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A Hopf-type Boundary Point Lemma for Pairs of Solutions to Quasilinear Equations

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Abstract. We present a Hopf boundary point lemma for the difference between two Hölder continuously differentiable functions, each weak solutions to a divergence-form quasilinear equation, under mild boundedness assumptions on the coefficients of this equation.

1 Introduction

In this work, we give a Hopf-type boundary point result for pairs of solutions to certain *quasilinear equations*. Our main theorem is roughly as follows.

***Theorem* 3.1** *Suppose $V \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ open set for some $\alpha \in (0,1)$ with $0 \in \partial V$, and suppose $A^1, \dots, A^n \in C^2(\overline{V} \times \mathbb{R} \times \mathbb{R}^n)$ and $B \in C^1(\overline{V} \times \mathbb{R} \times \mathbb{R}^n)$ where $A^1, \dots, A^n$ satisfy some mild boundedness assumptions. If $u_1, u_2 \in C^{1,\alpha}(\overline{V})$ are each weak solutions over V to the equation*

$$\sum_{i=1}^{n} D_i(A^i(x,u,Du)) + B(x,u,Du) = 0,$$

$u_1(0) = u_2(0) = 0$, and $u_1(x) \neq u_2(x)$ for all $x \in V$, then $Du_1(0) \neq Du_2(0)$.

See also Definition 2.1. The *mild boundedness assumptions* are given by (ii), (iii), and (iv) in the statement of Theorem 3.1.

The proof of Theorem 3.1 uses standard PDE techniques. For this, we show that $u = u_1 - u_2$ solves a *linear equation* over V of the form

$$\sum_{i,j=1}^{n} D_i(a^{ij} D_j u) + \sum_{i=1}^{n} c^i D_i u + du = 0 \tag{1.1}$$

(see Definition 2.2), where $a^{ij} \in C^{0,\alpha}(\overline{V})$, $c^i \in L^\infty(V)$, and $d \in L^q(V)$ for some $q > n$. We then apply a generalized Hopf boundary point lemma to u at the origin to conclude Theorem 3.1. More specifically, we apply the recent work of the author [8], given here as Lemma 2.5 for convenience to the reader. For a similar generalization of the Hopf boundary point lemma (which unfortunately is not quite sufficient for this paper), see [9].

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The main concern in order to apply Lemma 2.5 is showing that the coefficient d in (1.1) is in $L^q(V)$ for some $q > n$. This is not immediately evident, as d is defined in terms of the second derivatives of u_1, u_2, and while $u_1, u_2 \in C^{1,\alpha}(\overline{V})$, we can only conclude $u_1, u_2 \in C^2(V)$ by standard PDE arguments. Thus, $d \in L^q(V)$ for some $q > n$ must be carefully checked using the boundedness assumptions on $A^1, \dots, A^n$ (specifically, Theorem 3.1(iv)) and interior C^2 estimates for u_1, u_2. A natural question is whether we can circumvent this issue by showing that $u = u_1 - u_2$ solves a different linear equation (as in Definition 2.2) with coefficients that are not defined in terms of the second derivatives of u_1, u_2. However, this alternate strategy leads to much more cumbersome necessary assumptions on $A^1, \dots, A^n, B$; see Remark 3.2(i).

Before stating and proving our main result in Section 3, we need some basic definitions and results in order to prove Theorem 3.1, which we give in Section 2. In particular, we state Lemma 2.5 in Section 2, which is the version of the generalized Hopf boundary point lemma from [8] that we need. Throughout, we only assume knowledge of graduate-level real analysis, as well as access to the references [2, 8].

1.1 An Application

This work is a generalization of the argument used by the author in [7, Lemma 4.1] to study *co-dimension one area-minimizing currents with tangentially immersed boundary*.

To motivate [7], we consider a simple form of *Plateau's problem*: given a simple smooth loop γ in space, is there a smooth orientable surface $\mathcal{M}$ spanning γ with least area? The answer is in the affirmative. A naive solution is to take a sequence of smooth orientable surfaces $\{M_\ell\}_{\ell=1}^\infty$ spanning γ with $\operatorname{area}(M_\ell) \searrow \inf_{M \text{ spans } \gamma} \operatorname{area}(M)$, and then set $\mathcal{M} = \lim_{k\to\infty} M_\ell$. To show $\mathcal{M}$ exists and is a smooth orientable surface requires that we consider the theory of currents, which in space are heuristically the closure of the smooth orientable surfaces-with-boundary under bounded area and boundary length. In [7] the author considers Plateau's problem for more complicated boundaries; in space, we allow the loop γ to intersect itself tangentially.

We now describe [7, Lemma 4.1] in a simple form. Suppose $V \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ open set for some $\alpha \in (0,1)$ with $0 \in \partial V$. Also suppose $u_1, u_2 \in C^{1,\alpha}(\overline{V})$, $U \in C^\infty(\mathbb{R}^{n+1})$, and for $\ell = 1, 2$, let

$$\Sigma_\ell = \left\{ \big(x, u_\ell(x), U(x, u_\ell(x))\big) : x \in \overline{V} \right\} \subset \mathbb{R}^{n+2}.$$

Now let $\nu_\ell \colon \Sigma_\ell \to \mathbb{R}^{n+1}$ be the upward pointing unit normal of Σ_ℓ within the graph of U; thus, ν_ℓ is tangent to the graph of U, perpendicular to Σ_ℓ, while $\nu_\ell \cdot e_n > 0$. Finally, we suppose there is a Lipschitz function $H \colon \mathbb{R}^{n+1} \to \mathbb{R}$ so that

$$\int_{\Sigma_\ell} \operatorname{div}_{\Sigma_\ell} X = \int_{\Sigma_\ell} X \cdot H\nu_\ell$$

for all smooth vector fields $X \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with compact support in $V \times \mathbb{R}$; in [7] we say Σ_ℓ has *Lipschitz co-oriented mean curvature* with respect to the graph of U. In [7] we argue (and Theorem 3.1 implies) that $u_1(0) = u_2(0)$ while $u_1(x) \neq u_2(x)$ for all $x \in V$ implies $Du_1(0) \neq Du_2(0)$.

Most of the work in the proof of [7, Lemma 4.1] involves translating and rotating so that we are in a position to essentially apply Theorem 3.1. Also, the calculations of the

Appendix of [7] are done essentially to verify that assumption (iv) of Theorem 3.1 is satisfied. Lemma 4.1 [7] is used to study the asymptotic behavior near the boundary of solutions to Plateau's problem (that is, co-dimension one area-minimizing currents) with tangentially immersed boundary (such as loops in space with tangential self-intersections) having Lipschitz co-oriented mean curvature.

1.2 Classic Results

If we assume that $u_1, u_2 \in C^{1,1}(\overline{V})$, then Theorem 3.1 can be proved more directly using the classic Hopf boundary point lemma (see [2, Lemma 3]). For example, we refer the reader to the proof of [6, Lemma 5.1]. Another example is given by a classic result in differential geometry (see [1]).

Alexandrov's Theorem *A compact embedded constant mean curvature surface Σ in $\mathbb{R}^3$ must be a round sphere.*

Proving this requires using the now well-studied Alexandrov reflection method, as well as a geometric maximum principle, stated heuristically in the following form: suppose Σ_1, Σ_2 are both C^2 surfaces with the same constant mean curvature that meet tangentially at a point p and such that Σ_1 lies on one side of Σ_2 near p; then $\Sigma_1 = \Sigma_2$. The geometric maximum principle can be proved by writing Σ_1, Σ_2 locally near p as graphs of functions u_1, u_2, and applying the classic Hopf boundary point lemma to $u = u_1 - u_2$. For a modern exposition of the Alexandrov reflection method and the geometric maximum principle, see [3, Chapters 3,4].

2 Preliminaries

We will work in $\mathbb{R}^n$ with $n \geq 2$. We denote the volume of the open unit ball $B_1(0) \subset \mathbb{R}^n$ by $\omega_n = \int_{B_1(0)} \mathrm{dx}$. Standard notation for the various spaces of functions will be used; in particular, we note that $C^1_c(V; [0,\infty))$ shall denote the set of non-negative continuously differentiable functions with compact support in an open set $V \subseteq \mathbb{R}^n$.

Also, for $V \subseteq \mathbb{R}^n$ we will write functions $A\colon \overline{V} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by $A = A(x,z,p)$, where $x \in \overline{V}$, $z \in \mathbb{R}$, and $p \in \mathbb{R}^n$. For the convenience of the reader, we will let $D_i A$ denote the derivative of A with respect to the x_i-variable for $i \in \{1,\dots,n\}$, $\frac{\partial A}{\partial z}$ the derivative of A with respect to the z-variable, and $\frac{\partial A}{\partial p_j}$ the derivative of A with respect to the p_j-variable for $j \in \{1,\dots,n\}$.

We begin by defining the quasilinear equations we will consider.

Definition 2.1 Let $V \subseteq \mathbb{R}^n$ be an open set, and suppose $A^1,\dots,A^n, B \in C(V \times \mathbb{R} \times \mathbb{R}^n)$. We say $u \in C^1(V)$ is a *weak solution over V to the equation*

$$\sum_{i=1}^n D_i\big(A^i(x,u,Du)\big) + B(x,u,Du) = 0$$

if for all $\zeta \in C^1_c(V)$, we have

$$\int \sum_{i=1}^n A^i(x,u,Du) D_i\zeta - B(x,u,Du)\zeta \,\mathrm{dx} = 0.$$

This is [2, definition (13.2)]. We will also need to consider linear equations, in order to apply the results of [8].

Definition 2.2 Let $V \subset \mathbb{R}^n$ be an open set, and suppose $a^{ij}, b^i, c^i \in L^2(V)$ and $d \in L^1(V)$ for each $i, j \in \{1, \dots, n\}$. We say $u \in L^\infty(V) \cap W^{1,2}(V)$ is a *weak solution over* V of the equation

$$\sum_{i=1}^n D_i\Big(\sum_{j=1}^n a^{ij} D_j u + b^i u\Big) + \sum_{i=1}^n c^i D_i u + du \le 0$$

(or, more strictly, = 0) if for all $\zeta \in C^1_c(V; [0, \infty))$, we have

$$\int \sum_{i,j=1}^n a^{ij} D_j u D_i \zeta + \sum_{i=1}^n \big(b^i u D_i \zeta - c^i (D_i u)\zeta\big) - du\zeta \,\mathrm{dx} \ge 0$$

(resp., = 0).

The assumptions on the coefficients are merely to ensure integrability. We now introduce some terminology, in order to more conveniently state our results.

Definition 2.3 Let $V \subseteq \mathbb{R}^n$.

- Suppose we have functions $a^{ij}: V \to \mathbb{R}$ for $i, j \in \{1, \dots, n\}$. We say $\{a^{ij}\}_{i,j=1}^n$ are *uniformly elliptic over* V *with respect to* $\lambda \in (0, \infty)$ if

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i \xi_j \ge \lambda |\xi|^2$$

  for each $x \in V$ and $\xi \in \mathbb{R}^n$.
- Suppose we have functions $A^{ij}: V \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ for $i, j \in \{1, \dots, n\}$. We say $\{A^{ij}\}_{i,j=1}^n$ are *locally uniformly elliptic over* $\overline{V} \times \mathbb{R} \times \mathbb{R}^n$ if for each $R \in (0, \infty)$ there is $\lambda_R \in (0, \infty)$ so that

$$\sum_{i,j=1}^n A^{ij}(x, z, p)\xi_i \xi_j \ge \lambda_R |\xi|^2$$

  for each $(x, z, p) \in V \times [-R, R] \times \overline{B_R(0)}$ and $\xi \in \mathbb{R}^n$.

Before we give the version of the generalized Hopf boundary point lemma from [8] required, we first give the following *interior* C^2 *estimate*. We prove Lemma 2.4 using [2, Theorem 8.32]. The proof of Lemma 2.4 is standard, and known as the *difference quotient method*.

Lemma 2.4 *Suppose $V \subseteq \mathbb{R}^n$ is a bounded open set, and let $\alpha \in (0, 1)$. Also suppose that*

(i) $A^i \in C^2(\overline{V} \times \mathbb{R} \times \mathbb{R}^n)$ *for* $i \in \{1, \dots, n\}$ *and* $B \in C^1(\overline{V} \times \mathbb{R} \times \mathbb{R}^n)$,

(ii) $\left\{\frac{\partial A^i}{\partial p_j}\right\}_{i,j=1}^n$ *are locally uniformly elliptic over* $\overline{V} \times \mathbb{R} \times \mathbb{R}^{n+1}$.

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If $u \in C^{1,\alpha}(\overline{V})$ is a weak solution over V to the equation

$$\sum_{i=1}^{n} D_i\big(A^i(x,u,Du)\big) + B(x,u,Du) = 0, \tag{2.1}$$

then $u \in C^2(V)$. Furthermore, with $R = \|u\|_{C^{1,\alpha}(\overline{V})}$, let $\lambda_R \in (0,\infty)$ be as in Definition 2.3 applied to $\left\{\frac{\partial A^i}{\partial p_j}\right\}_{i,j=1}^{n}$. Then for each $x \in V$,

$$|D^2u(x)| \le \frac{C_{2.4}}{\min\{1, \operatorname{dist}(x,\partial V)\}},$$

where

$$C_{2.4} = C_{2.4}\Big(n,\alpha,R,\lambda_R,\big\{\|A^i\|_{C^2(\overline{V}\times[-R,R]\times\overline{B_R(0)})}\big\}_{i=1}^{n}, \|B\|_{C^1(\overline{V}\times[-R,R]\times\overline{B_R(0)})}\Big).$$

Proof Consider any $\widehat{x} \in V$ and let $\rho = \operatorname{dist}(\widehat{x},\partial V) \in (0,\infty)$. With fixed $h \in (-\frac{\rho}{2},\frac{\rho}{2})$ and $k \in \{1,\dots,n\}$, define for $x \in \overline{B_{\frac{1}{2}}(0)}$,

$$u_{h,k}(x) = \frac{u(\rho x + \widehat{x} + he_k) - u(\rho x + \widehat{x})}{h}, \quad u_{h,k} \in C^{1,\alpha}(\overline{B_{\frac{1}{2}}(0)}).$$

We wish to apply [2, Theorem 8.32] with $u = u_{h,k}$, $\Omega' = B_{\frac{1}{4}}(0)$, and $\Omega = B_{\frac{1}{2}}(0)$. We must thus compute that $u_{h,k}$ satisfies a linear equation as in Definition 2.2 over $B_{\frac{1}{2}}(0)$.

To do this, for any $\zeta \in C_c^1(B_{\frac{1}{2}}(0))$ we can input the test function

$$x \longrightarrow \frac{\zeta(\frac{x-\widehat{x}-he_k}{\rho}) - \zeta(\frac{x-\widehat{x}}{\rho})}{h} \text{ for } x \in B_\rho(\widehat{x})$$

(after extending ζ to be zero outside of $B_{\frac{1}{2}}(0)$) into the weak equation (2.1). After a change of variables we conclude that

$$\frac{1}{\rho^{n+1}}\int \sum_{i=1}^{n}\frac{1}{h}\Big(A^i(\mathbf{x},u(\mathbf{x},Du(\mathbf{x}))\Big|_{\mathbf{x}=\rho x+\widehat{x}}^{\rho x+\widehat{x}+he_k}\Big) D_i\zeta\,\mathrm{dx} - \frac{1}{\rho^n}\int \frac{1}{h}\Big(B(\mathbf{x},u(\mathbf{x},Du(\mathbf{x}))\Big|_{\mathbf{x}=\rho x+\widehat{x}}^{\rho x+\widehat{x}+he_k}\Big)\zeta\,\mathrm{dx} = 0.$$

Using single-variable calculus, we can compute that $u_{h,k}$ is a weak solution over $B_{\frac{1}{2}}(0)$ to the equation

$$\sum_{i=1}^{n} D_i\Big(\sum_{j=1}^{n} a_{h,k}^{ij} D_j u_{h,k} + b_{h,k}^{i} u_{h,k}\Big) + \sum_{i=1}^{n} c_{h,k}^{i} D_i u_{h,k} + d_h u_{h,k} = g_{h,k} + \sum_{i=1}^{n} D_i f_{h,k}^{i},$$

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where we define, for $x \in \overline{B_{\frac{1}{2}}(0)}$ and $i, j \in \{1, \dots, n\}$,

$$a^{ij}_{h,k}(x) = \int_0^1 \frac{\partial A^i}{\partial p_j}\big(P_{h,k}(t,x)\big)\,\mathrm{d}t, \qquad b^i_{h,k}(x) = \rho \int_0^1 \frac{\partial A^i}{\partial z}\big(P_{h,k}(t,x)\big)\,\mathrm{d}t,$$
$$c^i_{h,k}(x) = \rho \int_0^1 \frac{\partial B}{\partial p_i}\big(P_{h,k}(t,x)\big)\,\mathrm{d}t, \qquad d_{h,k}(x) = \rho^2 \int_0^1 \frac{\partial B}{\partial z}\big(P_{h,k}(t,x)\big)\,\mathrm{d}t,$$
$$g_{h,k}(x) = -\rho^2 \int_0^1 (D_k B)\big(P_{h,k}(t,x)\big)\,\mathrm{d}t,$$
$$f^i_{h,k}(x) = -\rho \int_0^1 (D_k A^i)\big(P_{h,k}(t,x)\big)\,\mathrm{d}t$$

with $P_{h,k}(t,x)$ for $t \in [0,1]$ and $h \in (-\frac{\rho}{2}, \frac{\rho}{2})$ defined by

$$P_{h,k}(t,x) = t\big(\rho x + \widehat{x} + he_k, u(\rho x + \widehat{x} + he_k), Du(\rho x + \widehat{x} + he_k)\big) + (1-t)\big(\rho x + \widehat{x}, u(\rho x + \widehat{x}), Du(\rho x + \widehat{x})\big).$$

Now let $L_{h,k}$ be the operator given by

$$L_{h,k}u = \sum_{i=1}^n D_i\Big(\sum_{j=1}^n a^{ij}_{h,k} D_j u + b^i_{h,k} u\Big) + \sum_{i=1}^n c^i D_i u + d_h u.$$

We now verify the hypothesis [2, Theorem 8.32] as follows:

- Let $R = \|u\|_{C^{1,\alpha}(\overline{V})}$; then $\{a^{ij}_{h,k}\}_{i,j=1}^n$ are uniformly elliptic over $B_{\frac{1}{2}}(0)$ with respect to λ_R by (ii).
- By (i), $u \in C^{1,\alpha}(\overline{V})$, and $\rho = \operatorname{dist}(\widehat{x}, \partial V)$, we have

  $$a^{ij}_{h,k}, b^i_{h,k}, f_{h,k} \in C^{0,\alpha}(\overline{B_{\frac{1}{2}}(0)}), \quad c^i_{h,k}, d_{h,k}, g_{h,k} \in L^\infty(B_{\frac{1}{2}}(0))$$

  for each $i, j \in \{1, \dots, n\}$. Furthermore, we have

  $$\max_{i,j=1,\dots,n} \Big\{\|a^{ij}_{h,k}\|_{C^{0,\alpha}(\overline{B_{\frac{1}{2}}(0)})}, \|b^i_{h,k}\|_{C^{0,\alpha}(\overline{B_{\frac{1}{2}}(0)})}, \|c^i_{h,k}\|_{L^\infty(B_{\frac{1}{2}}(0))}, \|d_{h,k}\|_{L^\infty(B_{\frac{1}{2}}(0))}\Big\} \le K_\rho,$$

  where we define, again with $R = \|u\|_{C^{1,\alpha}(\overline{V})}$,

  $$K_\rho = \max_{i=1,\dots,n} \Big\{(1 + \rho + \rho^\alpha R)\|A^i\|_{C^2(\overline{V}\times[-R,R]\times\overline{B_R(0)})}, \rho(1 + \rho + \rho^\alpha R)\|A^i\|_{C^2(\overline{V}\times[-R,R]\times\overline{B_R(0)})}, \rho\|B\|_{C^1(\overline{V}\times[-R,R]\times\overline{B_R(0)})}, \rho^2\|B\|_{C^1(\overline{V}\times[-R,R]\times\overline{B_R(0)})}\Big\}.$$

We conclude that the operator $L_{h,k}$ satisfies [2, (8.5), (8.85)] with $\lambda = \lambda_R$ and $K = K_\rho$. We can thus apply [2, Theorem 8.32] (with $a^{ij}, b^i, c^i, d, g, f^i$ replaced respectively by $a^{ij}_{h,k}, b^i_{h,k}, c^i_{h,k}, d_{h,k}, g_{h,k}, f^i_{h,k}$) over $\Omega = B_{\frac{1}{2}}(0)$ and with $\Omega' = B_{\frac{1}{4}}(0)$

(so that $d' = \operatorname{dist}(\Omega', \partial\Omega) = \frac{1}{4}$) to conclude (again with $R = \|u\|_{C^{1,\alpha}(\overline{V})}$) that

$$
\begin{aligned}
\text{(2.2)}\qquad &\|u_{h,k}\|_{C^{1,\alpha}(\overline{B_{\frac14}(0)})}\\
&\le C\Big(\|u_{h,k}\|_{L^\infty(B_{\frac12}(0))} + \|g_{h,k}\|_{L^\infty(B_{\frac12}(0))} + \sum_{i=1}^n \|f^i_{h,k}\|_{C^{0,\alpha}(\overline{B_{\frac12}(0)})}\Big)\\
&\le C\Big(R + \rho^2\|B\|_{C^1(\overline{V}\times[-R,R]\times\overline{B_R(0)})}\\
&\qquad + \rho\,(1+\rho+\rho^\alpha R)\sum_{i=1}^n \|A^i\|_{C^2(\overline{V}\times[-R,R]\times\overline{B_R(0)})}\Big),
\end{aligned}
$$

where $C = C(n, \alpha, \lambda_R, K_\rho) \in (0, \infty)$. In particular, the right-hand side is independent of $h \in (-\frac{\rho}{2}, \frac{\rho}{2})$. Letting $h \to 0$, we can show, using Arzela–Ascoli, that $u \in C^2(B_{\frac{\rho}{2}}(\widehat{x}))$.

This shows that $u \in C^2(V)$. We now prove the interior estimate for D^2u. Again with $\widehat{x} \in V$, now set $\rho = \min\{1, \operatorname{dist}(\widehat{x}, \partial V)\}$ and repeat the above calculations. However, still with $R = \|u\|_{C^{1,\alpha}(\overline{V})}$, we replace K_ρ with

$$K = \max_{i=1,\dots,n}\left\{(2+R)\|A^i\|_{C^2(\overline{V}\times[-R,R]\times\overline{B_R(0)})}, \|B\|_{C^1(\overline{V}\times[-R,R]\times\overline{B_R(0)})}\right\}.$$

Letting $h \to 0$ in (2.2), we conclude that

$$
\begin{aligned}
\rho|DD_k u(\widehat{x})| &= \lim_{h\to0}|Du_{h,k}(0)|\\
&\le C\lim_{h\to0}\Big(\|u_{h,k}\|_{L^\infty(B_{\frac12}(0))} + \|g_{h,k}\|_{L^\infty(B_{\frac12}(0))}\\
&\qquad + \sum_{i=1}^n \|f^i_{h,k}\|_{C^{0,\alpha}(\overline{B_{\frac12}(0)})}\Big)\\
&\le C\Big(R + \|B\|_{C^1(\overline{V}\times[-R,R]\times\overline{B_R(0)})}\\
&\qquad + \rho(2+R)\sum_{i=1}^n \|A^i\|_{C^2(\overline{V}\times[-R,R]\times\overline{B_R(0)})}\Big),
\end{aligned}
$$

where now $C = C(n, \alpha, \lambda_R, K) \in (0, \infty)$. ∎

We now state, for convenience, the version of the Hopf boundary point lemma from [8] we shall need. To do so, we introduce some notation: let $B^{n-1}_\rho(0)$ denote the ball of radius $\rho \in (0, \infty)$ centered at the origin in $\mathbb{R}^{n-1}$; $\overline{D}$ shall denote differentiation over $\mathbb{R}^{n-1}$. Also, we let $\mathbf{p}: \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection onto $\mathbb{R}^{n-1}$, and we will write points $y \in \mathbb{R}^{n-1}$.

Before we proceed, we note that the proof of Lemma 2.5 refers to the *Morrey space* $L^{1,\alpha}$. Suppose $\alpha \in (0,1)$ and $\mathcal{U} \subseteq \mathbb{R}^n$ is an open set; we say $d \in L^{1,\alpha}(\mathcal{U})$ if $d \in L^1(\mathcal{U})$ with finite $L^{1,\alpha}(\mathcal{U})$ norm, defined by

$$\|d\|_{L^{1,\alpha}(\mathcal{U})} := \sup_{x\in\mathbb{R}^n, \rho\in(0,\infty)} \frac{1}{\rho^{n-1+\alpha}}\int_{\mathcal{U}\cap B_\rho(x)} |d(y)|\,\mathrm{d}y$$

(see [8, Definition 2.1]). Indeed, [8] generalizes the Hopf boundary point lemma to linear equations (as in Definition 2.2) with lower-order coefficient $d \in L^{1,\alpha}$. Morrey spaces were introduced in [5] to study existence and regularity of solutions to elliptic systems, and since have been studied in and outside of partial differential equations.

See, for example, [4], which uses Morrey spaces to prove regularity results for solutions to non-linear divergence-form elliptic equations having inhomogeneous term consisting of a measure.

We will also use the fact that if $q > n$ and $\mathcal{U} \subset \mathbb{R}^n$ is a bounded open set, then $L^q(\mathcal{U}) \subseteq L^{1,\alpha}(\mathcal{U})$ for $\alpha = 1 - \frac{n}{q}$; see [8, Remark 2.2].

Lemma 2.5 *Let $\lambda \in (0,\infty)$ and $\alpha \in (0,1)$. Suppose*

$$w \in C^{1,\alpha}\big(B_1^{n-1}(0); [0,\infty)\big)$$

satisfies $w(0) = 0$ and $\overline{D}w(0) = 0$, and let

$$W = \{x \in B_1^{n-1}(0) \times (0,3) : x_n > w(\mathbf{p}(x))\}.$$

Also suppose

(i) $a^{ij} \in C^{0,\alpha}(\overline{W})$, $c^i \in L^\infty(W)$ *for* $i, j \in \{1,\dots,n\}$, *and* $d \in L^{\frac{n}{1-\alpha}}(W)$;
(ii) $\{a^{ij}\}_{i,j=1}^n$ *are uniformly elliptic over* W *with respect to* λ;
(iii) $d(x) \leq 0$ *for each* $x \in W$;
(iv) $a^{ij}(0) = a^{ji}(0)$ *for each* $i, j \in \{1,\dots,n\}$.

If $u \in C^1(\overline{W})$ is a weak solution over W to the equation

$$\sum_{i,j=1}^n D_i(a^{ij} D_j u) + \sum_{i=1}^n c^i D_i u + du \leq 0 \tag{2.3}$$

with $u(x) > u(0) = 0$ for all $x \in W$, then $D_n u(0) > 0$.

Proof Our goal is to apply the generalized Hopf boundary point lemma of [8] to u, after applying a change of variables. Choose $\rho \in (0,1)$ so that

$$\|w\|_{C^1(B_\rho^{n-1}(0))} < \max\left\{1, \sqrt{1 + \frac{\lambda/2}{\sum_{i,j=1}^n \|a^{ij}\|_{C(\overline{W})}}} - 1\right\}. \tag{2.4}$$

Define the map $\Psi_\rho \in C^{1,\alpha}(\overline{B_1(0)}; \overline{W})$ by

$$\Psi_\rho(x) = \rho(x + e_n) + w(\mathbf{p}(\rho x))e_n \text{ for } x \in B_1(0);$$

note that $\rho(x + e_n) + w(\mathbf{p}(\rho x))e_n \in W$ for $x \in B_1(0)$. Now define

$$u_\rho(x) = u(\Psi_\rho(x)) \text{ for } x \in \overline{B_1(0)}, \quad u_\rho \in C^1(\overline{B_1(0)}).$$

We derive a weak equation for u_ρ over $B_1(0)$, by applying Ψ_ρ as a change of variables to (2.3).

To this end, we compute for $x \in B_1(0)$

$$\begin{aligned} D_j u_\rho(x) &= \rho(D_j u)(\Psi_\rho(x)) + \rho(D_j w)(\mathbf{p}(\rho x))(D_n u)(\Psi_\rho(x)) \\ &= \rho(D_j u)(\Psi_\rho(x)) + (D_j w)(\mathbf{p}(\rho x)) D_n u_\rho(x) \\ &\qquad \text{for } j \in \{1,\dots,n-1\}, \\ D_n u_\rho(x) &= \rho(D_n u)(\Psi_\rho(x)). \end{aligned}$$

Likewise, we compute for $\zeta \in C^1_c(B_1(0))$ and $x \in B_1(0)$

$$\begin{aligned} D_i\big(\zeta(\Psi_\rho^{-1}(x))\big) &= \frac{1}{\rho}(D_i\zeta)(\Psi_\rho^{-1}(x)) - \frac{1}{\rho}(D_iw)(\mathbf{p}(x))(D_n\zeta)(\Psi_\rho^{-1}(x)) \\ &= \frac{1}{\rho}(D_i\zeta)(\Psi_\rho^{-1}(x)) - \frac{1}{\rho}(D_iw)(\mathbf{p}(\rho\Psi_\rho^{-1}(x)))(D_n\zeta)(\Psi_\rho^{-1}(x)) \\ &\quad \text{for } i \in \{1,\dots,n-1\}, \\ D_n\big(\zeta(\Psi_\rho^{-1}(x))\big) &= \frac{1}{\rho}(D_n\zeta)(\Psi_\rho^{-1}(x)). \end{aligned}$$

These calculations, and using $\Psi_\rho\colon B_1(0) \to W$ as a change of variables in (2.3), imply that u_ρ is a weak solution over $B_1(0)$ to the equation

$$\sum_{i,j=1}^n D_i\big(a_\rho^{ij}D_ju_\rho\big) + \sum_{i=1}^n c_\rho^i D_iu_\rho + d_\rho u_\rho \leq 0,$$

where we define $a_\rho^{ij}\colon \overline{B_1(0)} \to \mathbb{R}$, $c_\rho^i, d_\rho\colon B_1(0) \to \mathbb{R}$ for $i,j \in \{1,\dots,n\}$ by

$$\begin{aligned} a_\rho^{ij}(x) &= a^{ij}(\Psi_\rho(x)) \text{ for } i,j \in \{1,\dots,n-1\}, \\ a_\rho^{in}(x) &= a^{in}(\Psi_\rho(x)) - \sum_{\widehat{j}=1}^{n-1} a^{i\widehat{j}}(\Psi_\rho(x))(D_{\widehat{j}}w)(\mathbf{p}(\rho x)) \text{ for } i \in \{1,\dots,n-1\}, \\ a_\rho^{nj}(x) &= a^{nj}(\Psi_\rho(x)) - \sum_{\widehat{i}=1}^{n-1} a^{\widehat{i}j}(\Psi_\rho(x))(D_{\widehat{i}}w)(\mathbf{p}(\rho x)) \text{ for } j \in \{1,\dots,n-1\}, \\ a_\rho^{nn}(x) &= a^{nn}(\Psi_\rho(x)) + \sum_{\widehat{i},\widehat{j}=1}^{n-1} a^{\widehat{i}\widehat{j}}(\Psi_\rho(x))(D_{\widehat{i}}w)(\mathbf{p}(\rho x))(D_{\widehat{j}}w)(\mathbf{p}(\rho x)), \\ c_\rho^i(x) &= \rho c^i(\Psi_\rho(x)) \text{ for } i \in \{1,\dots,n-1\}, \\ c_\rho^n(x) &= \rho c^n(\Psi_\rho(x)) - \rho\sum_{\widehat{i}=1}^{n-1}\big(c^{\widehat{i}}(\Psi_\rho(x))\big)(D_{\widehat{i}}w)(\mathbf{p}(\rho x)), \\ d_\rho(z) &= \rho^2 d(\Psi_\rho(x)). \end{aligned}$$

We now verify the hypothesis of [8, Lemma 3.3].

- $a_\rho^{ij} \in C^{0,\alpha}(\overline{B_1(0)})$, $c_\rho^i \in L^\infty(B_1(0)) \subset L^{\frac{n}{1-\alpha}}(B_1(0))$ for $i,j \in \{1,\dots,n-1\}$, and
$$d_\rho \in L^{\frac{n}{1-\alpha}}(B_1(0)) \subset L^{\frac{n}{2(1-\alpha)}}(B_1(0)) \cap L^{1,\alpha}(B_1(0))$$
by $\Psi_\rho \in C^{1,\alpha}(\overline{B_1(0)};\overline{W})$, (i), [8, Definition 2.1], and [8, Remark 2.2] with $q = \frac{n}{1-\alpha}$.
- $\{a_\rho^{ij}\}_{i,j=1}^n$ are uniformly elliptic over $B_1(0)$ with respect to $\frac{\lambda}{2}$, by (ii) and (2.4).
- $\{0\}_{i=1}^n, d_\rho$ are *weakly non-positive* over $B_1(0)$, see [8, Definition 2.5] and [2, (8.8)]. In this case, we just mean that $\int d_\rho\zeta\,\mathrm{dx} \leq 0$ for each $\zeta \in C^1_c(B_1(0);[0,\infty])$. This is true by (iii).
- For each $i,j \in \{1,\dots,n\}$
$$a_\rho^{ij}(-e_n) = a^{ij}(0) = a^{ji}(0) = a_\rho^{ji}(-e_n)$$
by $w(0) = 0$ and $\overline{D}w(0) = 0$ (so that $\Psi_\rho(-e_n) = 0$).

Moreover, $w(0) = 0$ implies

$$u_\rho(x) = u(\Psi_\rho(x)) > u(0) = u_\rho(-e_n) = 0.$$

We conclude by [8, Theorem 4.1] that $0 < D_n u_\rho(-e_n) = \rho D_n u(0)$. ∎

3 Main Theorem

We are now ready to state and prove our main result.

Theorem 3.1 *Suppose $\alpha \in (0, \frac{n-1}{2n-1})$, and suppose $v \in C^{1,\alpha}(B_1^{n-1}(0))$ satisfies $v(0) = 0$, $\overline{D}v(0) = 0$, and $\|v\|_{C^{1,\alpha}(B_1^{n-1}(0))} \leq 1$. With*

$$V = \{x \in B_1^{n-1}(0) \times (-3,3) : x_n > v(\mathbf{p}(x))\},$$

suppose

(i) *$A^i \in C^2(\overline{V} \times \mathbb{R} \times \mathbb{R}^n)$ for each $i = 1, \dots, n$ and $B \in C^1(\overline{V} \times \mathbb{R} \times \mathbb{R}^n)$;*
(ii) *$\{\frac{\partial A^i}{\partial p_j}\}_{i,j=1}^n$ are locally uniformly elliptic over $\overline{V} \times \mathbb{R} \times \mathbb{R}^n$ (see Definition 2.3);*
(iii) *$\frac{\partial A^i}{\partial p_j}(0,0,p) = \frac{\partial A^j}{\partial p_i}(0,0,p)$ for each $i, j \in \{1, \dots, n\}$ and $p \in \mathbb{R}^n$;*
(iv) *for each $R \in (0,\infty)$, there is $C_R \in (0,\infty)$ so that*

$$\sup_{(x,z,p)\in\overline{V}\times[-R,R]\times\overline{B_R(0)}} \left| \frac{\partial^2 A^i}{\partial p_j \partial z}(x,z,p) \right| \leq C_R(|x| + |z|).$$

If $u_1, u_2 \in C^{1,\alpha}(\overline{V})$ are weak solutions over V to the equation

$$\sum_{i,j=1}^n D_i(A^i(x,u,Du)) + B(x,u,Du) = 0 \tag{3.1}$$

with $u_1(0) = u_2(0) = 0$ and $u_1(x) > u_2(x)$ for each $x \in V$, then $D_n u_1(0) > D_n u_2(0)$.

Proof Our goal is to apply Lemma 2.5 to $u = u_1 - u_2$.

First, we show that u solves a linear equation as in Definition 2.2 over V. Take any $\zeta \in C_c^1(V)$. Subtracting the weak equations (3.1) for u_1, u_2, we get

$$\int \left(A^i(x,u_1,Du_1) - A^i(x,u_2,Du_2)\right) D_i\zeta - \left(B(x,u_1,Du_1) - B(x,u_2,Du_2)\right)\zeta \, \mathrm{dx} = 0.$$

Using single-variable calculus, we can compute that $u = u_1 - u_2 \in C^{1,\alpha}(\overline{V})$ is a weak solution over V to the equation

$$\sum_{i,j=1}^n D_i(a^{ij} D_j u) + \sum_{i=1}^n c^i D_i u + du = 0,$$

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where we define for $x \in \overline{V}$,

$$a^{ij}(x) = \int_0^1 \frac{\partial A^i}{\partial p_j}(P(t,x))\,\mathrm{dt},$$

$$c^i(x) = \int_0^1 \frac{\partial B}{\partial p_i}(P(t,x)) + \frac{\partial A^i}{\partial z}(P(t,x))\,\mathrm{dt},$$

$$\begin{aligned}(3.2)\qquad d(x) = \int_0^1 &\frac{\partial B}{\partial z}(P(t,x)) \\ &+ \sum_{i=1}^n \Big\{ \frac{\partial D_i A^i}{\partial z}(P(t,x)) + \Big(\frac{\partial^2 A^i}{\partial z^2}(P(t,x))(tD_i u_1 + (1-t)D_i u_2)\Big) \\ &\qquad + \sum_{j=1}^n \Big(\frac{\partial^2 A^i}{\partial p_j \partial z}(P(t,x))(tD_iD_j u_1 + (1-t)D_iD_j u_2)\Big)\Big\},\end{aligned}$$

with

$$P(t,x) = \big(x, tu_1(x) + (1-t)u_2(x), tDu_1(x) + (1-t)Du_2(x)\big).$$

for $t \in [0,1]$ as well. To see this more clearly, note that after using one-dimensional calculus, we further apply integration by parts to the term:

$$\begin{aligned}\int\int_0^1 &\frac{\partial A^i}{\partial z}(P(t,x))\,\mathrm{dt}\,u D_i\zeta\,\mathrm{dx} \\ &= -\int D_i\bigg(\int_0^1 \frac{\partial A^i}{\partial z}(P(t,x))\,\mathrm{dt}\,u\bigg)\zeta\,\mathrm{dx} \\ &= -\int\int_0^1 \frac{\partial A^i}{\partial z}(P(t,x))\,\mathrm{dt}(D_i u)\zeta\,\mathrm{dx} - \int\int_0^1 \frac{\partial D_i A^i}{\partial z}(P(t,x))\,\mathrm{dt}\,u\zeta\,\mathrm{dx} \\ &\quad - \int\int_0^1 \frac{\partial^2 A^i}{\partial z^2}(P(t,x))(tD_i u_1 + (1-t)D_i u_2)\,\mathrm{dt}\,u\zeta\,\mathrm{dx} \\ &\quad - \int\int_0^1 \sum_{j=1}^n \frac{\partial^2 A^i}{\partial p_j \partial z}(P(t,x))(tD_iD_j u_1 + (1-t)D_iD_j u_2)\,\mathrm{dt}\,u\zeta\,\mathrm{dx}\end{aligned}$$

using $A^i \in C^2(\overline{V} \times \mathbb{R} \times \mathbb{R}^n)$ and $u_1, u_2 \in C^2(V)$ by Lemma 2.4, which explains the definition of c^i, d; see Remark 3.2(i).

Moreover, note that for each $x \in V$,

$$u(x) = u_1(x) - u_2(x) > u_1(0) - u_2(0) = 0.$$

This implies that u is a weak solution over V of the equation

$$\sum_{i,j=1}^n D_i(a^{ij}D_j u) + \sum_{i=1}^{n-1} c^i D_i u + d_- u \le 0$$

(see Definition 2.2), where a^{ij}, c^i for $i, j \in \{1, \ldots, n\}$ are as in (3.2), while

$$(3.3)\qquad d_-(x) = \min\{0, d(x)\} \text{ for } x \in V.$$

As noted before, our aim is to apply Lemma 2.5 to u. However, we will not apply Lemma 2.5 over the region V, but instead over W defined as follows.

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Using $\|v\|_{C^{1,\alpha}(B_1^{n-1}(0))} \le 1$, define and compute

$$
\begin{aligned}
(3.4) \qquad & w(y) = 2v(y) + 3|y|^{1+\alpha} \text{ for } y \in B_1^{n-1}(0) \\
& \qquad \text{where } w \in C^{1,\alpha}(B_1^{n-1}(0)), \\
& w(0) = 0 \quad \text{and} \quad \overline{D}w(0) = 0, \\
& W = \{x \in B_1^{n-1}(0) \times (-3,3) : x_n > w(\mathbf{p}(x))\}, \\
& W \subseteq V \cap \{x \in B_1^{n-1}(0) \times (0,3) : x_n > |\mathbf{p}(x)|^{1+\alpha}\}, \\
& \qquad \text{and } \overline{B_{\frac{x_n}{4}}(x)} \subset V \text{ when } x \in W.
\end{aligned}
$$

Let us show the last claim. Fix $\widehat{x} \in W$; then the fifth item in (3.4) implies $\widehat{x}_n > 0$. Moreover, for any $x \in \overline{B_{\frac{\widehat{x}_n}{4}}(\widehat{z})}$, we have by $\|v\|_{C^{1,\alpha}(B_1^{n-1}(0))} \le 1$ and the definition of w, W that

$$
\begin{aligned}
x_n &= v(\mathbf{p}(x)) + \widehat{x}_n + x_n - \widehat{x}_n + v(\mathbf{p}(\widehat{x})) - v(\mathbf{p}(x)) - v(\mathbf{p}(\widehat{x})) \\
&\ge v(\mathbf{p}(x)) + \widehat{x}_n - |x_n - \widehat{x}_n| - |\mathbf{p}(\widehat{x}) - \mathbf{p}(x)| - v(\mathbf{p}(\widehat{x})) \\
&\ge v(\mathbf{p}(x)) + \frac{\widehat{x}_n}{2} - v(\mathbf{p}(\widehat{x})) \\
&> v(\mathbf{p}(x)) + \frac{3}{2}|\mathbf{p}(x)|^{1+\alpha} \ge v(\mathbf{p}(x)).
\end{aligned}
$$

Thus, $\overline{B_{\frac{\widehat{x}_n}{4}}(\widehat{z})} \subset V$ when $\widehat{x} \in W$.

We now check that a^{ij}, c^i, d_- for $i,j \in \{1,\dots,n\}$ as in (3.2),(3.3) satisfy the hypothesis of Lemma 2.5 over W, in reverse order.

- Using (iii) and $u_1(0) = u_2(0) = 0$, we compute

$$
\begin{aligned}
a^{ij}(0) &= \int_0^1 \frac{\partial A^i}{\partial p_j}(0,0,tDu_1(0) + (1-t)Du_2(0))\,\mathrm{dt} \\
&= \int_0^1 \frac{\partial A^j}{\partial p_i}(0,0,tDu_1(0) + (1-t)Du_2(0))\,\mathrm{dt} = a^{ji}(0)
\end{aligned}
$$

  for each $i,j \in \{1,\dots,n\}$.
- $d_-(x) = \min\{0, d(x)\} \le 0$ for each $x \in W$.
- By (ii), we have that $\{a^{ij}\}_{i=1}^n$ are uniformly elliptic over W with respect to $\lambda_R \in (0,\infty)$, where we set

$$
R = \max\{\|u_1\|_{C^{1,\alpha}(\overline{V})}, \|u_2\|_{C^{1,\alpha}(\overline{V})}\}. \tag{3.5}
$$

- By (i) and $u_1, u_2 \in C^{1,\alpha}(\overline{V})$, we immediately conclude

$$
a^{ij} \in C^{0,\alpha}(\overline{W}) \text{ and } c^i \in L^\infty(W)
$$

  for each $i,j \in \{1,\dots,n\}$.

  We now show d, and hence d_-, is in $L^{\frac{n}{1-\alpha}}(W)$. For this, since $0 \in \partial V$, we conclude by Lemma 2.4 that for each $\ell = 1,2$,

$$
|D^2 u(x)| \le \frac{C_{2.4}}{\mathrm{dist}(x, \partial V)} \text{ for } x \in V \cap B_1(0)
$$

where $C_{2.4}$ depends on

$$n,\ \alpha,\ R,\ \lambda_R,\ \{\|A^i\|_{C^2(\overline{V}\times[-R,R]\times\overline{B_R(0)})}\}_{i=1}^n,\ \|B\|_{C^1(\overline{V}\times[-R,R]\times\overline{B_R(0)})}$$

Now suppose $x \in W$; then $\overline{B_{\frac{x_n}{4}}(x)} \subset V$ implies $\operatorname{dist}(x, \partial V) \geq \frac{x_n}{4}$ by (3.4). We thus conclude

$$|D^2 u(x)| \leq \frac{4C_{2.4}}{x_n} \text{ for each } x \in W \cap B_1(0). \tag{3.6}$$

We now consider each term in the definition of d given in (3.2), which we bound independently of $t \in [0,1]$ over W.

- By (i) and (3.5), we compute

$$\Big|\frac{\partial B}{\partial z}(P(t,x))\Big| \leq \|B\|_{C^1(\overline{W}\times[-R,R]\times\overline{B_R(0)})}$$

  for $t \in [0,1]$ and $x \in W$.
- Similarly, we have, for $t \in [0,1]$ and $x \in W$,

$$\Big|\frac{\partial D_i A^i}{\partial z}(P(t,x))\Big| \leq \|A^i\|_{C^2(\overline{W}\times[-R,R]\times\overline{B_R(0)})}$$

  and

$$\Big|\frac{\partial^2 A^i}{\partial z^2}(P(t,x))\cdot(tD_i u_1 + (1-t)D_i u_2)\Big| \leq \|A^i\|_{C^2(\overline{W}\times[-R,R]\times\overline{B_R(0)})} R$$

  for each $i \in \{1,\dots,n\}$.
- For $x \in W \cap B_1(0)$ we compute, using (3.6), (iv) with (3.5), $u_1(0) = u_2(0) = 0$, and

$$W \subseteq \{x \in B_1^{n-1}(0) \times (0,3) : x_n > |\mathbf{p}(x)|^{1+\alpha}\}$$

  by (3.4) that when $\mathbf{p}(x) \neq 0$,

$$\begin{aligned}
&\Big|\frac{\partial^2 A^i}{\partial p_j \partial z}(P(t,x)) \cdot (tD_iD_j u_1 + (1-t)D_iD_j u_2)\Big| \\
&\quad\leq \Big|\frac{\partial^2 A^i}{\partial p_j \partial z}(P(t,x))\Big| \frac{4C_{2.4}}{x_n} \\
&\quad\leq \frac{4C_{2.4}C_R(|x| + |tu_1(x) + (1-t)u_2(x)|)}{x_n} \\
&\quad\leq 4C_{2.4}C_R\Big(\frac{|\mathbf{p}(x)|}{x_n} + 1 + R\Big(\frac{|\mathbf{p}(x)|}{x_n} + 1\Big)\Big) \\
&\quad\leq 4C_{2.4}C_R(1+R)\Big(\frac{|\mathbf{p}(x)|}{|\mathbf{p}(x)|^{1+\alpha}} + 1\Big) \\
&\quad\leq \frac{8C_{2.4}C_R(1+R)}{|\mathbf{p}(x)|^{\alpha}}
\end{aligned}$$

for each $i, j \in \{1, \ldots, n\}$. Moreover, note that we can compute, using Tonelli's theorem, that

$$\int_{W \cap B_1(0)} \left(\frac{1}{|\mathbf{p}(x)|^{\alpha}} \right)^{\frac{n}{1-\alpha}} \mathrm{dx} < \infty,$$

since $\alpha \in (0, \frac{n-1}{2n-1})$.

Now suppose $x \in W \setminus B_1(0)$. In this case, either $x_n > \frac{1}{2}$ or $|\mathbf{p}(x)| > \frac{1}{2}$, both of which imply by (3.4) that $x_n > 2^{-1-\alpha}$. We thus compute by (3.6) and (iv) with (3.5) that

$$\begin{aligned}
&\left| \frac{\partial^2 A^i}{\partial p_j \partial z}(P(t,x)) \cdot (tD_iD_ju_1 + (1-t)D_iD_ju_2) \right| \\
&\quad \le \|A^i\|_{C^2(\overline{V} \times [-R,R] \times \overline{B_R(0)})} \frac{4C_{2.4}}{x_n} \\
&\quad \le 2^{3+\alpha} \|A^i\|_{C^2(\overline{V} \times [-R,R] \times \overline{B_R(0)})} C_{2.4}
\end{aligned}$$

for each $i, j \in \{1, \ldots, n\}$.

We conclude that d, and hence d_-, is in $L^{\frac{n}{1-\alpha}}(W)$.

Since $u(x) = u_1(x) - u_2(x) > u_1(0) - u_2(0) = 0$ for each $x \in W$, we conclude by Lemma 2.5 that $D_n u_1(0) > D_n u_2(0)$. ∎

Remark 3.2 We remark on the proof and statement of Theorem 3.1.

(i) Observe that in the proof of Theorem 3.1, we could instead show that $u = u_1 - u_2$ solves an equation over V of the form

$$\sum_{i=1}^{n} D_i \Big(\sum_{j=1}^{n} a^{ij} D_j u_\ell + b^i u_\ell \Big) + \sum_{i=1}^{n} c^i D_i u_\ell + d u_\ell = 0,$$

where, as opposed to (3.2), the coefficients are only defined in terms of u, Du and the first derivatives of $A^1, \ldots, A^n, B$. The idea is to try to avoid setting hypothesis (iv).

However, applying the generalized Hopf boundary point lemma of [8] in this case requires showing $\{b^i\}_{i=1}^n, d$ are weakly non-positive (see [8, Definition 2.5]). But in considering particular examples, this may be difficult (or impossible) to verify. Meanwhile, Theorem 3.1(iv) is far more accessible.

(ii) Note that we need not specifically assume $\alpha \in (0, \frac{n-1}{2n-1})$, since $\beta \in (0, \alpha)$ implies $C^{1,\alpha}(\Omega) \subset C^{1,\beta}(\Omega)$.

(iii) We need not assume $\|v\|_{C^{1,\alpha}(B_1^{n-1}(0))} \le 1$, by simply rescaling (as in the proofs of Lemmas 2.4, 2.5).

(iv) What we really need in order to prove Theorem 3.1 is

$$\left| \frac{\partial^2 A^i}{\partial p_j \partial z}(P(t,x)) \cdot (tD_iD_ju_1 + (1-t)D_iD_ju_2) \right| \le \varphi(x)$$

for each $t \in [0,1]$ and $x \in V$, where $\varphi \in L^q(V)$ for some $q > n$. Assumption (iv) merely guarantees this.

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