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ON PARTLY BILATERAL AND PARTLY UNILATERAL GENERATING FUNCTIONS

M. A. PATHAN AND YASMEEN[1]

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Abstract

The purpose of this work is to begin the development of a theory of generating functions that will not only include the generating functions which are partly bilateral and partly unilateral but also provide a set of expansions, by taking successive partial derivatives with respect to one of the variables of the generating relations. Our starting point is a result of Exton [4] on associated Laguerre polynomials whose application gives certain generating functions of the polynomials of Jacobi and Appell, and functions of n variables of Lauricella.

1. Introduction

An interesting double generating function for the associated Laguerre polynomials $L_n^{(m)}(x)$ was given by Exton [4; p. 147 (3)]

$$\exp(s + t - xt/s) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} s^m t^n F_n^m(x), \tag{1.1}$$

where $F_n^m(x) = {}_1F_1[-n; m + 1; x]/m!n! = L_n^{(m)}(x)/(m + n)!$.

The right hand side member of (1.1) is partly bilateral and partly unilateral.

The definition of $F_n^m(x)$ associated with Exton's result (1.1) can be modified by defining $m^* = \max\{0, -m\}$ and

$$F_n^m(x) = L_n^{(m)}(x)/(m + n)! = \frac{1}{n!} \sum_{r=m^*}^{n} \frac{(-n)_r x^r}{(m + r)!r!} \quad \text{if } n \geqslant m^*,$$

$$= 0 \quad \text{if } 0 \leqslant n < m^* \text{ (that is, if } m + n < 0 \leqslant n).$$

[1] Department of Mathematics, Aligarh Muslim University, Aligarh 202 001, India.

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No factorials of negative integers occur in this definition, so all the terms have meaning. (1.1) can now be rewritten in more enlightening form

$$\exp(s + t - xt/s) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} s^m t^n F_n^m(x) \tag{1.2}$$

by using the modified definition of $F_n^m(x)$.

The purpose of this note is to introduce the equation (1.2) as the main working tool to develop a theory of generating functions of special functions which are partly bilateral and partly unilateral. These generating relations also provide a set of expansions which may be obtained by taking successive partial derivatives with respect to one of the variables.

Section 2 shows how a Laplace transformation of (1.2) would yield a generating function of Jacobi polynomials $P_n^{(\alpha,\beta)}$ [6; p. 254 (1)] which is partly bilateral and partly unilateral. It serves as a motivation for section 3, which gives a number of generating functions of similar type for the Lauricella's hypergeometric function of $(k+1)$ variables $F_A^{(k+1)}$ [5; p. 41 (2.1.1)].

Associated Laguerre and Jacobi polynomials, Appell and Lauricella hypergeometric functions, their expansions and generating functions are of frequent occurence in quantum mechanics, statistics and other branches of applied mathematics. See Schiff [7; p. 84] and Exton [5; Chapters 7 and 8], for example. A great many unilateral and bilateral generating relations of special functions are known, and can be found in the literature. It seems astonishing that such simple generating functions which are partly bilateral and partly unilateral have been overlooked probably because of the nonavailability of the main working tool of the type of relation (1.2). Our note suffices to give an idea of the use of formula (1.2) and to support the contention that this work would help in obtaining the similar generating functions for other special functions.

2. Basic relations

We begin by replacing s, t and x in (1.2) by su, tu and xu respectively, multiply both the sides by u^{c-1} and take Laplace transforms with the help of the results [2; p. 137 (1)],

$$\int_0^\infty e^{-au} u^{c-1}\, du = \Gamma(c) a^{-c}, \qquad \operatorname{Re}(a) > 0, \operatorname{Re}(c) > 0,$$

and [2; p. 174 (29)]

$$\int_0^\infty e^{-au} u^{c-1} L_n^{(m)}(xu)\, du = \frac{\Gamma(c+n)}{n!} \frac{(a-x)^n}{a^{c+n}} {}_2F_1\left[\begin{matrix} -n, m-c+1; \\ 1-c-n; \end{matrix} \; \frac{a}{a-x}\right],$$

$$\operatorname{Re}(c) > 0, \operatorname{Re}(a) > 0, x > 0.$$

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Thus we arrive at the result

$$\left(\frac{a-s-t+xt/s}{a}\right)^{-c} = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(c)_{m+2n} s^m t^n (a-x)^n}{a^{m+2n}(m+n)!n!} {}_2F_1\left[\begin{matrix} -n, 1-n-c; \\ 1-m-2n-c; \end{matrix} \frac{a}{a-x}\right], \tag{2.1}$$

$\mathrm{Re}(a-s-t+xt/s) > 0$, $\mathrm{Re}(c) > 0$, $\mathrm{Re}(a) > 0$ and $x > 0$.

Equation (2.1) yields an interesting generating function for Jacobi polynomials [6; p. 254 (1)] given by

$$\left(\frac{a-s-t+xt/s}{a}\right)^{-c} = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(c)_{m+n} s^m t^n}{a^{m+n}(m+n)!} P_n^{(m,c-1)}\left(\frac{a-2x}{a}\right), \tag{2.2}$$

$\mathrm{Re}(a-s-t+xt/s) > 0$, $\mathrm{Re}(c) > 0$, $\mathrm{Re}(a) > 0$ and $x > 0$, which follows from [6; p. 255 (9)]

$$P_n^{(m,c)}(x) = \frac{(1+m+c)_{2n}}{n!(1+m+c)_n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n, -c-n; \\ -m-c-2n; \end{matrix} \frac{2}{x+1}\right].$$

For $s = t = x/2$, (2.2) reduces to

$$1 = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(c)_{m+n}(x/2a)^{m+n}}{(m+n)!} P_n^{(m,c-1)}\left(\frac{a-2x}{a}\right), \tag{2.3}$$

$\mathrm{Re}(a) > 0$, $\mathrm{Re}(c) > 0$ and $x > 0$.

Equations (2.2) establish an important formula whereby integral powers of x may be expanded as double series of Jacobi polynomials. Firstly, we notice that

$$V(x,s,t,a,c) = [(1/a)(a-s-t+xt/s)]^{-c}$$

gives $V(x, x/2, x/2, a, c) = 1$ and

$$\frac{\partial^r V}{\partial t^r} = (-1)^r (c)_r a^c \left(\frac{x}{s} - 1\right)^r (a-s-t+xt/s)^{-c-r}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} (-1)^r (-n)_r \frac{(c)_{m+n} s^m t^{n-r}}{a^{m+n}(m+n)!} P_n^{(m,c-1)}\left(\frac{a-2x}{a}\right), \tag{2.4}$$

$\mathrm{Re}(a) > 0$, $\mathrm{Re}(c) > 0$ and $x > 0$.

When $s = t = x/2$, (2.4) yields an expansion

$$x^r = \frac{(2a)^r}{(c)_r} \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(-n)_r (c)_{m+n} (x/2)^{m+n}}{a^{m+n}(m+n)!} P_n^{(m,c-1)}\left(\frac{a-2x}{a}\right), \tag{2.5}$$

for $r = 0, 1, 2, \ldots$, $\mathrm{Re}(a) > 0$, $\mathrm{Re}(c) > 0$ and $x > 0$.

This gives an effective technique for a second set of expansions which may be obtained in a similar manner by taking successive partial derivatives with respect to s of the generating relation (2.2) and letting $s = t = x/2$. We have, however, resisted the temptation of developing a general formula of these expansions as an application and a few expansions of the powers of x up to x^3 are given below

$$\frac{3cx}{2a} = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{m(c)_{m+n}(x/2)^{m+n}}{a^{m+n}(m+n)!} P_n^{(m,c-1)}\left(\frac{a-2x}{a}\right), \tag{2.6}$$

$$\frac{9c(c+1)x^2}{4a^2} - \frac{2c}{ax} = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{m(m-1)(c)_{m+n}(x/2)^{m+n}}{a^{m+n}(m+n)!} P_n^{(m,c-1)}\left(\frac{a-2x}{a}\right), \tag{2.7}$$

$$\frac{27c(c+1)(c+2)x^3}{8a^3} - \frac{9c(c+1)x^2}{a^2} + \frac{6cx}{a} = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{m(m-1)(m-2)(c)_{m+n}(x/2)^{m+n}}{a^{m+n}(m+n)!} P_n^{(m,c-1)}\left(\frac{a-2x}{a}\right), \tag{2.8}$$

where $\mathrm{Re}(a) > 0$, $\mathrm{Re}(c) > 0$ and $x > 0$.

3. Lauricella function

We shall now generalize these relations of section 2 and obtain a generating function for Lauricella's function of $(k+1)$ variables $F_A^{(k+1)}$. We recall

$$F_A^{(k)}(a, b_1, \ldots, b_k; c_1, \ldots, c_k; x_1, \ldots, x_k) = \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{(a)_{m_1+\cdots+m_k}(b_1)_{m_1}\cdots(b_k)_{m_k}}{(c_1)_{m_1}\cdots(c_k)_{m_k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_k^{m_k}}{m_k!},$$
$$|x_1| + \cdots + |x_k| < 1, \tag{3.1}$$

and [8; p. 260 (2(ii))]

$$\int_0^{\infty} e^{-pt} t^{a-1} L_{m_1}^{(\alpha_1)}(x_1 t) \cdots L_{m_k}^{(\alpha_k)}(x_k t)\, dt = \frac{\Gamma(a)}{p^a}\binom{\alpha_1+m_1}{m_1}\cdots\binom{\alpha_k+m_k}{m_k}$$
$$\times F_A^{(k)}\left[a, -m_1, \ldots, -m_k; \alpha_1+1, \ldots, \alpha_k+1; \frac{x_1}{p}, \ldots, \frac{x_k}{p}\right], \tag{3.2}$$

$\mathrm{Re}(a) > 0$, $\mathrm{Re}(p) > 0$.

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On multiplying both sides of (1.2) by $u^{c-1}L_{r_1}^{(m_1)}(x_1u)\cdots L_{r_k}^{(m_k)}(x_ku)$, replacing s, t and x by su, tu and xu and taking Laplace transforms with the help of (3.2), we get the following generating function

$$\left(\frac{a-s-t+xt/s}{a}\right)^{-c}F_A^{(k)}\Big(c,-r_1,\ldots,-r_k;m_1+1,\ldots,m_k+1;\frac{x_1}{a-s-t+xt/s},\ldots,\frac{x_k}{a-s-t+xt/s}\Big)$$
$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\binom{m+n}{n}\frac{(c)_{m+n}}{(m+n)!}\frac{s^mt^n}{a^{m+n}}F_A^{(k+1)}\Big(m+n+c,-n,-r_1,\ldots,-r_k;m+1,m_1+1;,\ldots,m_k+1;\frac{x}{a},\frac{x_1}{a},\ldots,\frac{x_k}{a}\Big),\tag{3.3}$$

$\operatorname{Re}(a-s-t+xt/s)>0$, $\operatorname{Re}(c)>0$, $|x_1|+\cdots+|x_k|<|a-s-t+xt/s|$. For $k=1$, $m_1=q$, $r_1=r$, $c=b+1$ and $x_1=y$, it reduces to

$$\left(\frac{a-s-t+xt/s}{a}\right)^{-b-1}{}_2F_1\left[\begin{matrix}-r,b+1;\\ q+1;\end{matrix}\ \frac{y}{a-s-t+xt/s}\right]$$
$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\binom{m+n}{n}\frac{(b+1)_{m+n}}{(m+n)!}\frac{s^mt^n}{a^{m+n}}F_2(m+n+b+1,-n,-r;m+1,q+1;x/a,y/a),\tag{3.4}$$

$\operatorname{Re}(a-s-t+xt/s)>0$, $\operatorname{Re}(b)>-1$, $|y|<|a-s-t+xt/s|$, where F_2 is Appell's function of two variables ([5; p. 23], see also [3; p. 224]).

Formula (3.4) is a generalization of the results (2.1) to (2.3). For example, if we put $y=0$ in (3.4) and apply the results [5; p. 215]

$${}_2F_1\left[\begin{matrix}-n,b;\\ c;\end{matrix}\ 1-x\right]=\frac{(c-b)_n}{(c)_n}{}_2F_1\left[\begin{matrix}-n,b;\\ b-c-n+1;\end{matrix}\ x\right]$$

and [5; p. 216]

$${}_2F_1\left[\begin{matrix}-n,b;\\ c;\end{matrix}\ x\right]=\frac{(b)_n}{(c)_n}(-x)^n{}_2F_1\left[\begin{matrix}-n,1-c-n;\\ 1-b-n;\end{matrix}\ \frac{1}{x}\right],$$

involving the hypergeometric polynomials, then we get (2.1).

In view of the definition of Jacobi polynomials [6; p. 254 (1)], equation (3.4) can be put in the form

$$\frac{r!}{(1+q)_r}\left(\frac{a-s-t+xt/s}{a}\right)^{-b-1}P_r^{(q,b-q-r)}\left(\frac{a-2y-s-t+xt/s}{a-s-t+xt/s}\right)$$
$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\binom{m+n}{n}\frac{(1+b)_{m+n}}{(m+n)!}\frac{s^mt^n}{a^{m+n}}F_2\Big(m+n+b+1,-n,-r;m+1,q+1;\frac{x}{a},\frac{y}{a}\Big).\tag{3.5}$$

Further, letting $s = t = x/2$, (3.3) gives

$$F_A^{(k)}(c, -r_1, \ldots, -r_k; m_1 + 1, \ldots, m_k + 1, x_1, \ldots, x_k)$$
$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \binom{m+n}{n} \frac{(c)_{m+n}(x/2)^{m+n}}{(m+n)!} F_A^{(k+1)}(m+n+c, -n, -r_1, \ldots, -r_k;$$
$$m+1, m_1+1, \ldots, m_k+1; x, x_1, \ldots, x_k), \qquad (3.6)$$

$\mathrm{Re}(c) > 0$, $|x_1| + \cdots + |x_k| < 1$.

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