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# ON VECTOR BUNDLES ON ALGEBRAIC SURFACES AND 0-CYCLES

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Let $X$ be an algebraic complex projective surface equipped with the euclidean topology and $E$ a rank 2 topological vector bundle on $X$. It is a classical theorem of Wu ([Wu]) that $E$ is uniquely determined by its topological Chern classes $c_1^{\mathrm{top}}(E) \in H^2(X, \mathbf{Z})$ and $c_2^{\mathrm{top}}(E) \in H^4(X, \mathbf{Z}) \cong \mathbf{Z}$. Viceversa, again a classical theorem of Wu ([Wu]) states that every pair $(a, b) \in (H^2(X, \mathbf{Z}), \mathbf{Z})$ arises as topological Chern classes of a rank 2 topological vector bundle. For these results the existence of an algebraic structure on $X$ was not important; for instance it would have been sufficient to have on $X$ a holomorphic structure. In [Sch] it was proved that for algebraic $X$ any such topological vector bundle on $X$ has a holomorphic structure (or, equivalently by GAGA an algebraic structure) if its determinant line bundle has a holomorphic structure. It came as a surprise when Elencwajg and Forster ([EF]) showed that sometimes this was not true if we do not assume that $X$ has an algebraic structure but only a holomorphic one (even for some two dimensional tori (see also [BL], [BF], or [T])). In the algebraic case the proof given in [Sch] showed at once a slightly stronger statement; not only every pair $(a, b) \in (NS(X), \mathbf{Z})$ arises as topological Chern classes of algebraic bundles, but also every pair $(L, b) \in (\mathrm{Pic}(X), \mathbf{Z})$. In algebraic geometry there are finer equivalence relations on the set of 0-cycles than just the "topological" one (or "homological" one), which is simply the degree of the given 0-cycle. By far, the most important such equivalence relation is the rational equivalence relation, which gives the Chow ring $A^*(X)$ of $X$ with $A^1(X) \cong \mathrm{Pic}(X)$ and $A^2(X)$ mapping surjectively onto $H^2(X, \mathbf{Z}) \cong \mathbf{Z}$ by the degree map. Mumford discovered that very often $A^2(X)$ is huge (see [Mu] or [B], Chapter 1). An algebraic vector bundle $E$ has Chern classes $c_i(E) \in A^i(X)$ (with $c_1(E) = \det(E)$). Thus it seems to be natural to ask if every pair $(c, d) \in (A^1(X), A^2(X))$ arises as "algebraic" Chern classes of some rank 2 algebraic vector bundle on $X$. In this note we prove that the answer is YES, i.e. we prove the following result.

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THEOREM 0.1. *Fix a projective complex algebraic surface $X$. For every pair $(L, c_2) \in (\mathrm{Pic}(X), A^2(X))$, there is a rank 2 algebraic vector bundle $E$ on $X$ with $(L, c_2)$ as Chern classes.*

Now fix a polarization $H$ on $X$, i.e. fix $H \in \mathrm{Pic}(X)$ with $H$ ample. There is a notion of stability (e.g. in the sense of Mumford-Takemoto) with respect to $H$. It is a natural question to see if the pair $(L, c_2)$ in the statement of 0.1 arises as Chern classes of some rank 2 $H$-stable vector bundle on $X$. Even for the corresponding "numerical" problem (with $c_i^{\mathrm{top}}$) there are numerical well-known restrictions on $c_2^{\mathrm{top}}$ (even on $\mathbf{P}^2$). By [BB], Prop. 1.2, for fixed $X$, $H$, and $L \in \mathrm{Pic}(X)$, this assertion $(\det, c_2^{\mathrm{top}}) \in (\mathrm{Pic}(X), \mathbf{Z})$ is true if the integer $c_2^{\mathrm{top}}$ is sufficiently large. We were unable to prove the corresponding result for all elements of $A^2(X)$ with sufficiently large degree (the construction which proves 0.1 gives very unstable vector bundles). We prove here (see 0.2) a far weaker statement replacing "rational equivalence" with the weaker "abelian equivalence" (see [Sa] or [Li], p. 127) in the following sense; fix a base point $P \in X$ so that the Albanese morphism $\alpha : X \to \mathrm{Alb}(X)$ is normalized by the condition $\alpha(\mathrm{P}) = 0$; extend by additivity (as in the case of curves) $\alpha$ to the set of 0-cycles of degree 0; then the Albanese class of a 0-cycle $D$ of degree $b$ is $\alpha(D - bP)$. Indeed the second result of this paper is the following theorem.

THEOREM 0.2. *Fix a projective complex algebraic surface $X$ and line bundles $H$, $L$ on $X$ with $H$ ample. Fix a base point $P \in X$. There is an integer $k_0$, depending on $X$, $H$ and $L$, such that for every $k \geq k_0$ and every $\mathbf{a} \in \mathrm{Alb}(X)$ there is a rank 2 $H$-stable vector bundle $E$ on $X$ with $c_1(E) = L$, $\deg(c_2(E)) = k$ and such that $\mathbf{a}$ is the Albanese class of the degree zero 0-cycle $c_2(E) - kP$.*

Note that if $X$ has Kodaira dimension $\kappa(X) < 0$, then "rational equivalence" and "Albanese equivalence" coincide.

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## §1. The proofs

Here we prove Theorems 0.1 and 0.2.

*Proof of* 0.1. Fix $L$ and $c_2$ (as a class in the Chow ring), with, say, $c_2$ represented by the cycle $A - B$ with $A$ and $B$ effective and disjoint. Let $H$ be a very

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ample line bundle. Just to fix the notations we assume $B$ reduced; it is easy to do the general case changing the notations in step (b) below; alternatively, it is easy to reduce the general case to the case in which $B$ is reduced. The proof will be divided in two parts.

(a) Let $F$ be a rank 2 vector bundle on $X$. For every integer $m$ the splitting principle shows that in the Chow ring $A^*(X)$ we have $c_1(F(mH)) = c_1(F). + 2mH$ and

$$c_2(F(mH)) = c_2(F) + c_1(F)\cdot(mH) + m^2H^2. \tag{1}$$

Hence to solve our problem it is sufficient to find an integer $z$ and a rank 2 vector bundle $Q$ on $X$ with $c_1(Q) = L + 2zH$ and $c_2(Q) = c_2 + zL\cdot H + z^2H^2$. We will find $z$ and $Q$ solving our problem and with $z$ very negative.

(b) Set $b := \mathrm{card}(B)$. Fix an integer $c \geq b$ and $c$ smooth curves $C_t \in |H|$ with $\mathrm{card}(C_i \cap B) = 1$ if $i \leq b$, $\mathrm{card}(C_i \cap B) = 0$ if $i > b$ and $C_i \cap C_j \cap B = \emptyset$ if $i \neq j$; set $x_t := B \cap C_t$, $i = 1, \ldots, b$. We assume that $(cH - L)\cdot H > 2p_a(C_i) := (K + H)\cdot H + 2$. Hence there are reduced disjoint effective divisors $F_i \subset C_i$, $1 \leq i \leq c$, with $x_i \in F_i$ if $i \leq b$, $F_t$ with $\mathbf{O}(cH - L) \mid C_t$ as associated line bundle on $C_t$ $(1 \leq i \leq c)$. Let $\mathbf{Z}$ be the union of $A$, $F_i \setminus \{x_i\}$ for all $i$ with $1 \leq i \leq b$, and $F_j$ for all $j$ with $b < j \leq c$. By construction and the fact that rational equivalence commutes with proper push-forward ([Fu], Th. 1.1.4), the rational equivalence class of $Z$ is $c_2 - zL\cdot H + z^2H^2$ with $z = -c$. Hence to prove 0.1 it is sufficient to prove the existence of a rank 2 vector bundle $Q$ which fits in the following exact sequence:

$$0 \to \mathbf{O}_x \to Q \to L \otimes H^{\otimes(-2c)} \otimes \mathrm{I}_z \to 0 \tag{2}$$

since $c_2(\mathbf{O}_z) = -Z$ by Riemann-Roch theorem. Furthermore, taking $c$ large enough, we may assume $h^0(X, K_X \otimes L \otimes H^{\otimes(-2c)}) = 0$. We will fix any $c \geq b$ with this property. By the choice of $c$ the pair $(L \otimes H^{\otimes(-2c)}, Z)$ satisfies trivially the Cayley-Bacharach property (see e.g. [Br] or [C]). Hence among the extensions of $L \otimes H^{\otimes(-2c)} \otimes \mathbf{I}_Z$ by $\mathbf{O}_X$ (i.e. like (2)) there is at least one with middle term, $\mathbf{Q}$, locally free, as wanted. □

*Proof of* 0.2. Fix the base point $P \in X$ to define uniquely the Albanese morphism $\alpha : X \to \mathrm{Alb}(X)$ with $0 = \alpha(P)$. Fix $H$ and $L$. We may assume $H$ very ample (taking if necessary a multiple depending only on $X$ of the given polarization). Twisting $L$ by $mH$ for some $m > 0$ depending only on $X$ and $H$, we may assume $h^0(K \otimes L^{-1}) = 0$ (a condition used in [BB], §1). We may assume $L$ and $K \otimes L$ very ample (twisting again $L$ by $mH$ for some $m > 0$ depending only on $X$

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and $H$). Set $q := \dim(\mathrm{Alb}(X)) = h^1(\mathbf{O})$. Fix the class $\mathbf{a} \in \mathrm{Alb}(X)$ as in the statement of 0.2. Fix an integer $t' > 0$ such that for every $t \geq t'$ the morphism $a_t$: $S^t(X) \to \mathrm{Alb}(X)$ from the $t$-th symmetric product of $X$, induced by the Albanese morphism $\alpha = a_1 : X \to A$ (with respect to $P$, i.e. with $a_t(D) := D - tP$ for every cycle $D \in S^t(X)$) is surjective. The proof will be divided into two steps.

(a) In this step we will show the existence of an integer $t'' \geq t'$ such that for every $t \geq t''$ there is a reduced $D \in S^t(X)$ such that for every $x \in D$ we have $h^0((K \otimes L) \otimes I_{D\setminus\{x\}}) = 0$ and such that $a_t(D)$ is the given class $\mathbf{a} \in \mathrm{Alb}(X)$. Fix any integer $z \geq t'$ with $z > h^0(K \otimes L)$ and a general $D \in S^z(X)$; in particular $D$ is reduced, $p \notin D$ and for every $x \in D$ we have $h^0((K \otimes L) \otimes I_{D\setminus\{x\}}) = 0$. Fix $z$ distinct smooth $C_i \in |H|$, $1 \leq i \leq z$, with $P \in C_i$, $\mathrm{card}(D \cap C_i) = 1$ for every $i$ and such that $C_i \cap C_j \cap D = \emptyset$ if $i \neq j$; set $x_i := D \cap C_i$. Set $g := p_a(C_i)$. Note that by Lefschetz theorem and the universal property of Albanese varieties the natural map $\mathrm{Alb}(C_i) \to \mathrm{Alb}(X)$ is surjective. We want to show that we may take $t'' := (2g+1)z$ (with $z := \max(t', h^0(K \otimes L) + 1)$ if we want). We fix a reduced cycle $D_i$ with $\deg(D_i) = 2g+1$, $x_i \in D_i$, $P \notin D_i$, $D_i - (2g+1)P$ linearly equivalent to zero in $C_i$ if $i < z$ (hence with $a_{2g+1}(D_i) = 0 \in \mathrm{Alb}(X)$) and with $D_z - (2g+1)P$ a class in $\mathrm{Alb}(C_i)$ mapped under the surjection $\mathrm{Alb}(C_i) \to \mathrm{Alb}(X)$ into the class $\mathbf{a}$. By construction we may take as $D$ the union of all $D_i$'s, $1 \leq i \leq z$.

(b) Fix an integer $k \geq t''$ (with $t''$ described in step (a)). Set $\mathbf{S} := \{D \subset S^k(X) : D$ is reduced and for every $x \in D$, $h^0((K \otimes L) \otimes I_{D\setminus\{x\}}) = 0\}$. For any $\mathbf{b} \in \mathrm{Alb}(X)$, let $\mathbf{S}(\mathbf{b}) := \{D \in \mathbf{S} : a_k(D) = \mathbf{b}\}$. Note that $\dim(\mathbf{S}) = 2k$ and that for every $\mathbf{b}$ every irreducible component of $\mathbf{S}(\mathbf{b})$ has codimension at most $q$ in $\mathbf{S}$. Note that every $D \in \mathbf{S}$ satisfies the Cayley-Bacharach property, hence define an extension (2) with $Q$ locally free with $c_1(Q) = L$ and $c_2(Q) = k$ (in $H^4(X, \mathbf{Z})$, i.e. $\deg(c_2(Q)) = k$); if $D \in \mathbf{S}(\mathbf{b})$, then $c_2(Q) - (k)P = \mathbf{b}$ in $\mathrm{Alb}(X)$. Hence it is sufficient to show that the set $\mathbf{S}^{\mathrm{un}} \subseteq \mathbf{S}$ giving unstable bundles has codimension at least $q+1$ in $\mathbf{S}$. Lemma 1.1 of [BB] states exactly the existence of a constant $C$ depending only on $X$, $H$ and $L$ but not $k$, such that every irreducible component of $\mathbf{S}^{\mathrm{un}}$ has dimension at most $C + q + k$. Thus it is sufficient to take $k > 2q + C$. □

We repeat that if $X$ has Kodaira dimension $\kappa(X) < 0$, then rational equivalence and abelian equivalence coincide. The proof of 0.2 works verbatim in positive characteristic $\neq 2$ ($\neq$ 2 just for the quotation of [BB]).

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