathbf{a})$, $i = 1, 2$, $n \geq 0$, we can easily see that $T_1\psi_n^1(x, \mathbf{a})$ and $T_2\psi_n^2(x, \mathbf{a})$ interchange each other through replacement;

$$(19) \qquad \begin{cases} \bar{\lambda}_+^1(a(x + \alpha e_1 + \beta e_2)) \longleftrightarrow \bar{\lambda}_+^2(a(x + \beta e_1 + \alpha e_2)) \\ \bar{\lambda}_-^1(a(x + \alpha e_1 + \beta e_2)) \longleftrightarrow \bar{\lambda}_-^2(a(x + \beta e_1 + \alpha e_2)), \end{cases}$$

$\alpha, \beta \in \mathbb{Z}$. Taking this into account, the next claim implies that the assertion of Lemma 6 is valid.

CLAIM 5. *Let φ be a polynomial of $\bar{\lambda}_\pm^1(a(x+\cdot))$, $\bar{\lambda}_\pm^2(a(x+\cdot))$. Suppose there exist two polynomials φ_+ and φ_- of $\bar{\lambda}_\pm^1(a(x + \cdot))$, $\bar{\lambda}_\pm^2(a(x + \cdot))$ such that*

(i) $\varphi(x, \mathbf{a}) = \varphi_+(x, \mathbf{a}) - \varphi_-(x, \mathbf{a})$

(ii) *by the replacement (19), we get $\varphi_-(x, \mathbf{a})$ from $\varphi_+(x, \mathbf{a})$.*

Then

$$\int_{E^{\mathbb{Z}^2}} L\varphi(x, \mathbf{a})\boldsymbol{\mu}(d\mathbf{a}) = 0,$$

and hence

$$\tilde{\varphi}(x, \mathbf{a}) = -\int_0^\infty E_{\mathbf{a}}[L\varphi(x, \mathbf{a}_t)]dt$$

is well defined. $\tilde{\varphi}(x, \mathbf{a})$ *is also a polynomial as in* (i), (ii). *And we have for the above φ, $\tilde{\varphi}$,*

$$\|\tilde{\varphi}\| \leq \frac{9}{\alpha}(p + q)\|\varphi\|$$

with respect to the same norm as in Claim 4.

The proof of Claim 5 and the rest of the proof of Lemma 6 are basically the same as before. Therefore we omit the details. However we only note that under the assumption (ii) in Claim 5, $L\varphi_+$ and $L\varphi_-$ interchange each

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other through the following replacement;

$$\begin{aligned}(20)\qquad & \lambda^1_{\underset{(-)}{+}}(a(x)) \prod_{j_1=1}^{t_1} \bar{\lambda}^1_+ \left(a\left(x+(\alpha^1_{j_1} \underset{(-)}{+} 1)e_1+\beta^1_{j_1}e_2\right)\right) \\ & \times \prod_{j_2=1}^{t_2} \bar{\lambda}^1_- \left(a\left(x+(\alpha^2_{j_2} \underset{(-)}{+} 1)e_1+\beta^2_{j_2}e_2\right)\right) \\ & \times \prod_{j_3=1}^{t_3} \bar{\lambda}^2_+ \left(a\left(x+(\alpha^3_{j_3} \underset{(-)}{+} 1)e_1+\beta^3_{j_3}e_2\right)\right) \\ & \times \prod_{j_4=1}^{t_4} \bar{\lambda}^2_- \left(a\left(x+(\alpha^4_{j_4} \underset{(-)}{+} 1)e_1+\beta^4_{j_4}e_2\right)\right) \\ & \longleftrightarrow \\ & \lambda^2_{\underset{(-)}{+}}(a(x)) \prod_{j_1=1}^{t_1} \bar{\lambda}^2_+ \left(a\left(x+\beta^1_{j_1}e_1+(\alpha^1_{j_1} \underset{(-)}{+} 1)e_2\right)\right) \\ & \times \prod_{j_2=1}^{t_2} \bar{\lambda}^2_- \left(a\left(x+\beta^2_{j_2}e_1+(\alpha^2_{j_2} \underset{(-)}{+} 1)e_2\right)\right) \\ & \times \prod_{j_3=1}^{t_3} \bar{\lambda}^1_+ \left(a\left(x+\beta^3_{j_3}e_1+(\alpha^3_{j_3} \underset{(-)}{+} 1)e_2\right)\right) \\ & \times \prod_{j_4=1}^{t_4} \bar{\lambda}^1_- \left(a\left(x+\beta^4_{j_4}e_1+(\alpha^4_{j_4} \underset{(-)}{+} 1)e_2\right)\right).\end{aligned}$$

Proof of Lemma 7. From the definition of ψ^1_n, $\psi^1_n(x+e_2,\mathbf{a})$ and $\psi^1_n(x-e_2,\mathbf{a})$ interchange one another through replacement;

$$(21)\qquad \begin{cases}\bar{\lambda}^1_{\underset{(-)}{+}}(a(x+\alpha e_1+\beta e_2)) \longleftrightarrow \bar{\lambda}^1_{\underset{(-)}{+}}(a(x+\alpha e_1-\beta e_2)) \\ \bar{\lambda}^2_+(a(x+\alpha e_1+\beta e_2)) \longleftrightarrow \bar{\lambda}^2_-(a(x+\alpha e_1-\beta e_2)),\end{cases}$$

so do $\lambda^2_+(a(x))\psi^1_n(x+e_2,\mathbf{a})$ and $\lambda^2_-(a(x))\psi^1_n(x-e_2,\mathbf{a})$.

Similarly we see that $\lambda^1_+(a(x))\psi^2_n(x+e_1,\mathbf{a})$ and $\lambda^1_-(a(x))\psi^2_n(x-e_1,\mathbf{a})$ interchange one another through replacement;

$$(22)\qquad \begin{cases}\bar{\lambda}^2_{\underset{(-)}{+}}(a(x+\alpha e_1+\beta e_2)) \longleftrightarrow \bar{\lambda}^2_{\underset{(-)}{+}}(a(x-\alpha e_1+\beta e_2)) \\ \bar{\lambda}^1_+(a(x+\alpha e_1+\beta e_2)) \longleftrightarrow \bar{\lambda}^1_-(a(x-\alpha e_1+\beta e_2)).\end{cases}$$

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This combined with the next claim implies that Lemma 7 is valid.

Claim 6. *Let* φ *be a polynomial of* $\bar{\lambda}^1_\pm(a(x+\cdot\,))$, $\bar{\lambda}^2_\pm(a(x+\cdot\,))$. *Suppose there exist two polynomials* φ_+ *and* φ_- *of* $\bar{\lambda}^1_\pm(a(x+\cdot\,))$, $\bar{\lambda}^2_\pm(a(x+\cdot\,))$ *such that*

(i) $\varphi(x,\mathbf{a}) = \varphi_+(x,\mathbf{a}) - \varphi_-(x,\mathbf{a})$

(ii) *by the replacement (20), we get* $\varphi_-(x,\mathbf{a})$ *from* $\varphi_+(x,\mathbf{a})$.

Then

$$\int_{E^{\mathbb{Z}^2}} L\varphi(x,\mathbf{a})\boldsymbol{\mu}(d\mathbf{a}) = 0,$$

hence

$$\tilde{\varphi}(x,\mathbf{a}) = -\int_0^\infty E_{\mathbf{a}}[L\varphi(x,\mathbf{a}_t)]dt$$

is well defined. $\tilde{\varphi}(x,\mathbf{a})$ *is also a polynomial as in (i), (ii). And we have for the above* φ, $\tilde{\varphi}$,

$$\|\tilde{\varphi}\| \le \frac{9}{\alpha}(p+q)\|\varphi\|,$$

with respect to the same norm $\|\cdot\|$ *as in Claim* 2.

Note that all the assertions in Claim 6 hold under another replacement (21) instead of (20).

The proof of Claim 6 is also similar to that of Claim 1. Therefore the detail is omitted.

We now complete the proof of Theorem 4. First note that by what has been proved above, the Lyapunov function V_4 satisfies all the conditions (17), (18) and that $V_4(x,\cdot\,)$ belongs to the domain of G for all sufficiently large α.

Define

$$\tau_{C_6} = \inf\{t > 0\,;\, |X_t| \le C_6\}$$

and

$$\sigma_N = \inf\{t > 0\,;\, |X_t| \ge N\}.$$

Apply Dynkin's formula to obtain,

$$E_{(x,\mathbf{a})}\left[V_4(X_{\tau_{C_6}\wedge\sigma_N}, \mathbf{a}_{\tau_{C_6}\wedge\sigma_N})\right] \le V_4(x,\mathbf{a}) \quad \text{for} \quad C_6 < |x| < N.$$

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Therefore

$$P_{(x,\mathbf{a})}(\tau_{C_6} < \sigma_N) \inf_{\substack{\mathbf{a}\in E^{\mathbb{Z}^2} \\ |x|\le C_6}} V_4(x,\mathbf{a}) + P_{(x,\mathbf{a})}(\tau_{C_6} > \sigma_N) \inf_{\substack{\mathbf{a}\in E^{\mathbb{Z}^2} \\ |x|\ge N}} V_4(x,\mathbf{a}) \le V_4(x,\mathbf{a}).$$

Letting $N \to +\infty$, we get

$$P_{(x,\mathbf{a})}(\tau_{C_6} = +\infty) = 0 \quad \text{for} \quad |x| > C_6.$$

Thus

$$P_{(x,\mathbf{a})}(\tau_{C_6} < +\infty) = 1.$$

On the other hand, it is easy to show

$$\inf_{\substack{|x|\le C_6 \\ \mathbf{a}\in E^{\mathbb{Z}^2}}} P_{(x,\mathbf{a})}(X_t = 0 \quad \text{for some} \quad t > 0) > 0.$$

This combined with an argument in the proof of Lemma 1.1, [2] shows

$$P_{(x,\mathbf{a})}(X_t = 0 \quad \text{for some} \quad t > 0) = 1.$$

Thus the proof of Theorem 4 is complete.

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