10

Operator-product expansion

In this chapter we will investigate two closely related problems. We work with ϕ^4 theory in $d=4$ dimensions and consider a time-ordered product of two fields, $T\phi(x)\phi(0)$, together with its Fourier transform

$$T\tilde{\phi}(q)\phi(0) = \int d^4x e^{iq\cdot x} T\phi(x)\phi(0).$$

(It is easiest to work with time-ordered products. The methods work with any pair of operators $TA(x)\,B(0)$ in any theory.)

The first problem is to ask how $T\phi(x)\phi(0)$ behaves as $x^\mu \to 0$. If the theory were totally finite then the result would just be $\phi^2(0)$. However, there are ultra-violet divergences that prevent the product from existing, so the limit does not exist. It was the idea of Wilson (1969) that $\phi(x)\phi(0)$ should behave like a singular function of x times the renormalized $[\phi^2]$ operator, as $x \to 0$. The full result is that we have an expansion of the form

$$T\phi(x)\phi(0) \sim \sum_{\mathcal{O}} C_{\mathcal{O}}(x^\mu)[\mathcal{O}(0)] \tag{10.0.1}$$

as $x \to 0$. Here the sum is over a set of local renormalized composite fields $[\mathcal{O}]$ and the $C_{\mathcal{O}}(x)$'s are c-number functions. This formula, or one of its generalizations, is called an operator product expansion (OPE), and the coefficients $C_{\mathcal{O}}$ are often called Wilson coefficients. Corrections to (10.0.1) are smaller by a power of x^2 than the terms given.

The second problem we wish to treat is the behavior of $T\tilde{\phi}(q)\phi(0)$ as $|q^2| \to \infty$. More precisely we will consider the momentum-space Green's function

$$\tilde{G}_{N+2} = \langle 0|T\tilde{\phi}(q)\phi(0)\tilde{\phi}(p_1)\dots\tilde{\phi}(p_N)|0\rangle, \tag{10.0.2}$$

when $q^\mu \to \infty$ along a fixed direction with $p_1, \dots, p_N$ fixed. In other words we scale the invariants $q^2 \to \kappa^2 q^2$, $p_i\cdot q \to \kappa p_i\cdot q$. There is an operator product expansion

$$\tilde{G}_{N+2} \sim \sum_{\mathcal{O}} \tilde{C}_{\mathcal{O}}(q)\langle 0|T\mathcal{O}(0)\tilde{\phi}(p_1)\dots\tilde{\phi}(p_N)|0\rangle. \tag{10.0.3}$$

The relation between the coordinate-space and momentum-space expan-

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

sions is elementary. Let us take the Fourier transform from x to q of a momentum-space Green's function of $T\phi(x)\phi(0)$. Then the large-q behavior is dominated by the singularities in x-space. The only relevant singularity is at $x = 0$. So $\tilde{C}_{\mathscr{O}}(q)$ is the large-q part of the Fourier transform of $C_{\mathscr{O}}(x)$. Conversely if one Fourier transforms $\tilde{G}_{N+2}$ to get

$$\langle 0|T\phi(x)\phi(0)\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_N)|0\rangle = \int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\tilde{G}_{N+2}, \qquad (10.0.4)$$

then the limit $x \to 0$ fails to exist if $\tilde{G}_{N+2}$ falls only as $1/q^4$ or slower as $q \to \infty$.

Thus knowing the large-q behavior is equivalent to knowing the singular small-x behavior, but the coordinate-space expansion also includes information on the leading non-singular part of the small-x region.

These expansions have a number of uses, particularly in an asymptotically free theory. There the perturbation theory when improved by the renormalization group gives an effective method of computing the Wilson coefficients. Among the uses are the following:

(1) The expansion (10.0.1) in coordinate space provides a definition of renormalized composite operators that does not involve any regularization (Brandt (1967)).
(2) Although there is no physically important process which directly uses the limit taken in the momentum-space expansion (10.0.3), it is used indirectly for deep-inelastic scattering of a lepton on a hadron. This involves a matrix element of the form
$$\langle p|\tilde{j}(q)j(0)|p\rangle.$$
Here $q^2 \to -\infty$, but with the ratio $q^2/q\cdot p$ fixed instead of $q^2/q\cdot p^2$ fixed. A dispersion relation relates this case to the limit used in (10.0.3), so the OPE is used indirectly, as we will see in Chapter 14.
(3) The form and the method of proof of short-distance operator-product expansion can be generalized to handle many interesting high-energy scattering processes. (See Buras (1981) and Mueller (1981) for a review.) The results in the present chapter form a prototype for these other results.

10.1 Examples

10.1.1 *Cases with no divergences*

We will mainly restrict our attention to Green's functions of $\phi(x)\phi(0)$ in which both $\phi(x)$ and $\phi(0)$ are connected to other external lines. This is the

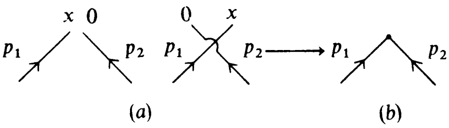

Fig. 10.1.1. Lowest-order graphs for operator-product expansion of $\phi(x)\phi(0)$.

case that is relevant to most applications. Our theory will be ϕ^4 theory

$$\mathscr{L} = \partial\phi^2/2 - m^2\phi^2/2 - g\phi^4/24 + \text{counterterms.} \tag{10.1.1}$$

First consider the tree graphs Fig. 10.1.1(*a*) for

$$\langle 0|T\phi(x)\phi(0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|0\rangle. \tag{10.1.2}$$

These give

$$\frac{\text{i}}{p_1^2 - m^2}\frac{\text{i}}{p_2^2 - m^2}[\exp(-\text{i}p_1\cdot x) + \exp(-\text{i}p_2\cdot x)]. \tag{10.1.3}$$

Expansion in a power series about $x = 0$ gives

$$\frac{\text{i}}{p_1^2 - m^2}\frac{\text{i}}{p_2^2 - m^2}[2 - \text{i}(p_1 + p_2)\cdot x - (p_1\cdot x^2 + p_2\cdot x^2)/2 + \cdots]. \tag{10.1.4}$$

This is equivalent to the replacement

$$T\phi(x)\phi(0) = \phi^2(0) + \tfrac{1}{2}x^\mu\partial_\mu\phi^2 + \tfrac{1}{2}x^\mu x^\nu\phi\partial_\mu\partial_\nu\phi + \cdots, \tag{10.1.5}$$

as illustrated in Fig. 10.1.1(*b*). This equation has the form of the operator-product expansion (10.0.1).

Thus the operator-product expansion in this case (free-field theory) is really a Taylor expansion of $\phi(x)$ about $x = 0$. The power of x in each term is just such that no dimensional coefficients are needed:

$$C_\mathscr{O}(x) = \text{constant} \times |x|^a, \text{ with } a = \dim(\mathscr{O}) - \dim[\phi(x)\phi(0)]. \tag{10.1.6}$$

This result also correctly gives the power-law behavior in the presence of renormalizable interactions, as we will see. But there will also be logarithmic corrections.

A feature which does not appear to survive inclusion of interactions is that the series on the right of (10.1.4) is convergent and sums to give $T\phi(x)\phi(0)$.

Consider next the graphs of Fig. 10.1.2 for the four-point function of $T\phi(x)\phi(0)$. The important factor comes from the lines carrying momentum q:

$$\text{i}g^2\int\frac{\text{d}^4q}{(2\pi)^4}\frac{\{\exp(-\text{i}q\cdot x) + \exp[\text{i}(q - p_A - p_B)\cdot x]\}}{(q^2 - m^2)[(p_A - q)^2 - m^2][(q - p_A - p_B)^2 - m^2]}. \tag{10.1.7}$$

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

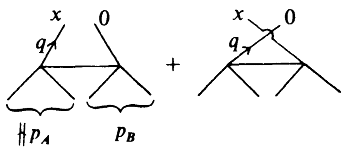

Fig. 10.1.2. Higher-order graphs for operator-product expansion of $\phi(x)\phi(0)$.

We now expand the integrand in powers of x to obtain

$$(10.1.7) \sim ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{2 - i(p_A + p_B)\cdot x}{(q^2 - m^2)[(p_A - q)^2 - m^2][(q - p_A - p_B)^2 - m^2]} + \cdots. \tag{10.1.8}$$

These first two terms are just those we would expect from (10.1.5). But the higher terms have at least two extra powers of q in the numerator and are therefore ultra-violet divergent. The divergences are those of the Green's function of the composite operators. They indicate that modification of the higher terms of the expansion is needed. For example, the behavior of the coefficient of $\phi\partial_\mu\partial_\nu\phi$ is modified by a logarithm of x.

Similar modifications will be needed for the coefficient of ϕ^2, when we consider higher-order corrections. Therefore we will find it convenient just to restrict our attention to the leading-power behavior, corresponding to the ϕ^2 term in (10.1.5).

10.1.2 *Divergent example*

Aside from trivial propagator corrections the contribution of order g to the two-point function of $T\phi(x)\phi(0)$ is given by Fig. 10.1.3(a), which gives

$$\frac{i^2}{(p_1^2 - m^2)(p_2^2 - m^2)} ig \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq\cdot x}}{(q^2 - m^2)[(q - p_1 - p_2)^2 - m^2]}. \tag{10.1.9}$$

When $x \to 0$ the integral diverges logarithmically. This is a symptom of the fact that there are two important regions of q that contribute. The first is

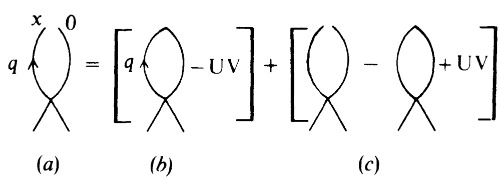

Fig. 10.1.3. Graph for operator-product expansion of $\phi(x)\phi(0)$ with divergences for the operator.

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

where q is finite as $x \to 0$; the contribution is correctly given by replacing $T\phi(x)\phi(0)$ by $\phi^2(0)$ in the graph. The second region is where q becomes large, up to $O(1/x)$ as $x \to 0$; in this region the interaction vertex in coordinate space is close to x and 0.

In the second region the loop is confined to a small region in coordinate space. From the point of view of p_1 and p_2 the loop is a point. So we should be able to represent the contribution of this region by an extra term in the Wilson coefficient of ϕ^2:

$$\left.\begin{aligned} T\phi(x)\phi(0) &\sim C_{\phi^2}(x)[\phi^2], \\ C_{\phi^2} &= 1 + (g/16\pi^2)c_1(x^2). \end{aligned}\right\} \tag{10.1.10}$$

Let us now calculate $c_1(x^2)$.

Now the contribution of the first region is given by replacing $T\phi(x)\phi(0)$ by $\phi^2(0)$. However, this operator has an ultra-violet divergence. So let us add and subtract the renormalized Green's function of $[\phi^2]$, i.e.,

$$\langle 0|T[\phi^2(0)]\tilde{\phi}(p_1)\tilde{\phi}(p_2)|0\rangle, \tag{10.1.11}$$

to give the equation depicted by Fig. 10.1.3. The contribution of order 1 from the first region is entirely contained in the term (b) representing (10.1.11):

$$\frac{\mathrm{i}^2}{(p_1^2 - m^2)(p_2^2 - m^2)} \frac{g}{16\pi^2} \left\{ \int_0^1 \mathrm{d}x \ln\left[\frac{m^2 - (p_1 + p_2)^2 x(1-x)}{4\pi\mu^2}\right] + \gamma \right\}. \tag{10.1.12}$$

Here we have used minimal subtraction. The remainder, term (c), is

$$\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)} \frac{-\mathrm{i}g}{(2\pi)^4} \times \\ \times \left\{ \int \mathrm{d}^4 q \frac{\mathrm{e}^{\mathrm{i}q\cdot x} - 1}{(q^2 - m^2)[(q - p_1 - p_2)^2 - m^2]} - \text{UV divergence} \right\}. \tag{10.1.13}$$

When $x \to 0$ the contribution from finite q is of order $|x|$. But there is a contribution of order 1 from large q: this is the contribution to the original graph minus whatever is taken care of by graph (b).

We can identify $c_1(x^2)$ in (10.1.10) as the $x \to 0$ behavior of the curly bracket factor of (10.1.13) (aside from a normalization factor), since to lowest order

$$\langle 0|T\phi^2\tilde{\phi}(p_1)\tilde{\phi}(p_2)|0\rangle = \frac{-2}{(p_1^2 - m^2)(p_2^2 - m^2)}.$$

Now the leading-power behavior of the curly-bracket factor is independent of p_1, p_2 and m. This is easily seen by differentiating with respect to any of these variables. The result is a convergent integral which goes to zero like a power of x when $x \to 0$. So we may define $c_1(x)$ by setting $m = p_1 = p_2 = 0$:

$$c_1(x) = \frac{1}{2\pi^2}\left\{\frac{i}{(2\pi\mu)^{d-4}}\int d^d q \frac{(e^{iq\cdot x} - 1)}{(q^2)^2} + \frac{2}{d-4}\right\}. \tag{10.1.14}$$

The integral is easily done by using

$$1/(q^2)^2 = \int_0^\infty dz z e^{-z(-q^2)}$$

to turn it into a Gaussian form, with the result

$$c_1(x) = \tfrac{1}{2}[\gamma + \ln(-\pi^2\mu^2x^2)]. \tag{10.1.15}$$

10.1.3 *Momentum space*

We now Fourier transform Fig. 10.1.3 to obtain the $O(g)$ contribution to

$$\langle 0|T\tilde{\phi}(q)\phi(0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|0\rangle.$$

As $q^2 \to \infty$ we find

$$\text{Fig. 10.1.3}(a) \sim \frac{-1}{(p_1^2 - m^2)(p_2^2 - m^2)}\frac{ig}{(q^2)^2} + O[1/(q^2)^3]. \tag{10.1.16}$$

This gives a contribution to the term in the operator-product expansion (10.0.3) with $\mathcal{O} = \phi^2$. The coefficient is

$$\tilde{C}_{\phi^2}(q^2) = \frac{ig}{2(q^2)^2} + O(g^2), \tag{10.1.17}$$

which is just the Fourier transform of the order g term in the coordinate-space expansion, $gc_1(x)/(16\pi^2)$. The g^0 term in the coordinate-space coefficient is independent of x, so that it gives a $\delta^{(4)}(q)$ in momentum space, and hence nothing at large q^2.

10.1.4 *Fig.* 10.1.3 *inside bigger graph*

The expansion (10.0.1) or (10.0.3) indicates that the same asymptotic behavior as $x \to 0$ (or as $q \to \infty$) is obtained independently of the Green's function considered. This happens because graphs like Fig. 10.1.3 can occur

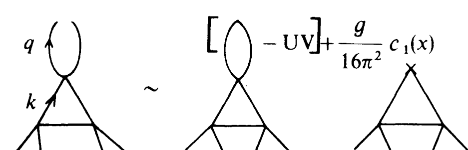

Fig. 10.1.4. Even higher-order graph for operator-product expansion of $\phi(x)\phi(0)$.

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

as subgraphs of graphs with more external lines. An example is given in Fig. 10.1.4.

10.2 Strategy of proof

First we will make precise the limits in which the operator-product expansion applies. If we are in Euclidean space (i.e., with imaginary time and energy) then there is really only one way in which we can take x^μ to zero or q^μ to infinity. However, in Minkowski space we can let $x^2 \to 0$ without each component going to zero, and we can let components of q^μ go to infinity without $q^2 \to \infty$. These cases are interesting physically. For example, the $q^\mu \to \infty$ limit with finite q^2 is the case of high-energy scattering. Much is known about these limits, but they are beyond the scope of this book.

We will prove the coordinate-space expansion (10.0.1) in the case that all components of x^μ go to zero with their ratios fixed. The corresponding momentum-space expansion (10.0.3) we will prove in the limit that all components of q^μ go to infinity with a fixed ratio, and with q^μ not light-like, so that $q^2 \to \infty$. These limits are essentially Euclidean.

Our proof will be in perturbation theory. The first step is to identify the regions of loop-momentum space that give the leading-power behavior in the $x \to 0$ or $q \to \infty$ limits. Then we generalize the arguments of the previous section, which applied to specific graphs.

The region of large q which we investigate in the momentum-space expansion (10.0.3) is precisely the one to which Weinberg's (1960) theorem applies. The theorem tells us to consider all subgraphs connected to the $\tilde{\phi}(q)$ and $\phi(0)$ in (10.0.2). For each such subgraph we let all its loop momentum be of order q, and count powers just as we did for UV divergences. The subgraph(s) with the largest power of q^2 correspond to the dominant regions of momentum space. Then as $q^2 \to \infty$, the complete graph is proportional to this power of q^2 times possible logarithms of q^2. Corrections are smaller by a power of q^2. Although Weinberg's result also tells us the highest power of $\ln(q^2)$ that appears, it is easier to determine this by first constructing the operator-product expansion and then applying renormalization-group methods (as in Section 10.5 below) to the coefficients.

Essentially the same method can be applied to obtain the short-distance behavior, (10.0.1). For example, in Fig. 10.1.4 we have leading contributions with q finite or with q large (of order $1/x$), but always with the lower loop momentum, k, finite. The leading power is $(x^2)^0$. The logarithm of x in

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

(10.1.15) for the corresponding Wilson coefficient comes from integrating over momenta intermediate between these two regions.

Suppose we have large momentum confined to a subgraph Γ. Then the power of q (for the momentum-space expansion) is exactly the dimension of Γ, since our theory is renormalizable, with dimensionless couplings. The leading power of q^2 comes from subgraphs with the largest dimension, i.e., subgraphs with the smallest possible number of external lines. This number is two (beyond $\tilde{\phi}(q)$ and $\phi(0)$), so that the leading power is $1/(q^2)^2$. The subgraphs have the form of the subgraph U in Fig. 10.2.1. In the ultra-violet subgraph U, all lines carry momentum of order q. This subgraph is 1PI in its lower two lines. All momenta in the infra-red subgraph I are finite.

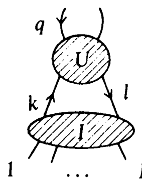

Fig. 10.2.1. General structure of leading regions of momentum space for N-point function of $\phi(x)\phi(0)$.

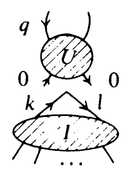

Fig. 10.2.2. Factorization of Fig. 10.2.1.

Now, to the leading power of q^2, the ultra-violet graph U is independent of the external momenta k and l flowing into it. Thus we may replace U by its value when $k = l = 0$ (and we may set the mass $m = 0$). We may also replace the infra-red subgraph I by an insertion of a vertex for $\phi^2/2$ in an N-point Green's function. This is illustrated in Fig. 10.2.2. This has the same structure as the operator-product expansion. But it should be emphasized that we are supposing that loop momenta are restricted to certain regions. These regions are not defined very precisely, and it is one of the tasks of the proof to remedy the impreciseness.

Schematically we have

$$\sum_{\text{graphs}} \tilde{G}_{N+2}(q, p_1, \ldots, p_N)$$

$$\sim \sum_U U(q, k=0, l=0) \sum_I \int_{\substack{\text{small } k \\ \text{small } l}} d^4k\, d^4l\, \delta(k + l - \sum p_i) I(k, l, p_1, \ldots, p_N). \tag{10.2.1}$$

To construct the expansion we generalize the technique that we applied to Fig. 10.1.3. We consider each graph U that could appear in Fig. 10.2.1, but we do not restrict its momenta. It can occur as a subgraph of some graph for the complete Green's function. If all momenta in U are of order q and if all

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

Fig. 10.2.3. A possible leading region for a subgraph of the form of U in Fig. 10.2.1.

momenta outside are small, then we get a leading contribution to the cross-section. We can also have a leading contribution where the large momenta occur inside a proper subgraph of U – as in Fig. 10.2.3. Suppose we subtract off all of these contributions. Then we integrate over all loop momenta of U and find that the result only gives a leading contribution when all its momenta are large. We therefore define the contribution of U to the Wilson coefficient as

$$C(U) = U - \text{subtractions for regions of form Fig. 10.2.3} \quad (10.2.2)$$

all evaluated at $k = l = m = 0$.

The resulting formula for $C(U)$ is very similar to that of the formula for renormalizing the ultra-violet divergences of a graph. In fact, as Zimmermann (1970, 1973b) explains, a good way to prove the operator-product expansion is to treat it exactly as a problem in renormalization. His method, used in the next section, is not to compute directly the Wilson coefficients but to define first a quantity which is a Green's function minus the leading terms in its operator-product expansion:

$$\tilde{G}_{N+2} - \sum_{\mathcal{O}} C_{\mathcal{O}} \tilde{G}_{\mathcal{O},N}.$$

This is constructed as a sum over graphs Γ for $\tilde{G}_{N+2}$. Each graph has subtracted from it not only counterterms to remove ultra-violet divergences but also counterterms to cancel the large-Q (or small-x) behavior. The result we call $R_W(\Gamma)$.

Now, the subtractions that remove the large-Q behavior are a sort of oversubtraction. So $R_W(\Gamma)$ is simply related to $R(\Gamma)$ in the style of a renormalization-group transformation. This transformation can then be written as the Wilson expansion, as we will see.

A disadvantage of Zimmermann's proof is that it uses BPHZ renormalization. He takes advantage of certain short-cuts available through the use of zero-momentum subtractions. We will choose not to take these shortcuts so that minimal subtraction can be applied to ultra-violet divergences. Our method of proof will be essentially the same as the one used for problems with large masses (Chapter 8).

The same techniques apply to the coordinate-space expansion. Here, the

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

momentum q is integrated over. When we do power-counting, large momenta are regarded as of order $1/x$ and small momenta as finite when $x \to 0$. The leading behavior again comes from graphs of the form of U in Fig. 10.2.1, and the power is $(x^2)^0$. There is a difference in the form of the possible graphs U that carry large momentum. All the graphs that we use in the momentum-space case are also used in coordinate space. But we can also have the graph consisting of the vertices for $\phi(x)$ and $\phi(0)$ and of nothing else.

10.3 Proof

We must now prove the operator product expansion. In ϕ^4 theory we consider the part of the Green's function,

$$\bar{G}_{N+2}(x, p_1, \ldots, p_N) = \langle 0 | T\phi(x)\phi(0)\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_N)|0\rangle, \qquad (10.3.1)$$

in which each of $\phi(x)$ and $\phi(0)$ is connected to some of the other external lines. We will now scale x by a factor κ and construct a decomposition of the form:

$$\begin{aligned}&\bar{G}_{N+2}(\kappa x, p_1, \ldots, p_N) \\ &\quad = C(\kappa^2 x^2)\langle 0 | T\tfrac{1}{2}[\phi^2](0)\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_N)|0\rangle + r_{N+2}(\kappa x, p_1, \ldots, p_N).\end{aligned} \qquad (10.3.2)$$

In every order of perturbation theory, the coefficient $C(\kappa^2 x^2)$ behaves like $(\kappa^2)^0$ times logarithms, when the scaling parameter κ goes to zero, while the remainder goes to zero like a power of κ.

Fourier transformation on x gives the result

$$\begin{aligned}&\tilde{G}_{N+2}(q/\kappa, p_1, \ldots, p_N) \\ &\quad = \tilde{C}(q^2/\kappa^2)\langle 0 | T\tfrac{1}{2}[\phi^2]\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_N)|0\rangle + \tilde{r}_{N+2}(q/\kappa, p_1, \ldots, p_N).\end{aligned} \qquad (10.3.3)$$

When $q/\kappa \to \infty$, $\tilde{C}(q^2/\kappa^2)$ behaves like κ^4/q^4 times logarithms, while $\tilde{r}_{N+2}(q/\kappa, p_1, \ldots)$ is smaller by a power.

Our proof is given for a specific Green's function in a specific theory. However, it can easily be generalized. Features specific to a gauge theory will be pointed out in Chapter 12. The particular case of QCD with the application to deep-inelastic scattering will be treated in Chapter 14.

10.3.1 *Construction of remainder*

We consider the set of graphs for G_{N+2}. For each graph Γ, we will construct its contribution $r(\Gamma)$ to the remainder $\tilde{r}_{N+2}$. Each graph Γ we consider to be an unrenormalized (but regulated) graph containing only basic interaction vertices. These are derived from

$$\mathscr{L}_{\text{basic}} = \partial\phi^2/2 - m^2\phi^2/2 - \mu^{4-d}g\phi^4/24,$$

where g and m are the renormalized coupling and mass.

As usual the renormalized value of the graph is $R(\Gamma)$, which is Γ, plus a series of counterterm graphs to cancel its ultra-violet divergences. For our proof we will use a renormalization prescription in which the theory is finite when the renormalized mass m is set to zero. The renormalized value $R(\Gamma)$ is then the contribution of Γ to the Green's function $\tilde{G}_{N+2}$.

The remainder term $r(\Gamma)$ is equal to Γ plus a somewhat different series of counterterm graphs. These counterterms will be constructed so that they cancel not only the ultra-violet divergences but also the leading $x \to 0$ or $q \to \infty$ behavior of Γ.

Now $r(\Gamma)$ is in effect a oversubtracted form of Γ. The oversubtractions are of the form of an insertion of the operator $[\phi^2]$ times a coefficient. Thus $R(\Gamma) - r(\Gamma)$ is the Wilson expansion, i.e., the first term on the right of (10.3.2) or (10.3.3).

The coefficient $C(x^2)$ (or $\tilde{C}(q^2)$) depends on the coupling g and on the renormalization mass μ. It must be independent of all the momenta. In order to be able to neglect m in the ultra-violet limit $x \to 0$ or $q \to \infty$, we must use a renormalization prescription in which the counterterms do not become infinite when $m \to 0$ (with fixed regulator). For concreteness we will use minimal subtraction in what follows.

In order to define $r(\Gamma)$, let us recall the definition of the ordinary renormalization $R(\Gamma)$. This starts from the fact that the divergences of Γ come from regions of loop momenta where all lines in some set of 1PI subgraphs carry a momentum that approaches infinity. We label each region by the subgraph consisting of all the lines carrying large momentum. Then

$$R(\Gamma) = \Gamma - \sum_{\gamma} C_\gamma(\Gamma). \tag{10.3.4}$$

The sum is over all subgraphs γ of Γ, and $C_\gamma(\Gamma)$ is essentially Γ with the subgraph γ replaced by its large-momentum divergence. We define $C_\gamma(\Gamma)$ to be non-zero only if γ is a disjoint union of one or more 1PI graphs $\gamma_1, \ldots, \gamma_n$. In that case each γ_i is replaced by a counterterm vertex $C(\gamma_i)$, which is the divergent part of γ_i. To avoid double-counting, the subdivergences are subtracted off first:

$$C(\gamma_i) = \mathscr{P}\left[\gamma_i - \sum_{\delta \subsetneq \gamma} C_\delta(\gamma_i)\right]. \tag{10.3.5}$$

Here '$\mathscr{P}$' denotes 'pole part at $d=4$' and the sum is over all proper subgraphs δ of γ_i. As usual, if we use some other renormalization prescription than minimal subtraction the operator $\mathscr{P}$ is replaced by the operator appropriate to the renormalization prescription.

The definition of the remainder $r(\Gamma)$ is almost identical to the definition of $R(\Gamma)$. Now, the leading short-distance (i.e., $x \to 0$) behavior of Γ comes from the following regions:

(1) where the momentum q is finite,
(2) where q gets large and the momenta in a graph of the form of U in Fig. 10.2.1 also get large.

Further leading contributions come from regions where in addition momenta get infinite in some set of divergent 1PI graphs. These extra contributions correspond to the ultra-violet divergences. In the momentum-space expansion (10.3.3) the same regions are leading except for the region of finite q.

We define $r(\Gamma)$ to be Γ with all ultra-violet divergences subtracted and then with all the leading small-x behavior subtracted:

$$r(\Gamma) = R(\Gamma) - \sum_{\delta} \sum_{\gamma} L_{\gamma \cup \delta}(\Gamma). \tag{10.3.6}$$

Here the sum over δ is over all graphs of the form of U in Fig. 10.2.1 and the sum over γ is over all subgraphs γ of Γ that do not intersect δ. We use $L_{\gamma \cup \delta}$ to symbolize a subtraction operation defined below. It is used to extract the contribution that comes from the region where the momenta in graph δ are of order $1/x$ and the momenta in γ go to infinity. The case of finite q in Fig. 10.2.1 is covered by the case that δ consists of the vertices for $\phi(x)$ and $\phi(0)$ only.

We define the subtraction $L_{\gamma \cup \delta}(\Gamma)$ to be zero unless γ is a disjoint union of 1PI $\gamma_1, \ldots, \gamma_n$. In that case each γ_i is replaced by its overall counterterm $C(\gamma_i)$ defined in (10.3.5) while the graph δ is replaced by a quantity $L(\delta)$. $L(\delta)$ is to contain the leading behavior of δ when all internal lines have large momenta. This is the same idea as that $C(\gamma_i)$ is the overall divergence of γ_i. Now there are regions where a subgraph δ' of δ carries momenta of order q and other lines in δ carry small momenta. To avoid double-counting we subtract them first. So we write:

$$L(\delta) = T\left[\delta - \sum_{\delta' \subsetneq \delta} \sum_{\gamma \cap \delta = \varnothing} L_{\gamma \cup \delta'}(\delta) \right]. \tag{10.3.7}$$

Here T is to be an operator that picks out the leading $x \to 0$ behavior of its argument. Now this behavior is independent of m, and of the finite external

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

momenta. (We prove this by differentiating with respect to the variables m and p_i^μ.) So we define T to set the values of m and the finite external momenta to zero.

We now have a complete definition of $r(\Gamma)$.

10.3.2 *Absence of infra-red and ultra-violet divergences in* $r(\Gamma)$.

In Fig. 10.1.1(*a*) the only possible graph of the form of U is the one that consists of $\phi(x)\phi(0)$ alone. We call it δ_1. Its value is $e^{-ip_1\cdot x} + e^{-ip_2\cdot x}$. It has no subgraphs so that

$$L(\delta_1) = (e^{-ip_1\cdot x} + e^{-ip_2\cdot x})_{p_1=p_2=0} = 2. \qquad (10.3.8)$$

The remainder is therefore

$$r(\text{Fig. 10.1.1}) = \text{Fig. 10.1.1} - \text{Fig. 10.1.1}(b). \qquad (10.3.9)$$

Here we regarded $L(\delta_1)$ as a $[\phi^2]$ vertex. It is manifest that this remainder is exactly the graph minus its Wilson expansion.

Now we turn to Fig. 10.1.3(*a*). It has δ_1 as a subgraph of form U and also the loop, which we call δ_2. Then by (10.3.7)

$$\begin{aligned} L(\delta_2) &= \{\delta_2 - L_{\delta_1}(\delta_2)\}_{m=p_1=p_2=0} \\ &= \left\{\frac{\mu^{4-d} ig}{(2\pi)^d}\int d^d q \frac{e^{-iq\cdot x} - 1}{(q^2 - m^2)[(p_1+p_2-q)^2 - m^2]}\right\}_{m=p_1=p_2=0} \\ &= \frac{ig\mu^{4-d}}{(2\pi)^d}\int d^d q \frac{(e^{-iq\cdot x} - 1)}{(q^2 + i\varepsilon)^2}. \end{aligned} \qquad (10.3.10)$$

Hence

$$\begin{aligned} r(\Gamma) &= R(\Gamma) - L_{\delta_1}(\Gamma) - L_{\delta_2}(\Gamma) \\ &= \frac{i^2}{(p_1^2 - m^2)(p_2^2 - m^2)}\frac{ig\mu^{4-d}}{(2\pi)^d}\left\{\int d^d q \frac{e^{-iq\cdot x}}{(q^2 - m^2)[(p_1+p_2-q)^2 - m^2]}\right. \\ &\quad \left. - \int d^d q \frac{1}{(q^2 - m^2)[(p_1+p_2-q)^2 - m^2]} - \int d^d q \frac{(e^{-iq\cdot x} - 1)}{(q^2)^2}\right\}, \end{aligned} \qquad (10.3.11)$$

where we use Γ to denote Fig. 10.1.3(*a*).

The following properties hold:

(1) $L_{\delta_2}(\Gamma)$ is infra-red convergent even though it has zero mass and zero external momentum. Although δ_2 has an infra-red divergence when m, p_1, and p_2 approach zero, the subtraction term $L_{\delta_1}(\delta_2)$ exactly cancels the divergence.

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

(2) $L_{\delta_1}(\Gamma)$ is ultra-violet divergent, since in replacing the vertex $\delta_1 = e^{-iq\cdot x}$ by 1 we remove the ultra-violet cut off. However, $L_{\delta_2}(\Gamma)$ contains all the large-q behavior of Γ and of subtractions for subgraphs of δ_2. Thus $L_{\delta_2}(\Gamma)$ cancels the ultra-violet divergence of $L_{\delta_1}(\Gamma)$.

(3) When m, p_1, p_2 approach zero, we find that

$$\begin{aligned} r(\delta_2) &\equiv R(\delta_2) - L_{\delta_1}(\delta_2) - L(\delta_2) \\ &= \delta_2 - L_{\delta_1}(\delta_2) - L(\delta_2) \\ &\to 0. \end{aligned}$$

This is just the statement that $L(\delta_2)$ is the value of δ_2 at $m = p_1 = p_2 = 0$, after subtractions on subgraphs are made.

The explanations of these properties are convoluted, but with the aim of demonstrating that they are true in general. Refer now to the general definition of $r(\Gamma)$, viz., (10.3.6), and refer to Fig. 10.2.1 instead of Fig. 10.1.3. Then the above properties get replaced by:

(1) $L(\delta)$ is infra-red convergent for any graph of form U in Fig. 10.2.1: the only regions that could give infra-red problems are cancelled by subtractions.

(2) $r(\Gamma)$ is ultra-violet and infra-red convergent if m, p_1^2 and p_2^2 are non-zero. The individual terms $L_{\gamma\cup\delta}(\Gamma)$ are IR finite. The subtractions remove all ultra-violet behavior.

(3) $r(\delta) = 0$ when $m = p_1 = p_2 = 0$.

10.3.3 $R(\Gamma) - r(\Gamma)$ *is the Wilson expansion*

Although

$$W(\Gamma) \equiv R(\Gamma) - r(\Gamma) = \sum_{\delta} L_\delta\left(\sum_{\substack{\gamma \\ \gamma\cap\delta=\varnothing}} C_\gamma(\Gamma)\right) \tag{10.3.12}$$

contains the leading $x \to 0$ behavior of $R(\Gamma)$, it is not yet in the form of the operator-product expansion, which is

$$W(\Gamma) = \sum_{\delta} \bar{C}(\delta) R(\Gamma/\delta). \tag{10.3.13}$$

Here $\bar{C}(\delta)$ is the contribution of a subgraph δ (of form U) to the Wilson coefficient, while Γ/δ is Γ with δ contracted to a point, i.e., replaced by a ϕ^2 vertex. $R(\Gamma/\delta)$ will now include pole-part subtractions for the divergent Green's functions of ϕ^2.

Summing (10.3.13) over Γ can be done by independently summing over δ and Γ/δ. This gives the operator-product expansion (first term of the right-

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

hand side of (10.0.1)) with

$$C_{\phi^2}(x^2) = \sum_{\delta} \bar{C}(\delta). \tag{10.3.14}$$

(We have used the same symbol for the complete Wilson coefficient $C_{\phi^2}(x)$ as for the contribution from a particular graph.) Hence to prove the expansion (10.0.1) or (10.0.3), we have to prove that $W(\Gamma)$ as defined by (10.3.12) equals the right-hand side of (10.3.13).

Any graph Γ for the Green's function G_{N+2} can be written in the form of Fig. 10.2.1. In general there will be several possibilities for the upper subgraph U. For the following argument we will choose U to be the biggest possible graph. Then, in (10.3.12) for $W(\Gamma)$, all the subgraphs δ are contained in U. So we have

$$\begin{aligned} W(\Gamma) &= \int \frac{d^4k d^4l}{(2\pi)^8} W(U) R(I) \\ &\equiv W(U) \otimes R(I), \end{aligned} \tag{10.3.15}$$

since no divergent 1PI subgraphs include parts of both U and I. Suppose we prove (10.3.13) when Γ is replaced by U. Then from (10.3.15) it follows that

$$\begin{aligned} W(\Gamma) &= \sum_{\delta} \bar{C}(\delta) R(U/\delta) \otimes R(I) \\ &= \sum_{\delta} \bar{C}(\delta) R(\Gamma/\delta). \end{aligned} \tag{10.3.16}$$

The last line follows because (*a*) the graphs for Γ/δ are of the form U/δ times I, and (*b*) ultra-violet divergent 1PI subgraphs are entirely within U/δ or within I. Notice that the extra divergences for Γ/δ as compared with Γ are due to the presence of the ϕ^2 vertex. These divergences are confined to graphs of the form of U/δ.

Rather than prove (10.3.13) for the case $\Gamma = U$, we use it as a *definition* of $\bar{C}(U)$, recursive in terms of $\bar{C}(\delta)$ for smaller δ:

$$2\bar{C}(U) = W(U) - \sum_{\delta \subsetneq U} \bar{C}(\delta) R(U/\delta). \tag{10.3.17}$$

The factor 2 multiplying $\bar{C}(U)$ is the lowest-order 1PI Green's function of the operator ϕ^2. It is evidently much too good to be true that we have reduced what ought to be a deep proof to a mere definition. The important physics was at the previous step, (10.3.16). By implicit use of power-counting arguments, to restrict ultra-violet divergences to within U/δ, we proved (10.3.13) given its truth for the case $\Gamma = U$. The Wilson coefficient $C_{\phi^2}(x)$ as it occurs in the expansion (10.0.1) is then the same no matter what Green's function we take of $T\phi(x)\phi(0)$. This universality is an important feature of the Wilson expansion.

The other important feature of the expansion is that the Wilson coefficient is a purely ultra-violet object. This will enable renormalization-group techniques to be useful in its calculation. For this purpose we must now prove that $\bar{C}(U)$ is independent of m, k and l.

The proof is done by differentiating the right-hand side of (10.3.17) with respect to the mass m or with respect to one of the external momenta k or l. Let Δ represent this operation. It is applied in turn to each propagator in $W(U)$ and in $R(U/\delta)$. Inductively, we assume $\bar{C}(\delta)$ satisfies $\Delta\bar{C}(\delta) = 0$. This is true for the lowest-order graph, δ_1, for which $\bar{C}(\delta_1) = 1$ follows from (10.3.8). In the general case $\Delta\bar{C}(U)$ is therefore given by

$$2\Delta\bar{C}(U) = \Delta W(U) - \sum_{\delta \subsetneq U} \bar{C}(\delta)\Delta R(U/\delta), \qquad (10.3.18)$$

as illustrated in Fig. 10.3.1. We have a series of terms in each of which one particular propagator of U is differentiated. The differentiation improves the ultra-violet behavior, so the differentiated propagator cannot be a part of a leading large-momentum region connected to the vertices $\phi(x)\phi(0)$. We therefore factor out a maximal two-particle graph δ', just as we factored out U in (10.3.15).

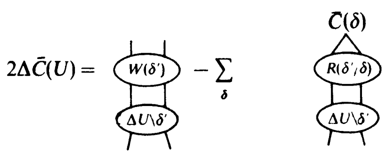

Fig. 10.3.1. Differentiation of Wilson coefficient with respect to a mass or an external momentum.

Now we can use the result

$$W(\delta') = \sum_{\delta''} \bar{C}(\delta'')R(\delta'/\delta'')$$

to give zero for $\Delta\bar{C}(U)$. (This argument is analogous to (10.3.16).) The operator-product expansion is now proved.

10.3.4 *Formula for* $\bar{C}(U)$

From the above results we may deduce

$$\bar{C}(U) = L\left[R(U) - \sum_{\delta \subsetneq U} \bar{C}(\delta)R(U/\delta)\right]. \qquad (10.3.19)$$

Heuristically this says that $\bar{C}(U)$ is that part of U arising when all its internal

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

lines have momenta of order $1/x$, less contributions taken care of by terms $\bar{C}(\delta)$ in the operator-product expansion with smaller graphs δ contained in U. The term in square brackets is the unrenormalized U minus (a) all contributions involving momenta not all of order q, and (b) contributions with all momenta of order q taken care of by the subtractions under (a). This equation reproduces exactly the calculation performed in Section 10.1 of the $O(g)$ Wilson coefficient, viz.,

$$\bar{C}(\delta_2) = [\delta_2 - \bar{C}(\delta_1)R(\delta_2/\delta_1)]_{m=p_1=p_2=0}.$$

10.4 General case

The most fundamental form of the operator-product expansion is (10.0.1), which is proved by obtaining its matrix elements from the Green's functions (10.0.2). We proved the Green's function expansion restricted to connected graphs in ϕ^4 theory, and restricted to the leading power of x^2. The only operator that then contributes is $\mathcal{O} = \phi^2$. Our proof generalizes. We may take the full Green's functions in any theory and include non-leading powers of x.

In the general case the operator L extracts the appropriate number of terms in a Taylor expansion in the mass m^2 and in the external momenta. The result is that each Wilson coefficient $C_{\mathcal{O}}(x)$ behaves in each order of perturbation theory like a power of x^2 times logarithms of $(x^2\mu^2)$ times a polynomial in x^μ and m^2 with dimensionless coefficients. If $\mathcal{O}$ is a tensor operator then the coefficient is also a tensor, as illustrated by the lowest-order example (10.1.5).

The leading operator for ϕ^4 theory is in fact the unit operator since it has lowest dimension. We have

$$T\phi(x)\phi(0) = C_1(x^2)1 + C_{\phi^2}(x^2)[\phi^2] + O(x^\mu \cdot \text{logs}). \qquad (10.4.1)$$

Operators linear in ϕ are prohibited by the $\phi \to -\phi$ symmetry.

To compute the coefficient, $C_1(x^2)$, of the unit operator to lowest order, we use the graph of Fig. 10.4.1. We will extract terms up to $O(x^0)$, so that we

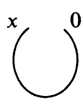

Fig. 10.4.1. Lowest-order term for $\langle 0|T\phi(x)\phi(0)|0\rangle$.

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

find:

$$\begin{aligned} C_1(x^2) &= C_1(x^2)\langle 0|1|0\rangle \\ &= \{\text{Fig. 10.4.1} - C_{\phi^2}(x^2)\langle 0|[\phi^2(0)]|0\rangle\}_{\substack{\text{expanded in powers}\\ \text{of } m^2 \text{ to order } m^2}} \\ &= \left\{ \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{i}e^{\mathrm{i}q\cdot x}}{q^2 - m^2 + \mathrm{i}\varepsilon} - \left[\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\varepsilon} \right.\right. \\ &\qquad \left.\left. - \frac{\mu^{d-4} m^2}{8\pi^2 (d-4)} \right] \right\}_{\text{expanded}} \\ &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{i}(e^{\mathrm{i}q\cdot x} - 1) \left[\frac{1}{q^2} + \frac{m^2}{(q^2)^2} \right] + \frac{\mu^{d-4} m^2}{8\pi^2(d-4)} \\ &= -\frac{1}{4\pi^2 x^2} + \frac{m^2}{16\pi^2}[\gamma + \ln(-\pi^2\mu^2 x^2)] \quad \text{at } d = 4. \end{aligned} \tag{10.4.2}$$

The Wilson coefficient is obtained by expanding in powers of mass and external momentum up to an appropriate degree, which is two here. We have no external momenta, so the expansion is in powers of m^2.

In the next section, when we apply the renormalization group to compute the Wilson coefficients we will find that each coefficient is given as a series in the effective coupling with the renormalization mass μ set to $(-x^2)^{1/2}$. The main application of these methods will be to asymptotically free theories. If we truncate the perturbation expansion, then the error will be of the order of the first omitted term, and hence the fractional error is of order

$$1/[\ln(-x^2)]^{p+1},$$

where p is the number of loops in the highest-order graph. This error dominates any positive power of x.

Consequently, it is difficult to use the power-law corrections to the leading power in a Wilson coefficient. This would suggest we cannot properly use anything but the coefficient of the unit operator. However, in applications we will normally work with connected Green's functions. For these the leading coefficient is of the two-particle operator ϕ^2.

10.5 Renormalization group

To make calculations for a Wilson coefficient, we must use the renormalization group to obtain maximum information from a low-order calculation in the way we will now explain.

The coefficient has the functional dependence

$$C(x^2, g, \mu).$$

To use this we must set μ^2 to be of order $|1/x^2|$ to avoid large logarithms, just as in calculating Green's functions when all their external momenta get large (Chapter 7). The RG equations are simple, since, to the leading power, there is no mass dependence. (If we use non-leading powers of x then we have polynomial dependence on masses.) The renormalization-group equation for the Wilson coefficient can be derived most easily from the renormalization-group equations for Green's functions in the following fashion.

In the ϕ^4 theory we have the renormalization-group operator:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} = \mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - \gamma_m m^2\frac{\partial}{\partial m^2}. \tag{10.5.1}$$

The renormalization-group equations we need are

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_{N+2}^{(\mathrm{conn})} = -(N/2+1)\gamma G_{N+2}^{(\mathrm{conn})}, \tag{10.5.2}$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_N([\phi^2],\dots)^{(\mathrm{conn})} = (\gamma_m - \tfrac{1}{2}N\gamma)G_N([\phi^2],\dots)^{(\mathrm{conn})}. \tag{10.5.3}$$

Here, $G_{N+2}^{(\mathrm{conn})}$ denotes the Green's function of $N+2$ external ϕ-fields, restricted to connected graphs. $G_N([\phi^2],\dots)^{(\mathrm{conn})}$ denotes the connected Green's function of the renormalized $[\phi^2]$ operator with N ϕ-fields. To derive (10.5.3) we used the fact that

$$m^2[\phi^2] = m_0^2\phi_0^2 + \text{coefficient times } 1,$$

and $m_0^2 = Z_m m^2$.

We apply the operator $\mu\mathrm{d}/\mathrm{d}\mu + (N/2+1)\gamma$ to both sides of

$$\begin{aligned}&\langle 0|T\phi(x)\phi(0)\tilde{\phi}(p_1)\dots\tilde{\phi}(p_N)|0\rangle\\&\quad = C_{\phi^2}(x)\langle 0|T[\phi^2]\tilde{\phi}(p_1)\dots\tilde{\phi}(p_N)|0\rangle + O(x^2),\end{aligned} \tag{10.5.4}$$

to obtain

$$0 = \left[\mu\frac{\mathrm{d}C_{\phi^2}}{\mathrm{d}\mu} + (\gamma+\gamma_m)C_{\phi^2}\right]\langle 0|T[\phi^2]\tilde{\phi}(p_1)\dots\tilde{\phi}(p_N)|0\rangle + O(x^2). \tag{10.5.5}$$

Since the $O(x^2)$ terms are order x^2 independently of g, m, and μ, they remain of order x^2 after $\mu\mathrm{d}/\mathrm{d}\mu$ is applied. (As usual, 'order x^2' means order x^2 modulo possible logarithms.) Immediately we obtain

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}C_{\phi^2} = (\gamma_m + \gamma)C_{\phi^2}. \tag{10.5.6}$$

https://doi.org/10.1017/9781009401807.010 Published online by Cambridge University Press

This equation can be solved – it is effectively in the $m = 0$ theory. The anomalous dimension of C_{ϕ^2} is the anomalous dimension of $\phi(x)\phi(0)$ minus the anomalous dimension of $[\phi^2]$. This is the same relation as for the engineering dimensions. The signs in (10.5.6) arise from peculiarities of our definitions of γ_m and γ.