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CONSTANT CURVED MINIMAL CR 3-SPHERES IN $\mathbb{C}P^n$

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Abstract

In this paper we prove that minimal 3-spheres of CR type with constant sectional curvature c in the complex projective space $\mathbb{C}P^n$ are all equivariant and therefore the immersion is rigid. The curvature c of the sphere should be $c = 1/(m^2 - 1)$ for some integer $m \geq 2$, and the full dimension is $n = 2m^2 - 3$. An explicit analytic expression for such an immersion is given.

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1. Preliminary

In [1], Bejancu established the concept of CR-submanifold M in a Kähler manifold N. Namely, if there is a decomposition $TM = V_1 \oplus V_2$ with V_i a subbundle of TM, $i = 1, 2$, such that $JV_1 \subset T^{\perp}M$ and $JV_2 = V_2$, where J is the complex structure of N and $T^{\perp}M$ is the normal bundle on M, then M is called a *CR-submanifold* of N.

In this paper, we assume that N is the complex projective space $\mathbb{C}P^n$ with constant holomorphic sectional curvature 4.

The minimal surface theory in $\mathbb{C}P^n$ has made a great progress over the past thirty years. For constant curved minimal 2-spheres in $\mathbb{C}P^n$, the immersion $\varphi : S^2 \to \mathbb{C}P^n$ is uniquely determined by the induced metric, and φ can be constructed from its directrix $\varphi_0 : S^2 \to \mathbb{C}P^n$ by using arithmetical procedure [2].

Up to now merely a few examples have been known for higher dimensional minimal submanifolds in $\mathbb{C}P^n$. There are some examples of holomorphic submanifolds and Lagrangian minimal submanifolds [3, 4, 6]. In [5] we studied equivariant minimal 3-spheres with constant (sectional) curvature c immersed in $\mathbb{C}P^n$. Here the terminology

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'equivariant' means that the immersion $\varphi : S^3 \to \mathbb{C}P^n$ is compatible with the Lie group structure on $S^3 = \mathrm{SU}(2)$, that is, there exists a homomorphism $E : S^3 \to \mathrm{U}(n+1)$ of Lie group such that $\varphi = A \circ \pi_2 \circ E$, where

$$\pi_2 : \mathrm{U}(n+1) \to \mathbb{C}P^n = \mathrm{U}(n+1)/\,\mathrm{U}(1) \times \mathrm{U}(n)$$

is the natural projection and $A : \mathbb{C}P^n \to \mathbb{C}P^n$ is a holomorphic isometry. In [5], we provided two examples of minimal immersions from S^3 into $\mathbb{C}P^n$. One of these examples is below.

EXAMPLE 1. For a given integer $m \geq 2$, put $k = (m-2)(m+1), l = (m-1)(m+2)$,

$$\cos^2 t = \frac{1}{2} - \frac{1}{2m} = \frac{m-1}{2m}, \quad \sin^2 t = \frac{1}{2} + \frac{1}{2m} = \frac{m+1}{2m},$$

where $t \in (0, \pi/2)$. Let

$$f = \sum_{j=0}^{k} \sqrt{\binom{k}{j}} z^j w^{k-j} \varepsilon_j, \quad g = \sum_{j=0}^{l} \sqrt{\binom{l}{j}} z^j w^{l-j} \varepsilon'_j,$$

where $(z, w) \in S^3 = \{(z, w) \in \mathbb{C}^2 \mid z\bar{z} + w\bar{w} = 1\}$, and $\{\varepsilon_0, \ldots, \varepsilon_k, \varepsilon'_0, \ldots, \varepsilon'_l\}$ is the natural basis of $\mathbb{C}^{k+l+2} = \mathbb{C}^{k+1} \oplus \mathbb{C}^{l+1}$. Let $\pi : S^{2n+1} \to \mathbb{C}P^n$ be the Hopf fibration. Define $\varphi = \pi \circ e_0 : S^3 \to \mathbb{C}P^{k+l+1}$, where

$$e_0 = (\cos tf, \sin tg) : S^3 \to S^{2(k+l)+3} \subset \mathbb{C}^{k+l+2}. \tag{1.1}$$

Then

(a) φ is an equivariant minimal immersion with respect to the induced metric ds^2;
(b) φ is of CR type, that is, $\varphi(S^3)$ is a CR-submanifold of $\mathbb{C}P^n$;
(c) The sectional curvature of the induced metric ds^2 is a constant $c = 1/(m^2 - 1)$.

Since k and l are all even, the immersion φ in Example 1 induces an embedding $\psi : \mathbb{R}P^3 \to \mathbb{C}P^n$.

We will always assume that (S^3, ds^2) has constant sectional curvature c and we will identify S^3 with the Lie group SU(2). Up to an isometry of S^3 we may consider the metric ds^2 as a bi-invariant metric on SU(2). Two maps $\varphi, \psi : S^3 \to \mathbb{C}P^n$ are said to be *equivalent* if there is a holomorphic isometric $A : \to CP^n\mathbb{C}P^n$ such that $\psi = A \circ \varphi$. We have the following results from [5].

THEOREM 1.1 ([5]). *Let $\varphi : S^3 \to \mathbb{C}P^n$ be an equivariant minimal immersion of CR type with constant curvature c. If φ is linearly full, then $c = 2/(n+1)$ where $n = 2m^2 - 3$ for some integer $m \geq 2$. Moreover, up to an isometry of S^3, φ is equivalent to the immersion defined in Example 1.*

THEOREM 1.2 ([5]). *Let $\varphi : S^3 \to \mathbb{C}P^n$ be a minimal immersion of CR type. Suppose that the induced metric is bi-invariant. If $\varphi^*\Omega$ is left-invariant, where Ω is the Kähler form of $\mathbb{C}P^n$, then φ is equivariant.*

In the present paper we will prove the following

THEOREM 1.3. *Up to an isometry of S^3, a minimal immersion $\varphi : S^3 \to \mathbb{C}P^n$ of CR type with constant curvature c is equivariant.*

Theorem 1.3 together with Theorem 1.1 implies that a compact minimal CR-submanifold M of dimension 3 with constant curvature $c > 0$ in $\mathbb{C}P^n$ is an embedded $\mathbb{R}P^3$, since the universal covering space of M is the 3-sphere S^3 with constant curvature c. It has rigidity. And the curvature $c = 1/(m^2 - 1)$ for some integer $m \geq 2$. If the immersion is full, then $n = 2m^2 - 3$. Up to a holomorphic isometry of $\mathbb{C}P^n$ and an isometry of S^3, (1.1) is the unique analytic expression of the embedding.

2. Local formulae

Identify S^3 with the Lie group SU(2) with metric ds^2 of constant curvature c as follows

$$S^3 \ni (z, w) \longleftrightarrow \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in \mathrm{SU}(2),$$

where the metric ds^2 is bi-invariant and is given by $ds^2 = c^{-1}\sum_{j=1}^{3}\omega'_j \otimes \omega'_j$ with $\{\omega'_j \mid j = 1, 2, 3\}$ being determined by

$$(2.1)\quad \begin{pmatrix} i\omega'_1 & -\omega'_2 + i\omega'_3 \\ \omega'_2 + i\omega'_3 & -i\omega'_1 \end{pmatrix} = \begin{pmatrix} \bar{z} & \bar{w} \\ -w & z \end{pmatrix}\begin{pmatrix} dz & -d\bar{w} \\ dw & d\bar{z} \end{pmatrix}, \quad (i = \sqrt{-1}).$$

Denote by $\mathfrak{su}(2)$ the set of all left-invariant vector fields on $S^3(= \mathrm{SU}(2))$. It is well known that $\mathfrak{su}(2)$ is a real vector space of dimension 3 and the dual space $\mathfrak{su}(2)^*$ consists of all left-invariant 1-forms on S^3. The bi-invariant metric ds^2 defines an inner product on $\mathfrak{su}(2)$ in a natural way, and induces the inner product on $\mathfrak{su}(2)^*$ and $\mathfrak{su}(2)^* \wedge \mathfrak{su}(2)^*$ respectively.

Let $\varphi : S^3 \to \mathbb{C}P^n$ be an isometric immersion of CR type. That is to say, $\varphi(S^3)$ is a CR-submanifold of $\mathbb{C}P^n$. Denote by g, Ω and J the metric, the Kähler form and the complex structure of $\mathbb{C}P^n$ respectively. The tensor field $\varphi^*\Omega$ defines a bundle endomorphism $F : TM \to TM$ by

$$(2.2)\quad ds^2(FX, Y) = -\varphi^*\Omega(X, Y) = g(J\varphi_*X, \varphi_*Y), \quad \forall X, Y \in T_pS^3,\ p \in S^3.$$

Since $\varphi^*\Omega$ is skew symmetric and φ is of CR type, $F_p : T_pS^3 \to T_pS^3$ has rank 2 for all $p \in S^3$. We then have a decomposition $TS^3 = V_1 \oplus V_2$ of F-invariant subbundle such that

$$V_1 = \ker F, \quad (F|_{V_2})^2 = -I. \tag{2.3}$$

Here $F|_{V_2}$ determines an orientation of V_2. Thus V_1 is orientable, and there is a unit section X_1 of V_1. By definition $FX_1 = 0$. Take a local orthonormal frame $\{X_2, X_3\}$ of V_2 defined on some open subset U such that

$$FX_2 = X_3, \quad FX_3 = -X_2. \tag{2.4}$$

Let $\{\omega_1, \omega_2, \omega_3\}$ be the dual frame of $\{X_1, X_2, X_3\}$. We then have

$$ds^2 = \varphi^*g = \sum_{j=1}^{3} \omega_j \otimes \omega_j, \quad -\varphi^*\Omega = \omega_2 \otimes \omega_3 - \omega_3 \otimes \omega_2 = \omega_2 \wedge \omega_3 \tag{2.5}$$

on U by (2.2)–(2.4).

Denote by $\langle\ ,\ \rangle$ the canonical symmetric scalar product of $\mathbb{C}^{n+1}$. Choose a local unitary frame $\{e_0, e_1, \dots, e_n\}$ of the trivial bundle $\underline{\mathbb{C}}^{n+1} = S^3 \times \mathbb{C}^{n+1}$ such that $\varphi = \pi \circ e_0$ on U. Set

$$\begin{cases} de_0 = i\rho_0 e_0 + \sum_A \theta_A e_A; \\ de_A = -\bar{\theta}_A e_0 + \sum_B \theta_{AB} e_B, \quad (A, B = 1, \dots, n), \end{cases}$$

where $i = \sqrt{-1}$ and $\rho_0 = -i\langle de_0, \bar{e}_0\rangle$ is a real 1-form. From (2.5) we get (see, for example, [5])

$$\begin{cases} \varphi^*g = \frac{1}{2}\sum_A (\theta_A \otimes \bar{\theta}_A + \bar{\theta}_A \otimes \theta_A) = \sum_{j=1}^{3} \omega_j \otimes \omega_j; \\ -\varphi^*\Omega = \frac{i}{2}\sum_A \theta_A \wedge \bar{\theta}_A = \omega_2 \wedge \omega_3. \end{cases} \tag{2.6}$$

Set $\theta_A = \sum_{j=1}^3 a_{Aj}\omega_j$ and $e'_j = \sum_{A=1}^n a_{Aj} e_A$. By (2.6), we have

$$\sum_A a_{Aj}\bar{a}_{Ak} = \delta_{jk} - iJ_{jk},$$

where $J_{jk} = ds^2(FX_j, X_k) = g(J\varphi_* X_j, \varphi_* X_k)$. It follows that

$$\langle e'_1, \bar{e}'_2\rangle = \langle e'_1, \bar{e}'_3\rangle = 0, \quad \langle e'_2, \bar{e}'_3\rangle = -i, \quad |e'_1|^2 = |e'_2|^2 = |e'_3|^2 = 1,$$

and from $|e'_2 + ie'_3|^2 = 0$, we obtain $e'_3 = ie'_2$. We have proved

LEMMA 2.1. *Let* $\varphi : S^3 \to \mathbb{C}P^n$ *be an isometric immersion of CR type. Then* $TS^3 = V_1 \oplus V_2$ *where* $V_1 = \ker F$, V_2 *is perpendicular to* V_1 *and* $(F|_{V_2})^2 = -\mathrm{id}$. *Furthermore, locally we have an orthonormal frame* $\{\omega_1, \omega_2, \omega_3\}$ *of* T^*S^3 *with* ω_1 *a section of* V_1^*, *and a unitary frame* $\{e_0, e_1, \dots, e_n\}$ *of* $\underline{\mathbb{C}}^{n+1}$ *such that* $\varphi = \pi \circ e_0$ *satisfying* $de_0 = i\rho_0 e_0 + \omega_1 e_1 + \omega e_2$, *where* $\omega = \omega_2 + i\omega_3$.

Exterior differentiating (2.1) gives

$$d\omega_1' = 2\omega_2' \wedge \omega_3', \quad d\omega_2' = 2\omega_3' \wedge \omega_1', \quad d\omega_3' = 2\omega_1' \wedge \omega_2', \tag{2.7}$$

which implies that $d : \mathfrak{su}(2)^* \to \mathfrak{su}(2)^* \wedge \mathfrak{su}(2)^*$ is an isomorphism between vector spaces.

A local section σ of T^*S^3 is said to be *left-invariant* if there are real numbers a_1, a_2, a_3 such that $\sigma = a_1\omega_1' + a_2\omega_2' + a_3\omega_3'$. Note that a left-invariant local 1-form σ can be extended uniquely as a left-invariant 1-form $\tilde{\sigma} \in \mathfrak{su}(2)^*$. We have the following

LEMMA 2.2. *Suppose* $\{\omega_1, \omega_2, \omega_3\}$ *is a local orthonormal frame of* T^*S^3 *defined on an open subset* U *of* S^3. *If*

$$d\omega_1 = 2a\omega_2 \wedge \omega_3, \quad d\omega_2 = 2a\omega_3 \wedge \omega_1, \quad d\omega_3 = 2a\omega_1 \wedge \omega_2 \tag{2.8}$$

for some constant a, *then* $a^2 = c$ *and* ω_j *is left-invariant for* $j = 1, 2, 3$.

PROOF. Let $\{\omega_{jk} \mid j, k = 1, 2, 3\}$ be the connection forms satisfying

$$\nabla X_j = -\sum_k \omega_{jk} X_k, \tag{2.9}$$

where $\{X_j\}$ is the dual of $\{\omega_j\}$ and ∇ is the Levi-Civita connection. The structure equations for S^3 are

$$\begin{cases} d\omega_j = -\sum_k \omega_{jk} \wedge \omega_k, \quad \omega_{jk} + \omega_{kj} = 0; \\ d\omega_{jk} = -\sum_l \omega_{jl} \wedge \omega_{lk} + c\omega_j \wedge \omega_k. \end{cases} \tag{2.10}$$

From (2.8) we know that $\omega_{12} = a\omega_3$, $\omega_{23} = a\omega_1$, $\omega_{31} = a\omega_2$, $c = a^2$.

Since $\{\tilde{\omega}_j = (1/\sqrt{c})\omega_j' \mid j = 1, 2, 3\}$ is a global orthonormal frame of T^*S^3, we set $\omega_j = \sum_k u_{jk}\tilde{\omega}_k$, where u_{jk} $(j, k = 1, 2, 3)$ are defined on U and satisfy

$$\sum_l u_{lj} u_{lk} = \delta_{jk} = \sum_l u_{jl} u_{kl}. \tag{2.11}$$

Without loss of generality we set $a = \sqrt{c}$ and

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = *1 = \tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge \tilde{\omega}_3,$$

where $*1$ denotes the the volume element of S^3.

The Hodge's star operator induces a bundle homomorphism $* : T^*U \to (T^*U) \wedge (T^*U)$ by

$$*\sigma \wedge \tau = \langle \sigma, \tau \rangle * 1, \quad \forall \sigma, \tau \in T_p^*U,\ p \in U.$$

It is clear that $*(f\sigma) = f * \sigma$ for all $\sigma \in C^\infty(T^*U)$ and $f \in C^\infty(U)$. From (2.8) we see that $d\omega_j = 2\sqrt{c} * \omega_j$ for $j = 1, 2, 3$. On the other hand, $d\tilde{\omega}_j = 2\sqrt{c} * \tilde{\omega}_j$ ($j = 1, 2, 3$) by (2.7). Thus

$$\text{(2.12)} \qquad \begin{aligned} \sum_k u_{jk} d\tilde{\omega}_k &= 2\sqrt{c} \sum_k u_{jk} * \tilde{\omega}_k = 2\sqrt{c} * \omega_j \\ &= d\omega_j = \sum_k (du_{jk} \wedge \tilde{\omega}_k + u_{jk} d\tilde{\omega}_k) \quad (j = 1, 2, 3). \end{aligned}$$

If we set $du_{jk} = \sum_l u_{jk,l}\tilde{\omega}_l$, then $u_{jk,l} = u_{jl,k}$ ($j, k, l = 1, 2, 3$) by (2.12). From (2.11) we see that $\sum_j u_{jm} u_{jk,l} = 0$ for $m, k, l = 1, 2, 3$. Consequently u_{jk} are constants for $j, k = 1, 2, 3$. □

LEMMA 2.3. *Suppose $\{\omega_1, \omega_2, \omega_3\}$ is a local orthonormal frame of T^*S^3 defined on an open subset U of S^3. If $d\omega_1 = 2a\omega_2 \wedge \omega_3$ for some constant $a \neq 0$, then ω_1 is left-invariant.*

PROOF. Let $\{X_j\}$ be the dual of $\{\omega_j\}$. Set

$$\text{(2.13)} \qquad Z = (X_2 - iX_3)/2, \quad \omega = \omega_2 + i\omega_3, \quad \sigma = \omega_{12} + i\omega_{13},$$

where $\{\omega_{jk}\}$ are the connection forms determined by (2.9). Then (2.10) can be rewritten as

$$\text{(2.14)} \qquad \begin{cases} d\omega_1 = -(\bar{\sigma} \wedge \omega + \sigma \wedge \bar{\omega})/2, & d\omega = \sigma \wedge \omega_1 + i\omega_{23} \wedge \omega, \\ d\sigma = i\omega_{23} \wedge \sigma + c\,\omega_1 \wedge \omega, & d\omega_{23} = i(\sigma \wedge \bar{\sigma} + c\,\omega \wedge \bar{\omega})/2. \end{cases}$$

By assumption we have $-(\bar{\sigma} \wedge \omega + \sigma \wedge \bar{\omega})/2 = 2a\omega_2 \wedge \omega_3 = ia\omega \wedge \bar{\omega}$. This forces $\sigma(X_1) = 0$. We set $\sigma = \lambda\omega + \mu\bar{\omega}$ with $\lambda - \bar{\lambda} = -2ia$. Thus $\lambda = \lambda_1 - ia$ for some real λ_1. Using (2.14) we get

$$\text{(2.15)} \qquad \begin{aligned} &[d\lambda_1 - (\lambda_1 - ia)^2\omega_1 - (|\mu|^2 + c)\omega_1] \wedge \omega \\ &\quad + [d\mu - 2\mu\lambda_1\omega_1 - 2i\mu\omega_{23}] \wedge \bar{\omega} = 0, \end{aligned}$$

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which implies that $X_1(\lambda_1) - (\lambda_1 - ia)^2 - (|\mu|^2 + c) = 0$. Since X_1 is real and $a \neq 0$, we get $\lambda_1 = 0$ and

$$|\mu|^2 = a^2 - c. \tag{2.16}$$

If $a^2 \neq c$, we set $\mu = be^{it}$ with $b = \sqrt{a^2 - c}$. From (2.15) we know that $d\mu = 2i\mu\omega_{23} + \nu\bar{\omega}$. This gives $idt = 2i\omega_{23} + \mu^{-1}\nu\bar{\omega}$. Thus $\nu = 0$ and $\omega_{23} = dt/2$. Using (2.14) we get $|\mu|^2 = |\lambda|^2 + c = a^2 + c$, contradicting (2.16).

Therefore, we have $a^2 = c$ and $\mu = 0$. Then $\sigma = -ia\omega$, that is, $\omega_{12} = a\omega_3$ and $\omega_{31} = a\omega_2$. Now we have $d\omega_{23} = ia^2\omega \wedge \bar{\omega} = 2a^2\omega_2 \wedge \omega_3 = a\, d\omega_1$ by (2.14). Locally we set $\omega_{23} = a\omega_1 + df$ for some $f \in C^\infty(U)$. Choose frame

$$\left\{\widetilde{X}_1 = X_1,\ \widetilde{X}_2 = \cos f\, X_2 + \sin f\, X_3,\ \widetilde{X}_3 = -\sin f\, X_2 + \cos f\, X_3\right\}$$

and let $\{\tilde{\omega}_j\}$ be the dual frame of $\{\widetilde{X}_j\}$. Then $\tilde{\omega}_1 = \omega_1$, $\tilde{\omega}_2 = \cos f\,\omega_2 + \sin f\,\omega_3$ and $\tilde{\omega}_3 = -\sin f\,\omega_2 + \cos f\,\omega_3$.

From (2.10) we have

$$d\tilde{\omega}_1 = 2a\tilde{\omega}_2 \wedge \tilde{\omega}_3, \quad d\tilde{\omega}_2 = 2a\tilde{\omega}_3 \wedge \tilde{\omega}_1, \quad d\tilde{\omega}_3 = 2a\tilde{\omega}_1 \wedge \tilde{\omega}_2.$$

Thus $\omega_1 = \tilde{\omega}_1$ is left-invariant by virtue of Lemma 2.2. □

3. Proof of Theorem 1.3

Let $\varphi : S^3 \to \mathbb{C}P^n$ be a minimal immersion of CR type with induced metric $ds^2 = c^{-1}\sum_j \omega'_j \otimes \omega'_j$. It is sufficient to prove that $\varphi^*\Omega$ is left-invariant by virtue of Theorem 1.2.

Since $V_1 = \ker F$ is orientable, we have a unit section ω_1 of V_1^*. Using Lemma 2.1, for any $p \in S^3$ we have a local unitary frame $\{e_0, e_1, \dots, e_n\}$ of $\underline{\mathbb{C}}^{n+1}$ and a local orthonormal frame $\{\omega_2, \omega_3\}$ of V_2^* defined on an open neighbourhood U of p such that

$$\begin{cases} de_0 = i\rho_0 e_0 + \omega_1 e_1 + \omega e_2; \\ de_1 = -\omega_1 e_0 + i\rho_1 e_1 + \theta_{12} e_2 + \sum_{A=3}^{n} \theta_{1A} e_A; \\ de_2 = -\bar{\omega} e_0 - \bar{\theta}_{12} e_1 + i\rho_2 e_2 + \sum_{A=3}^{n} \theta_{2A} e_A; \\ de_A = -\bar{\theta}_{1A} e_1 - \bar{\theta}_{2A} e_2 + \sum_{B=3}^{n} \theta_{AB} e_B \quad (A = 3, \dots, n), \end{cases} \tag{3.1}$$

where $\omega = \omega_2 + i\omega_3$ and $\theta_{AB} + \bar{\theta}_{BA} = 0$ for $3 \leq A, B \leq n$.

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The exterior differential of (3.1) gives

$$id\rho_0 = -\omega \wedge \bar{\omega} = -2i\varphi^*\Omega \tag{3.2}$$

by (2.6) and

$$d\omega_1 = i(\rho_0 - \rho_1) \wedge \omega_1 + \bar{\theta}_{12} \wedge \omega, \quad d\omega = i(\rho_0 - \rho_2) \wedge \omega - \theta_{12} \wedge \omega_1, \tag{3.3}$$

$$\omega_1 \wedge \theta_{1A} + \omega \wedge \theta_{2A} = 0 \quad (A = 3, \dots, n), \tag{3.4}$$

$$id\rho_1 = -\theta_{12} \wedge \bar{\theta}_{12} - \sum_{A=3}^{n} \theta_{1A} \wedge \bar{\theta}_{1A}, \tag{3.5}$$

$$id\rho_2 = \omega \wedge \bar{\omega} + \theta_{12} \wedge \bar{\theta}_{12} - \sum_{A=3}^{n} \theta_{2A} \wedge \bar{\theta}_{2A},$$

$$d\theta_{12} = -\omega_1 \wedge \omega + i(\rho_1 - \rho_2) \wedge \theta_{12} - \sum_{A=3}^{n} \theta_{1A} \wedge \bar{\theta}_{2A}, \tag{3.6}$$

$$d\theta_{1A} = i\rho_1 \wedge \theta_{1A} + \theta_{12} \wedge \theta_{2A} + \sum_{B=3}^{n} \theta_{1B} \wedge \theta_{BA}, \quad (A \geq 3). \tag{3.7}$$

Comparing (3.3) with (2.14) and noting that ω_1, ω_{23}, ρ_0, ρ_1 and ρ_2 are all real-valued 1-forms, we get

$$\begin{cases} i(\rho_0 - \rho_1) \wedge \omega_1 = (\theta_{12} \wedge \bar{\omega} - \bar{\theta}_{12} \wedge \omega)/2; \\ (\sigma + \theta_{12}) \wedge \bar{\omega} + (\bar{\sigma} + \bar{\theta}_{12}) \wedge \omega = 0; \\ i(\rho_0 - \rho_2 - \omega_{23}) \wedge \omega = (\sigma + \theta_{12}) \wedge \omega_1. \end{cases} \tag{3.8}$$

The third equation in (3.8) gives $(\rho_0 - \rho_2 - \omega_{23})(\overline{Z}) = 0$, where $\overline{Z} = \frac{1}{2}(X_2 + iX_3)$. Thus $\rho_0 - \rho_2 - \omega_{23} = \lambda_0\omega_1$ for some real-valued function λ_0 and therefore $\sigma + \theta_{12} = -i\lambda_0\omega + \mu_0\omega_1$.

From (3.8) one gets $\lambda_0 = \mu_0 = 0$. Therefore,

$$\sigma = -\theta_{12}, \quad \omega_{23} = \rho_0 - \rho_2 \quad \text{and} \tag{3.9}$$

$$i(\rho_0 - \rho_1) \wedge \omega_1 = (\bar{\sigma} \wedge \omega - \sigma \wedge \bar{\omega})/2. \tag{3.10}$$

Since φ is minimal, we have [5]

$$\operatorname{tr}\{\nabla\omega_1 - i\rho_0 \otimes \omega_1 + i\rho_1 \otimes \omega_1 - \bar{\theta}_{12} \otimes \omega\} = 0, \tag{3.11}$$

$$\operatorname{tr}\{\theta_{1A} \otimes \omega_1 + \theta_{2A} \otimes \omega\} = 0 \quad (A = 3, \dots, n). \tag{3.12}$$

According to (2.9), $\nabla\omega_1 = -\omega_{12} \otimes \omega_2 - \omega_{13} \otimes \omega_3 = -(\bar{\sigma} \otimes \omega + \sigma \otimes \bar{\omega})/2$. Substituting it into (3.11) and using (3.9), we have

$$\operatorname{tr}\left\{ -i(\rho_0 - \rho_1) \otimes \omega_1 + (\bar{\sigma} \otimes \omega - \sigma \otimes \bar{\omega})/2 \right\} = 0. \tag{3.13}$$

If we set

$$i(\rho_0 - \rho_1) = 2i\lambda\omega_1 + \mu\omega - \bar{\mu}\bar{\omega} \tag{3.14}$$

for some real-valued function λ defined on U, then

$$\sigma = -2\bar{\mu}\omega_1 - i\lambda\omega + \nu\bar{\omega} \tag{3.15}$$

by (3.10) and (3.13). Similarly, we may set

$$\theta_{1A} = \lambda_A\omega \quad \text{and} \quad \theta_{2A} = \lambda_A\omega_1 + \mu_A\omega \quad (A = 3, \dots, n) \tag{3.16}$$

by (3.4) and (3.12).

Exterior differentiating (3.9) and using (2.14), (3.6), (3.9), (3.14) and (3.15) we get

$$\bar{\mu}^2 = i\lambda\nu, \tag{3.17}$$

$$\sum_{A=3}^{n} \lambda_A\bar{\mu}_A = \mu\nu - i\lambda\bar{\mu}, \tag{3.18}$$

$$\sum_{A=3}^{n} |\lambda_A|^2 + 2(\lambda^2 + |\mu|^2) = 1 - c. \tag{3.19}$$

Now we claim that (1) $\mu \equiv 0$ and (2) $\nu \equiv 0$ on U.

In fact, if $\mu \neq 0$ at a point $q \in U$, then $\bar{\mu}^2 = i\lambda\nu \neq 0$ near q. Thus $\sum_A \lambda_A\bar{\mu}_A = \nu(\lambda^2 + |\mu|^2)/\bar{\mu} \neq 0$ by (3.18) and therefore $\sum_A \lambda_A e_A \neq 0$ locally. By taking new frame

$$\left\{ e_0, e_1, e_2, e_3' = \frac{\sum_A \lambda_A e_A}{\left|\sum_A \lambda_A e_A\right|}, e_4' \dots, e_n' \right\}$$

we set $\lambda_3 \neq 0$ and $\lambda_4 = \cdots = \lambda_n = 0$ in (3.16). From (3.7) we have

$$[d\lambda_3 + i\lambda_3(\omega_{23} - \rho_1 + \rho_3) + \mu_3\sigma] \wedge \omega = -2\lambda_3\sigma \wedge \omega_1 = 2\lambda_3(i\lambda\omega - \nu\bar{\omega}) \wedge \omega_1$$

by (3.15), where $\rho_3 = -i\theta_{33}$. This gives $\nu = 0$, a contradiction by (3.17). Thus $\mu \equiv 0$ on U.

It follows that if $\nu \neq 0$ at some point $p_1 \in U$, then $\lambda = 0$ near p_1 by (3.17). Locally we have $\rho_0 = \rho_1$ by (3.14) and $\sigma = \nu\bar{\omega}$ by (3.15), then

$$-\omega \wedge \bar{\omega} = id\rho_0 = id\rho_1 = -\sigma \wedge \bar{\sigma} - \sum_{A=3}^{n} \theta_{1A} \wedge \bar{\theta}_{1A} = \left(|\nu|^2 - \sum_{A=3}^{n} |\lambda_A|^2\right) \omega \wedge \bar{\omega}$$

by (3.2), (3.5) and (3.9). This together with (3.19) leads to

$$|\nu|^2 = \sum_{A=3}^{n} |\lambda_A|^2 - 1 = -c,$$

a contradiction. So we have $\nu \equiv 0$ on U.

Now $\sigma = -i\lambda\omega$ and therefore

$$\lambda\omega_{23} \wedge \omega + c\omega_1 \wedge \omega = d\sigma = -id\lambda \wedge \omega + \lambda(\lambda\omega_1 \wedge \omega + \omega_{23} \wedge \omega)$$

by (2.14). It follows that $[id\lambda + (c - \lambda^2)\omega_1] \wedge \omega = 0$. Set $id\lambda + (c - \lambda^2)\omega_1 = \mu_1\omega$. We then get $\lambda^2 = c$ since λ is real.

From (2.14) we see that $d\omega_1 = i\lambda\omega \wedge \bar{\omega} = 2\lambda\omega_2 \wedge \omega_3$, where $\lambda = \pm\sqrt{c} \neq 0$. Thus ω_1 is left-invariant by virtue of Lemma 2.3. Since ω_1 is a global section of V_1^*, we know $\omega_1 \in \mathfrak{su}(2)^*$. Then by (3.2) we finally obtain

$$\varphi^*\Omega = -\frac{i}{2}\omega \wedge \bar{\omega} = -\frac{1}{2\lambda}d\omega_1 \in d(\mathfrak{su}(2)^*) = \mathfrak{su}(2)^* \wedge \mathfrak{su}(2)^*.$$

This completes the proof of Theorem 1.3. □

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