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Higher Dimensional Asymptotic Cycles

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Abstract. Given a p-dimensional oriented foliation of an n-dimensional compact manifold M^n and a transversal invariant measure τ, Sullivan has defined an element of $H_p(M^n, R)$. This generalized the notion of a μ-asymptotic cycle, which was originally defined for actions of the real line on compact spaces preserving an invariant measure μ. In this one-dimensional case there was a natural 1–1 correspondence between transversal invariant measures τ and invariant measures μ when one had a smooth flow without stationary points.

For what we call an oriented action of a connected Lie group on a compact manifold we again get in this paper such a correspondence, provided we have what we call a positive quantifier. (In the one-dimensional case such a quantifier is provided by the vector field defining the flow.) Sufficient conditions for the existence of such a quantifier are given, together with some applications.

1 Introduction

Let M^n be a smooth compact oriented manifold and suppose we are given a smooth action of the additive group of the real line on M^n. We will denote by v the velocity field on M^n corresponding to this flow and assume that we are given a finite measure μ on the Borel subsets of M^n that is invariant with respect to the flow. If ω is any smooth one-form on M^n and we let $\lambda_v(\omega)$ be defined to be $\int_{M^n} \omega \rfloor v \, d\mu$ (where $\omega \rfloor v$ is the interior product of ω with v), then λ_v is a one-dimensional current in the sense of De Rham. If $\omega = df$ then $(\omega \rfloor v)(x) = (\frac{df}{dt})(x) = \lim_{\Delta t \to 0} \frac{f(x\Delta t) - f(x)}{\Delta t}$. Since μ is assumed to be invariant, $\int_{M^n} \frac{f(x\Delta t)-f(x)}{\Delta t} \, d\mu = 0$, so $\int df \rfloor v \, d\mu = 0$.

This just says that λ_v is closed; that is to say that if ω is closed $\int_{M^n} \omega \rfloor v \, d\mu$ just depends on the class in the De Rham group $H^1(M^n, R)$ to which ω belongs. Thus we get from v and μ an element of $\operatorname{Hom}\big(H^1(M^n, R), R\big)$ which corresponds to the element of $H_1(M^n, R)$ that was called the asymptotic cycle A_μ in [5]. Given v and μ, to evaluate A_μ the most direct method would be to choose a basis for $H^1(M^n, R)$, let $\omega_1, \dots, \omega_k$ be one-forms corresponding to this basis, and evaluate $\int_{M^n} \omega_i \rfloor v \, d\mu$. If μ comes from a positive n-form α then it is known that $\alpha \rfloor v$ is a closed $(n-1)$-form and A_μ is the one-dimensional homology class arising from the $(n-1)$-dimensional cohomology class of $\alpha \rfloor v$ by Poincaré duality. If there is no point on M^n at which v vanishes, the orbits of our action of R^1 on M^n yield a one-dimensional oriented foliation of M^n. Using the notion of a transversal invariant measure as defined in [6], a generalization of the μ asymptotic cycle was given that applies to arbitrary smooth oriented foliations of a compact manifold [6].

We will sketch the definition of a transversal invariant measure. Suppose we are given a smooth p-dimensional oriented foliation of M^n and that on each closed $(n-p)$-dimensional disc D in M^n that is transverse to the foliation we are given

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a finite measure τ_D on the Borel subsets of the interior of D. If D_1 and D_2 are two such discs with $x_1 \in$ interior D_1 and $x_2 \in$ interior D_2 linked by a path A lying in a single leaf of the foliation, then A determines the germ of a homeomorphism from a neighborhood of x_1 in D_1 to a neighborhood of x_2 in D_2. If this always makes the germ of τ_{D_1} at x_1 correspond to the germ of τ_{D_2} at x_2 we say that our system τ of measures is a transversal invariant measure. If we are given any collection $\{D_\alpha\}$ of closed transversal discs such that each leaf intersects the interior of at least one of the D_α and we are given a system of finite measures $\{\tau_\alpha\}$ on the interiors of the D_α that satisfy the above compatibility condition, this is enough to determine a transversal invariant measure.

If $D_1, \ldots, D_k$ and $\tau_1, \ldots, \tau_k$ determine a transversal invariant measure and if $F_1, \ldots, F_k$ are flow boxes of the foliation centered on $D_1, \ldots, D_k$ whose interiors cover all of M^n, and if in addition ω is any smooth p-form on M^n, we can find forms $\omega_1, \ldots, \omega_k$ such that the support of ω_i is contained in the interior of the image of F_i and $\omega = \omega_1 + \cdots + \omega_k$. (By a flow box we mean a map F of $B^{n-p} \times B^p$, the cartesian product of the closed unit $(n-p)$ ball in R^{n-p} centered at the origin with the closed unit p-ball in R^p, homeomorphically onto a subset of M^n such that for any $a \in B^{n-p}$, $F(a \times B^p)$ is contained in a single leaf of our foliation. We say that F is centered on the closed transversal disc $F\big(B^{n-p} \times (0, \ldots, 0)\big)$.

If $q = F_i\big(a, (0, \ldots, 0)\big)$ let L_q be $F_i(a \times B^p)$. Denote by $f_i(q)$ the number $\int_{Lq} \omega_i$. Then $\sum_{i=1}^k \int_{D_i} f_i(q)\, d\tau_i(q)$ turns out to be unchanged if we substitute for $D_1, \ldots, D_k$, $\tau_1, \ldots, \tau_k$ and $\omega_1, \ldots, \omega_k$ any other collection $D'_1, \ldots, D'_k$ with measures $\tau'_1, \ldots, \tau'_k$ determining the same transversal invariant measure τ and forms $\omega'_1, \ldots, \omega'_l$ with the same $\sum \omega'_i$. Thus if τ is a transversal invariant measure, τ determines a current (which we will denote by λ_τ such that $\lambda_\tau(\omega) = \sum_{i=1}^K \int_{D_i} f_i(q)\, d\tau_i(q)$. This current can be shown to be closed and thus yields an element A_τ of $H_p(M^n, R)$. This element is called the Ruelle-Sullivan class of τ. Let us now go back to the situation where we had a one-dimensional oriented foliation associated with a smooth flow. As before let v be the velocity field associated with the flow. On any orbit O of the flow the one-form dual to v determines a measure μ_O on O.

Suppose that F is any flow box, that F is centered on the $(n-1)$ disc D, and that μ is any finite measure defined on the Borel subsets of M^n. For any $x \in D$ let $O(x)$ be the orbit through x and let $a \in B^{n-1}$ be such that $x = F(a, 0)$. Then there is one and only one measure τ_D on the Borel subsets of the interior of D such that if f is any continuous function whose support is contained in the interior of the image of F,

$$\int_{M^n} f(x)\, d\mu(x) = \int_D \left(\int_{F(a \times B^1)} f(y)\, d\mu_{O(p)}(y) \right) d\tau_D(a).$$

Since $F(a, B^1)$ is contained in $O(p)$, the integral $\int_{F(a \times B^1)} f(y)\, d\mu_{O(p)}(y)$ has an obvious meaning. If μ is an invariant measure then τ_D depends only on D and not on the particular flow box F we used. Moreover in this case we get a transversal invariant measure τ, and given a transversal invariant measure τ, we can go backwards and get a finite invariant measure μ defined on the Borel subsets of M^n. If μ is any finite invariant measure and τ is the-corresponding transversal invariant measure, then $A_\mu = A_\tau$.

The Ruelle-Sullivan class plays an essential role in the index theorem Connes has given for families of elliptic operators acting along the leaves of a p-dimensional oriented foliation of a smooth manifold M^n [3]. However for $p > 1$ it is not easy to describe a specific transversal invariant measure for a concretely given p-dimensional foliation. In the one-dimensional case, where the foliation can be specified by giving a smooth vector field v (which determines a flow) it is much easier to give specific examples, because of the connection between transversal invariant measures and measures invariant with respect to the flow, and because in many cases one can specify an invariant measure by giving an n-form on M^n.

If we are given a smooth action of a connected Lie group L on a compact oriented manifold M^n we will say that the action is oriented provided all the orbits are of the same dimension and we are given a continuously varying orientation of the tangent spaces to the orbits. Obviously an oriented action determines an oriented foliation. In this paper we will see how, under favorable conditions, one can get a 1–1 correspondence between transversal invariant measures τ and finite measures μ defined on the Borel subsets of M^n that are invariant under the action of L such that $A_\tau = A_\mu$, where A_μ is given a suitable definition.

In what follows we will assume that M^n is a smooth compact oriented manifold and that we are given a smooth oriented action of a connected Lie group L on M^n whose orbits have dimension p. (The action will be on the right.)

Definition A *quantifier* is a continuous field of p-vectors on M^n, everywhere tangent to the orbits and invariant under the action of L. A quantifier is said to be *positive* if it is nowhere zero and at each point of M^n determines an orientation of the tangent space to the orbit through that point that agrees with the orientation associated with our oriented action.

Definition A *preferred action* is an oriented action of a connected Lie group L such that for any $x \in M^n$ the isotropy group D_x of x (the set of elements of L leaving x fixed) is a normal subgroup of L and L/D_x is unimodular. Thus a free action of a unimodular group is preferred as is any oriented action of a commutative group, assuming these groups are connected.

2 Statement of Results

We will prove:

Theorem 1 *Every preferred action possesses a positive quantifier.*

Given a positive quantifier v, every quantifier is of the form fv, where f is a continuous invariant realvalued function. Thus if our action possesses no non-constant continuous invariant functions and a positive quantifier exists, the vector space of quantifiers is one-dimensional.

Suppose we are given a positive quantifier v.

Theorem 2 *We can, given v, define a canonical 1–1 correspondence between finite invariant measures μ defined on the Borel subsets of M^n and transversal invariant measures τ.*

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For any invariant measure μ define a linear functional λ_μ^ν on the space of C^∞ p-forms ω by $\lambda_\mu^\nu(\omega) = \int_{M^n} \omega \rfloor \nu \, d\mu$. Then we will show that λ_μ^ν is a closed current in the sense of De Rham and therefore defines an element A_μ^ν in $H_p(M^n, R)$ which we will call the *asymptotic cycle* associated with the pair (μ, ν). If τ is the transversal invariant measure associated with μ the Ruelle-Sullivan class $A_\tau = A_\mu^\nu$.

For smooth actions of the additive group of the real line without stationary points we recover the situation described in the opening paragraphs of this introduction by taking ν to be the velocity field of the flow.

We will also prove

Theorem 3 *If ν is a positive quantifier and the invariant measure μ arises from a positive n-form ω, then $\omega \rfloor \nu$ is closed and A_μ^ν can be gotten by Poincaré duality from the element of $H^{n-p}(M^n, R)$ determined by $\omega \rfloor \nu$.*

Next we will assume we are given a preferred action that preserves some smooth Riemannian metric. Suppose that ν_1 and ν_2 are positive quantifiers and that μ_1 and μ_2 are invariant measures for this action. Then we will prove:

Theorem 4 *There exists a positive constant λ such that $A_{\mu_2}^{\nu_2} = \lambda A_{\mu_1}^{\nu_1}$.*

Corollary *If O_1 and O_2 are compact orbits and a_1 and a_2 are the images in $H_p(M^n, R)$ of their fundamental homology classes, then a_1 is a scalar times a_2.*

Now suppose that G is a connected Lie group and that K is a closed subgroup of G such that the space G/K of right cosets is compact.

Definition A p-dimensional jacket for K is a closed normal subgroup H of G containing K such that the natural map of $H^p(G/H, R)$ into $H^p(G/K, R)$ is surjective.

Let L be a subgroup of G corresponding to some subalgebra ℓ of the Lie algebra g of G, and allow L to act on the right on G/K in the usual way. Suppose that there are no non-constant continuous invariant functions for this action of L on G/K and suppose further that this is an oriented action with p-dimensional orbits and that ν is a positive quantifier for this action.

Theorem 5 *If μ_1 and μ_2 are any two invariant probability measures for the action of L on G/K, $A_{\mu_1}^\nu = A_{\mu_2}^\nu$, provided K possesses a p-dimensional jacket.*

Moreover, as we will see, the asymptotic cycle can be found in this case without performing any integrations.

3 The Main Theorems

We are now ready to prove:

Theorem 1 *A preferred action possesses a positive quantifier.*

Proof We will say that a quantifier ν on M^n is *semipositive* provided that at each $x \in M^n$ such that $\nu(x) \neq 0$ the orientation of the tangent space at x given by $\nu(x)$ agrees with that associated with our oriented action. Since M^n is compact, to prove that our action possesses a positive quantifier it is enough to show that for each $x \in M^n$ there is a semipositive quantifier ν such that $\nu(x) \neq 0$.

Suppose that for a given $x_0 \in M^n$ we have any quantifier ν_0 such that $\nu_0(x_0) \neq 0$. Define $\epsilon(x)$ to equal one if $\nu_0(x) \neq 0$ and the orientation of the tangent space to the orbit at x induced by $\nu_0(x)$ agrees with that associated with our oriented action. Otherwise if $\nu_0(x) \neq 0$ define $\epsilon(x)$ to equal minus one, and if $\nu_0(x) = 0$ define $\epsilon(x)$ to equal zero. Let $\nu(x) = \epsilon(x)\nu_0(x)$. Clearly $\nu(x)$ is continuous wherever $\epsilon(x) \neq 0$. Put a Riemannian metric on M^n and introduce the associated norm on the space of p-vectors at each point. Then $|\nu(x)| = |\nu_0(x)|$ for all x. Since $\lim_{x\to a} |\nu(x)| = \lim_{x\to a} |\nu_0(x)| = |\nu_0(a)|$, if $\epsilon(a) = 0$, $lim_{x\to a}|\nu(x)| = 0$. Therefore if $\epsilon(a) = 0$, $\lim_{x\to a} \nu(x) = 0$ so $\lim_{x\to a} \nu(x) = \nu(a)$. Thus $\nu(x)$ is continuous everywhere and is a semipositive quantifier. Hence to prove the existence of a positive quantifier, it is enough to show that for any $x \in M^n$ there is a quantifier ν such that $\nu(x) \neq 0$.

Next let ω_0 be any element in the space of p-vectors over the tangent space at the identity element e in our Lie group L. For any $x \in M^n$ let f_x be the map of L into M^n that sends $\ell \in L$ into $x\ell$. Define $\nu(x)$ to be the image of ω_0 under the map of p-vectors induced by f_x. Then for any $x \in M^n$, we can choose ω_0 so that $\nu(x)$ is a non-zero p-vector at x tangent to the orbit through x. To prove that any preferred action possesses a positive quantifier it is enough to prove the following:

Lemma 1 *If we are given a preferred action of L on M^n and all the orbits have dimension p, for any ω_0 the corresponding $\nu(x)$ is invariant under the action of L.*

Proof We will first establish this in the case in which M^n consists of a single orbit. We therefore assume that we have a connected Lie group G with identity element e and a closed normal subgroup K such that G/K is unimodular. G will act on the right on the space of right cosets mod K.

We will need some notations. For any vector space V, $\wedge^p(V)$ will denote the vector space of p-vectors over V. For any linear map T of a vector space V_1 into a vector space V_2 we will denote by $\bar{T}$ the induced map of $\wedge^p(V_1)$ into $\wedge^p(V_2)$. For any $g_0 \in G$ we will let d_{g_0} be the differential at g_0 of the projection of G onto G/K sending g into Kg. For any Lie group and any element a of that group R_a will be the differential at the identity element e of the map sending g into ga and L_a will be the differential at e of the map sending g into ag.

Given ω_0 we now define ν to be the p-vector field whose value at any $Kg \in G/K$ is $\bar{R}_{Kg}\big(\bar{d}_e(\omega_0)\big)$. This p-vector field is obviously invariant under the action of G on G/K. To complete the proof that in the special case of a homogeneous space our lemma holds we need to show that for any $x \in G$, $\nu(Kx)$ is the image of ω_0 under the map of p-vectors induced by the function f_{Kx} sending $g \in G$ into Kxg. We see that this means we have to show that $\nu(Kx) = \bar{d}_x\bar{L}_x(\omega_0)$.

However $R_{Kx} = L_{Kx}\,\mathrm{Ad}_{G/K}(Kx)$, so $\bar{R}_{Kx} = \bar{L}_{Kx}\overline{\mathrm{Ad}}_{G/K}(Kx)$. Since we are assuming G/K is unimodular, $\overline{\mathrm{Ad}}_{G/K}(Kx)$ is the identity map for any x so $\bar{R}_{Kx} = \bar{L}_{Kx}$. Then $\nu(Kx) = \bar{R}_{Kx}\bar{d}_e(\omega_0) = \bar{L}_{Kx}\bar{d}_e(\omega_0)$. However $L_{Kx}d_e = d_xL_x$, so we get $\nu(Kx) = \bar{d}_x\bar{L}_x(\omega_0)$ as we wished.

To complete the proof of Lemma 1 and therefore that of Theorem 1, we note that for any $x \in M^n$ there is a homogeneous space L/D_x of the kind we have considered

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and a 1–1 equivariant continuous map of L/D_x onto the orbit containing x. The p-vector ω_0 induces a p-vector field both on L/D_x and on M^n and the fact that the field on L/D_x is invariant implies that the field on M^n is invariant.

This establishes Lemma 1 and therefore concludes the proof of Theorem 1.

If we are given a preferred action of L on M^n such that no non-constant invariant continuous function exists, then if we choose ω_0 such that the corresponding ν is not identically zero, either ν or $-\nu$ will be a positive quantifier. Thus in this case we can hope to construct the (essentially unique) positive quantifier.

Before proceeding to the proof of Theorem 2 we need to cite a result that appears in [6, Theorems I.12 and I.13].

Assume we have given a particular positive quantifier ν. If μ is any finite non-negative measure on the Borel subsets of M^n, the map which assigns to any p-form ω the value $\int_{M^n} \omega \rfloor \nu \, d\mu$ is called a structure current in [6].

It is a current in the sense of De Rham and we will denote it by λ_μ^ν. Recall also that in the introduction we associated with any transversal invariant measure τ a current that we denoted by λ_τ.

Using a partition of unity it is easy to see that every continuous real valued function on M^n is of the form $\omega \rfloor \nu$, so two different measures μ_1 and μ_2 determine different structure currents, and moreover if $\tau_1 \neq \tau_2$ then $\lambda\tau_1 \neq \lambda\tau_2$.

The result we need from [6] asserts that there is a 1–1 correspondence between closed structure currents λ_μ^ν and transversal invariant measures τ such that $\lambda_\mu^\nu = \lambda_\tau$ when λ_μ^ν and τ correspond.

What we are going to do is prove that λ_μ^ν is closed if and only if μ is an invariant measure. This will give us a canonical 1–1 correspondence between transversal invariant measures and finite invariant measures defined on the Borel subsets of M^n given a positive quantifier ν.

First we will prove:

Theorem 2A *λ_μ^ν is closed if and only if μ is an invariant measure.*

Proof Assume λ_μ^ν is closed; let τ be the corresponding transversal invariant measure. If f is a given real valued continuous function on M^n, then as was noted previously we can find a p-form ω such that $f = \omega \rfloor \nu$. Then

$$\lambda_\tau(\omega) = \int \omega \rfloor \nu \, d\mu = \int f(x) \, d\mu(x).$$

For any function or tensor on M^n we will indicate its translate by an element $\ell \in L$ by adjoining a subscript ℓ. Thus $f_\ell(x)$ will equal $f(x\ell)$.

We next note that simce τ is a transversal invariant measure, $\lambda_\tau(\omega) = \lambda_\tau(\omega\ell) = \int \omega_\ell \rfloor \nu \, d\mu$ and since $\nu = \nu_\ell$, we see that $\int f(x) \, d\mu(x) = \int f(x\ell) \, d\mu(x)$. Thus μ is an invariant measure.

Next we want to show that if μ is an invariant measure λ_μ^ν is closed. This just says that for any $(p-1)$ form α, $\int_{M^n} d\alpha \rfloor \nu \, d\mu$ is zero.

If $F_1, \dots, F_k$ are flow boxes the union of whose interiors is all of M^n we can get smooth functions $f_1, \dots, f_k$ such that the support of each f_i is contained in the interior of F_i and $\sum f_i = 1$. Then $d\alpha = \sum d(f_i\alpha)$. Therefore it is enough to show that

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$\int_{M^n} d\alpha \rfloor \nu \, d\mu = 0$ whenever α is a smooth $(p-1)$ form whose support is contained in the interior of a flow box.

Suppose then that we have a smooth closed $(n-p)$ disc D transverse to our foliation and let $F\colon B^{n-p} \times B^p \to M^n$ be a smooth map sending $B^{n-p} \times B^p$ diffeomorphically into M^n. Assume further that for any $a \in B^{n-p}$, $F(a \times B^p)$ is contained in a single orbit, and that $F\big(B^{n-p} \times (0,\dots,0)\big)$ is D.

For any $x = F\big(a,(0,\dots,0)\big)$ in D let L_x equal $F(a, B^p)$. We let Π be the map of $F(B^{n-p} \times B^p)$ onto D sending $F(a,b)$ onto $F\big(a,(0,\dots,0)\big)$. Define the measure τ_D on D to equal $\Pi^\star \bar{\mu}$, where $\bar{\mu}$ is the restriction of μ to $F(B^{n-p} \times B^p)$. Then we will need the following standard result:

We can associate with each x in the interior of D a probability measure $\bar{\mu}_x$ whose support is contained in L_x so that

(a) For each continuous function f whose support is contained in the interior of $F(B^{n-p} \times B^p)$ the function $k\colon x \to \int_{L_x} f(q)\, d\bar{\mu}_x(q)$ is a Borel measurable function on the interior of D.

(b)
$$\int_{F(B^{n-p} \times B^p)} f(q)\, d\bar{\mu}(q) = \int k(x)\, d\tau_D(x).$$

Moreover if $\bar{\mu}_x^1$ and $\bar{\mu}_x^2$ are two such sets of measures there is a set N contained in the interior of D such that $\tau_D(N) = 0$ and for any x in the interior of D and outside N, $\bar{\mu}_x^1 = \bar{\mu}_x^2$.

The space of p-vectors tangent to any L_x is one dimensional so there is an unambiguous meaning attached to the p-form $\nu_x^\star$ defined on L_x and dual to ν along L_x.

We are going to show that for x in D (except for a set of τ_D-measure zero) $\bar{\mu}_x$ is a scalar multiple of the measure on the interior of L_x arising from $\nu_x^\star$.

If α is a smooth $(p-1)$ form whose support is contained in the interior of our flow box,
$$\int_{L_x} (d\alpha \rfloor \nu)\nu_x^\star = \int_{L_x} d\alpha = 0.$$

Therefore we will have showed that for x outside a set of τ_D measure zero,
$$\int_{L_x} (d\alpha \rfloor \nu)\, d\bar{\mu}_x = 0$$
and hence that
$$\int_{F(B^{n-p} \times B^p)} (d\alpha \rfloor \nu)\, d\bar{\mu} = 0.$$

By what we have said previously this will suffice to prove that λ_μ^ν is closed.

We will need the following:

Lemma 2 *Suppose $\ell \in L$ and f is a continuous function such that for q outside a compact subset of the interior of $F(B^{n-p} \times B^p)$ both $f(q)$ and $f(q\ell)$ vanish. Then there is a set N_ℓ^f in the interior of D such that*

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(a) $\tau_D(N_\ell^f) = 0$.
(b) *If* $x \notin N_\ell^f$ *but* x *is in the interior of* D

$$\int_{L_x} f(q)\, d\bar{\mu}_x(q) = \int_{L_x} f(q\ell)\, d\bar{\mu}_x(q).$$

Proof To prove this we note that since μ is an invariant measure,

$$\int_{F(B^{n-p}\times B^p)} f(q)\, d\bar{\mu}(q) = \int_{F(B^{n-p}\times B^p)} f(q\ell)\, d\bar{\mu}(q).$$

By the same token if g is any continuous function on the interior of D

$$\int_{F(B^{n-p}\times B^p)} g(\Pi q) f(q)\, d\bar{\mu}(q) = \int_{F(B^{n-p}\times B^p)} g(\Pi q) f(q\ell)\, d\bar{\mu}(q).$$

Therefore

$$\int_D \left(\int_{L_x} \big(f(q) - f(q\ell)\big)\, d\bar{\mu}_x(q) \right) g(x)\, d\tau_D(x) = 0.$$

Thus if $h(x) = \int_{L_x} \big(f(q) - f(q\ell)\big)\, d\bar{\mu}_x(q)$, $h(x) = 0$ except on a set of τ_D measure zero, which proves our lemma.

Now let $C(M^n)$ be the Banach space of continuous real valued functions on M^n and let S be the set of all pairs (f, ℓ) in $C(M^n) \times L$ such that both $f(x)$ and $f(x\ell)$ vanish for x outside a compact subset of the interior of $F(B^{n-p} \times B^p)$. Let $\{(f_i, \ell_i)\}$ be a countable dense subset of S. If we let $N_D = \bigcup N_{\ell_i}^{f_i}$, then $\tau_D(N_D) = 0$.

Lemma 3 *For any* f *and any* ℓ, *if* x *is outside* N_D *and both* $f(q)$ *and* $f(q\ell)$ *vanish for* q *outside a compact subset of the interior of* $F(B^{n-p} \times B^p)$ *then*

$$\int_{L_x} f(q)\, d\bar{\mu}_x(q) = \int_{L_x} f(q\ell)\, d\bar{\mu}_x(q).$$

Proof Obvious.

We are now ready to prove Theorem 2A. We need only show that for x in the interior of D but outside N_D the measure $\bar{\mu}_x$ is a scalar multiple of the measure on L_x determined by $v_x^\star$.

If we put the leaf topology on the orbit $O(x)$ containing x, $O(x)$ becomes a homogeneous space of the Lie group L. The p-form $v_x^\star$ on $O(x)$ determined a σ-finite measure on $O(x)$ invariant under the action of L. However a σ-finite invariant measure on a homogeneous space of a connected Lie group is unique up to a multiplicative constant. Therefore we need only show that $\bar{\mu}_x$ on L_x extends to a σ-finite invariant measure on $O(x)$.

We first note that for any $\ell \in L$ the collection of all bounded functions g on $O(x)$ such that $g(y)$ and $g(y\ell)$ both vanish outside the interior of L_x and such that $\int g(y)\, d\bar{\mu}_x(y)$ and $\int g(y\ell)\, d\bar{\mu}_x(y)$ are defined and equal is closed under bounded

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pointwise convergence. It follows from Lemma 3 that if S is any Borel subset of the interior of L_x such that $S\ell$ is contained in the interior of L_x then $\bar{\mu}_x(S) = \bar{\mu}_x(S\ell)$.

By virtue of the homeomorphism $x \to x\ell$ we can put a measure on $(L_x)\ell$ that is the translate by ℓ of $\bar{\mu}_x$. By what we established above, for any Borel subset of the interior of $(L_x)\ell_1 \cap (L_x)\ell_2$, the measures we get are the same.

Now suppose K is any compact subset of $O(x)$. We can choose a finite set $\ell_1, \ldots, \ell_n$ in L such that K is contained in $\bigcup(\text{interior } L_x)\ell_i$. Consider the collection of all sets $K \cap B_1 \cap \cdots \cap B_n$ where each B_i equals either (interior $L_x)\ell_i$ or its complement. If we exclude the case where each B_i is the complement of (interior $L_x)\ell_i$, we get $2^n - 1$ sets that cover K and on each of these sets we have a countably additive measure gotten by translating $\bar{\mu}_x$. Thus we get a countably additive measure on K.

If $\ell_1', \ldots, \ell_n'$ is another set such that K is contained in $\bigcup(\text{interior } L_x)\ell_i'$, we need to show that the measure we get on K from this set is the same as that which we got from $\ell_i, \ldots, \ell_n$. However if we take the union of these two finite sets and consider the measure we get on K from this finite set, it is clear that this measure coincides both with the one we got from $\ell_1, \ldots, \ell_n$ and the one we got from $\ell_1', \ldots, \ell_n'$.

It is also clear that if S is a Borel subset of K and $S\ell$ is also a subset of K their measures are equal.

Finally, choose an increasing sequence $K_1, \ldots, K_n, \ldots$ of compact sets whose interiors cover $O(x)$. This enables us to define an extension of $\bar{\mu}_x$ from the collection of Borel subsets of L_x to the collection of all Borel sets with compact closure. This can be seen to extend to a countably additive measure on $O(x)$ which is invariant under the action of L and is σ-finite. Thus the proof of Theorem 2A is completed.

By what was said previously this establishes:

Theorem 2 *There is a 1–1 correspondence between transversal invariant measures τ and finite invariant measuers μ on the Borel subsets of M^n such that $\lambda_\tau = \lambda_\mu^\nu$ if τ and μ correspond.*

Corollary *An oriented action of a connected commutative Lie group always possesses a transversal invariant measure.*

Proof It follows from Theorem 1 that such a action always has a positive quantifier.

It is well known that any commutative group of homeomorphisms of a compact metric space possesses an invariant Borel measure. Thus our corollary follows from Theorem 2.

4 The Applications

We are now ready to prove:

Theorem 3 *If ν is a positive quantifier and the invariant measure μ arises from a positive n-form ω then $\omega \rfloor \nu$ is closed and A_μ^ν can be gotten by Poincaré duality from the element of $H^{n-p}(M^n, R)$ determined by $\omega \rfloor \nu$.*

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Proof Suppose we are given a smooth flow box centered at the transversal D. If α is any p-form whose support is contained in the interior of the image of F and τ'_D is the transversal invariant measure associated with (ν, μ) then $\int_{F(B^{n-p}\times B^p)}(\alpha\rfloor\nu)\omega = \int_D(\int_{L_x}\alpha)\,d\tau'_D(x)$. However, since $\omega\rfloor\nu$ is invariant under the action of our Lie group, by introducing coordinates we see that $\int_D(\int_{L_x}\alpha)\,d\tau'_D(x) = \int(\omega\rfloor\nu)\wedge\alpha$.

Any p-form α is a finite sum of p-forms whose supports are contained in the interiors of the images of smooth flow boxes and in our discussion of the Ruelle-Sullivan class we saw that we could use this to define $\lambda(\alpha)$ for any p-form α. We said that if we started with any transverse invariant measure τ, the $\lambda_\tau(\alpha)$ we got was zero for any bounding p-form α. Thus $\int_{M^n}(\omega\rfloor\nu)\wedge\alpha = 0$ for any bounding p-form α, which implies that $\omega\rfloor\nu$ is closed. Moreover the equality $\lambda(\alpha) = \int_{M^n}(\omega\rfloor\nu)\wedge\alpha$ precisely tells us that the Ruelle-Sullivan class A_τ and consequently the asymptotic cycle A^ν_μ arises from $\omega\rfloor\nu$ by Poincaré duality.

Theorem 4 *Suppose we are given a preferred action that preserves a Riemannian metric and that ν_1 and ν_2 are positive quantifiers for this action. Then if μ_1 and μ_2 are finite invariant measures, there is a positive constant λ such that $A^{\nu_2}_{\mu_2} = \lambda A^{\nu_1}_{\mu_1}$.*

We are going to associate with a suitably chosen p-vector w_0 over the tangent space at the identity of our original Lie group L a positive quantifier ν_0. For a positive quantifier ν_0 gotten in this way we will be able to establish two properties that, taken together, will imply Theorem 4. First we will show that if μ_1 and μ_2 are any two finite invariant measures then

$$A^{\nu_0}_{\mu_1}/\mu_1(M^n) = A^{\nu_0}_{\mu_2}/\mu_2(M^n).$$

We will also see that if ν is any positive quantifier and μ is any finite invariant measure there exists a positive constant α such that $A^\nu_\mu = \alpha A^{\nu_0}_\mu$. It is clear that once we have established these two properties of ν_0, Theorem 4 will have been proved.

Proof Recall that the group of isometries of the compact manifold M^n is a Lie group that acts smoothly on M^n. The action of L on M^n gives a 1–1 continuous homomorphism of L into this group. Let the closure of the image of L be denoted by $\bar{L}$. It is a compact Lie group that acts smoothly on M^n.

Any orbit under the action of $\bar{L}$ on M^n determines a conjugacy class of subgroups of $\bar{L}$, namely the isotropy groups of points in the orbit. Theorem 3.1 of [2] asserts:

> There exists a maximum orbit type $\bar{L}/H$ for $\bar{L}$ on M^n (*i.e.*, H is conjugate to a subgroup of each isotropy group). The union $M^n_{(H)}$ of the orbits of type $\bar{L}/H$ is open and dense in M^n and its image $M^\star_{(H)}$ in the orbit space $M^\star = M^n/\bar{L}$ is connected.

An orbit of type $\bar{L}/H$ is called a *principal orbit*.

We will also need the following consequence of Theorem 5.8 of [2].

> If $q \in M^\star_{(H)}$ there is an open neighborhood U of q in $M^\star_{(H)}$ for which there exists an equivariant diffeomorphism of $U \times \bar{L}/H \to$ onto $F^{-1}(U)$, where F is the projection of M^n onto $M^n/\bar{L}$. (Here $\bar{L}$ acts on $U \times \bar{L}/H$ so that $g \in \bar{L}$ sends (q, Hg_0) into (q, Hg_0g).)

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Now suppose that ω_0 is any element in the space of p-vectors over the tangent space at the identity e in our original Lie group L. For any $x \in M^n$ let f_x be the map of L into M^n that sends $\ell \in L$ into $x\ell$. Define $v_0(x)$ to be the image of ω_0 under the map of p-vectors induced by f_x. Then Lemma 1 tells us that v_0 is a quantifier for the action of L, because this is a preferred action.

Since the image of L is dense in $\bar{L}$, v_0 is invariant under the action of $\bar{L}$. By the Theorem of [2] cited above it follows that if v_0 is zero on $F^{-1}(z)$ where $z \in M^{\star}_{(H)}$, there is an open set V in $M^{\star}_{(H)}$ containing z such that v_0 is zero on $F^{-1}(V)$. However the set of all $q \in M^{\star}_{(H)}$ such that v_0 is zero on $F^{-1}(q)$ is closed in $M^{\star}_{(H)}$. Since $M^{\star}_{(H)}$ is connected it would follow that v_0 is zero on $F^{-1}(M^{\star}_{H})$, and since $M^{\star}_{(H)}$ is dense in $M^n/\bar{L}$ it would follow that v_0 is identically zero.

Since v_0 is invariant under the action of $\bar{L}$ and $\bar{L}$ is connected, it follows that if v_0 is not identically zero it vanishes nowhere on $F^{-1}(M^{\star}_{(H)})$. By the same theorem from [2] we used above we see that if $z \in M^{\star}_{(H)}$, either there is an open set $W \subseteq M^{\star}_{(H)}$ such that $z \in W$ and v_0 is a positive quantifier on $F^{-1}(W)$ or the same holds for $-v_0$. Since $M^{\star}_{(H)}$ is connected and v_0 vanishes nowhere on $F^{-1}(M^{\star}_{(H)})$ it follows that either v_0 is a positive quantifier on $F^{-1}(M^{\star}_{(H)})$ or the same holds for $-v_0$.

Since $M^{\star}_{(H)}$ is dense in $M^n/\bar{L}$, it follows that there are only three possibilities:

(a) v_0 is identically zero,
(b) v_0 is semipositive,
(c) $-v_0$ is semipositive.

Now we wish to prove:

Lemma 4 *We can choose ω_0 so that the corresponding v_0 is a positive quantifier.*

Proof By the compactness of M^n and the fact that v_0 depends linearly on ω_0, we see that it is enough to show that for any $x \in M^n$ we can choose ω_0 so that v_0 is semipositive and $v_0(x) \neq 0$. We can certainly pick ω_0 so that $v_0(x)$ is positive.

Since possibilities (a) and (c) above cannot hold, v_0 is semipositive. Thus our lemma is proved.

Next let f be any continuous real valued function on M^n and let m be a probability measure on $\bar{L}$.

Lemma 5 $\int_{M^n} f(x)\, d\mu(x) = \int_{M^n} \big(\int_{\bar{L}} f(xg)\, dm(g) \big)\, d\mu(x)$.

Proof By the fact that μ is an invariant measure this must hold if m is concentrated at a single point. It follows that it is still true if the support of μ is finite. However if we let $h_x(g) = f(xg)$, the family of function h_x is equiuniformly continuous and uniformly bounded. It follows that we can get a sequence m_i of probability measures such that the support of each m_i is finite and $\int h_x(g)\, dm_i(g)$ converges uniformly in x to $\int h_x(g)\, dm(g)$. Our lemma follows.

If we let m be Haar measure on $\bar{L}$ with $m(\bar{L}) = 1$ and if λ_x is the invariant measure on $x\bar{L}$ such that $\lambda_x(x\bar{L}) = 1$, we note that $\int_{\bar{L}} f(xg)\, dm(g) = \int_{x\bar{L}} f(y)\, d\lambda_x(y)$. (We know that such an invariant measure λ_x exists because $\bar{L}$ is compact.)

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Now suppose ω is a closed p-form on M^n and $q \in M_H^\star$. We know there is equivariant diffeomorphism between $F^{-1}(U)$ and $U \times \bar{L}/H$ for some connected open set U containing q. Via this diffeomorphism ω corresponds to a closed form $\bar{\omega}$ on $U \times \bar{L}/H$. If ν_0 is a positive quantifier on M^n arising from a p-vector ω_0 over the tangent space to L at its identity element, the restriction of ν_0 to $F^{-1}(U)$ corresponds via this diffeomorphism to the positive quantifier $\bar{\nu}_0$ on the L space $U \times \bar{L}/H$ arising from ω_0. The form ω_1 on $\bar{L}/H$ that arises from the imbedding of $\bar{L}/H$ into $U \times \bar{L}/H$ that sends Hg into (q_1, Hg) for any $q_1 \in U$ is cohomologous to the form ω_2 we get using any other $q_2 \in U$. Here we are assuming, as we may, that U is arcwise connected so that these two imbeddings are homotopic. Therefore the integral of $\omega_1 \rfloor \bar{\nu}_0$ with respect to any invariant measure λ on the L space $\bar{L}/H$ is the same as the integral of $\omega_2 \rfloor \bar{\nu}_0$, as follows from the fact that these integrals depend only on the cohomology class determined by our forms, a fact that we learn from Lemma 2.

Thus if on each $\bar{L}$ orbit $x\bar{L}$ we place the invariant measure λ_x, then $\int_{x\bar{L}} \omega \rfloor \nu_0 d\lambda_x$ is a function on $M/\bar{L}$ that is locally constant on the connected set $M_H^\star$. Thus $\int_{x\bar{L}} \omega \rfloor \nu_0\, d\lambda_x$ is constant on $M_H^\star$. Since $\int_{\bar{L}} (\omega \rfloor \nu_0)(xg)\, dm(g) = \int_{x\bar{L}} \omega \rfloor \nu_0\, d\lambda_x$ is a continuous function of $x \in M^n$ that is constant on the dense subset $F^{-1}(M_{(H)}^\star)$ it is constant on all of M^n. By Lemma 5 we see that for any finite invariant measure μ on M^n,

$$\int_{M^n} (\omega \rfloor \nu_0)(x)\, d\mu(x) = \int_{M^n} k(\omega)\, d\mu(x) = k(\omega)\mu(M^n)$$

where $k(\omega) = \int_{\bar{L}} (\omega \rfloor \nu_0)(xg)\, dm(g)$ for any $x \in M^n$. Thus if we identify $A_\mu^{\nu_0}$ with the element of $\operatorname{Hom}\bigl(H^p(M^n, R), R\bigr)$ that it determines, $k(\omega) = \frac{1}{\mu(M^n)}$ times the value of $A_\mu^{\nu_0}$ at the cohomology class of ω. Therefore $A_{\mu_1}^{\nu_0} = \frac{\mu_1(M^n)}{\mu_2(M^n)} A_{\mu_2}^{\nu_0}$ for any two invariant measures μ_1 and μ_2. This establishes the first of the two properties of ν_0 that we need to have in order to prove Theorem 4.

Now if ν_2 is any positive quantifier on M^n, there is a continuous function β on $M^n/\bar{L}$ such that $\nu_2(x) = \beta\bigl(F(x)\bigr)\, \nu_0(x)$.

Thus by Lemma 5, for any invariant measure μ

$$\begin{aligned}\int_{M^n} \omega \rfloor \nu_2\, d\mu &= \int_{M^n} \left(\int_{\bar{L}} (\omega \rfloor \nu_2)(xg)\, dm(g) \right) d\mu(x) \\ &= \int_{M^n} \beta\bigl(F(x)\bigr)\, k(\omega)\, d\mu(x) = k(\omega) \int_{M^n} \beta\bigl(F(x)\bigr)\, d\mu(x).\end{aligned}$$

From this it follows that $A_\mu^{\nu_2}$ equals a positive constant times $A_\mu^{\nu_0}$. This establishes the second property of ν_0 that we needed and therefore the proof of Theorem 4 is completed.

In this connection it is worth proving the following:

Proposition *If ν is a positive quantifier for an oriented flow and O is any compact orbit, there is an invariant measure μ such that A_μ^ν is the element of $H_p(M^n, R)$ arising from the fundamental homology class of the oriented orbit O.*

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Proof We need only show that there is a transversal invariant measure τ such that this is true for A_τ.

Any transversal disc intersects O in only a finite number of points. For any transversal D of our action and any Borel subset of the interior of D we define $\tau_D(S)$ to be the number of points of O that lie in S. Then it follows from the way we defined A_τ that for any closed p-form ω the element of $\operatorname{Hom}\big(H^p(M^n), R\big)$ determined by A_τ, when applied to the cohomology class of ω, is $\int_O \omega$. This proves our proposition.

Finally, suppose G is a connected Lie group and K is a closed subgroup such that the space of right cosets G/K is compact. Let L be a subgroup of G corresponding to a Lie subalgebra ℓ of the Lie algebra g of G. Suppose that there are no non-constant continuous invariant functions for the action of L on the right on G/K. Suppose further that v is a positive quantifier for the action and that K possesses a p-dimensional jacket H, where p is the dimension of each orbit under the action of L on G/K.

Theorem 5 *If μ_1 and μ_2 are two invariant probability measures, $A^v_{\mu_1} = A^v_{\mu_2}$.*

Proof The compact group G/H is acted on by L and the projection of G/K onto G/H is equivariant. Each p-dimensional cohomology class over the reals on G/H is represented by a p-form that is invariant under the action of G/H on itself on the right and therefore is invariant under the action of L. Given a p-dimensional cohomology class λ on G/K, choose such an invariant form in a cohomology class on G/H that lifts to λ. If ω is the lifting of this form to G/K, ω is invariant under the action of L because the map of G/K to G/H is equivariant. Then $\omega \rfloor v$ is an invariant function on G/K and therefore is a constant. Hence $\int_{G/K} \omega \rfloor v \, d\mu$ is the same for all invariant probability measures μ. This proves our theorem.

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