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ABSTRACT

We study families of rational curves on irreducible holomorphic symplectic varieties. We give a necessary and sufficient condition for a sufficiently ample linear system on a holomorphic symplectic variety of $K3^{[n]}$ -type to contain a uniruled divisor covered by rational curves of primitive class. In particular, for any fixed n , we show that there are only finitely many polarization types of holomorphic symplectic variety of $K3^{[n]}$ -type that do not contain such a uniruled divisor. As an application, we provide a generalization of a result due to Beauville–Voisin on the Chow group of 0-cycles on such varieties.

1. Introduction

Let S be a $K3$ surface and H an ample divisor on S . By a theorem of Bogomolov and Mumford [MM83], the linear system $|H|$ contains an element with irreducible components that are rational. A simple, yet striking application of the existence of ample rational curves on any projective $K3$ surface S has been given by Beauville and Voisin in [BV04]. They remarked that any point on any rational curve on the $K3$ determines the same (canonical) 0-cycle c_S of degree 1, and proved that the image of the intersection product $\text{Pic}(S) \otimes \text{Pic}(S) \rightarrow CH_0(S)$ is contained in $\mathbb{Z} \cdot c_S$. Tensoring with \mathbb{Q} we may restate these two results as follows as an equality between the following three groups:

$$\begin{aligned} \text{Im}((j_1)_* : CH_0(R_1)_{\mathbb{Q}} \rightarrow CH_0(S)_{\mathbb{Q}}) &= \text{Im}((j_2)_* : CH_0(R_2)_{\mathbb{Q}} \rightarrow CH_0(S)_{\mathbb{Q}}) \\ &= \text{Im}(\text{Pic}(S)_{\mathbb{Q}} \otimes \text{Pic}(S)_{\mathbb{Q}} \rightarrow CH_0(S)_{\mathbb{Q}}), \end{aligned}$$

where $j_i : R_i \hookrightarrow S$, $i = 1, 2$, are any two rational curves on the $K3$ surface S .

The goal of this paper is to investigate the extent to which Bogomolov and Mumford’s and Beauville and Voisin’s results can be generalized to the higher-dimensional setting.

Let X be a compact Kähler manifold. We say that X is irreducible holomorphic symplectic (in the text, we often simply refer to such manifolds as holomorphic symplectic) if X is simply connected and $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate form of degree 2. These objects were introduced by Beauville in [Bea83]. The holomorphic symplectic surfaces are the $K3$ surfaces.

The cohomology group $H^2(X, \mathbb{Z})$ is endowed with a natural quadratic form q , the Beauville–Bogomolov form. We denote it by q .

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It should be noted that holomorphic symplectic varieties with $b_2 > 3$ are not hyperbolic. This has been proved by Verbitsky, cf. [Ver15, Ver17], using among other things his global Torelli theorem [Ver13]. Much less seems to be known on the existence of rational curves on (projective) holomorphic symplectic varieties.

In order to investigate those, we make the following definition.

DEFINITION 1.1. Let C be a stable curve of genus 0, and let $f : C \rightarrow X$ be a morphism that is unramified at all the generic points of C . We say that the curve $f(C)$ in X is *ruling* if there exists a family

$$p : \mathcal{C} \rightarrow S$$

of stable curves over an irreducible, quasi-projective base S , a point $0 \in S$, and a morphism

$$\phi : \mathcal{C} \rightarrow X$$

such that $C = p^{-1}(0)$, $\phi|_C = f$, and $\phi(\mathcal{C})$ has codimension 1 in X .

We say that $\phi(\mathcal{C})$ is *uniruled of codimension 1*, that it is ruled by the stable curve $f : C \rightarrow X$ – or, for short, ruled by $f(C)$.

A stable genus 0 curve in X is by definition a morphism $f : C \rightarrow X$ as above.

Given a curve R in X , we may consider the class $[R]$ of R in $H_2(X, \mathbb{Q})$. Let $[R]^\vee \in H^2(X, \mathbb{Q})$ be the Poincaré dual of $[R]$. Then $[R]^\vee$ is the class of a divisor in X . We say that R is *positive* if $q([R]^\vee) > 0$, and that R is *ample* if $[R]^\vee$ is an ample class.

Recall that a holomorphic symplectic manifold is said to be of $K3^{[n]}$ -type if it is a deformation of the Hilbert scheme that parametrizes zero-dimensional subschemes of length n on some $K3$ surface. Our main result is the following: we prove it in a slightly more precise and technical form in Theorem 4.5 and Proposition 4.6.

THEOREM 1.2. *Let $n \geq 1$ be an integer. Let $\mathfrak{M} = \bigcup_{d>0} \mathfrak{M}_{2d}$ be the union of the moduli spaces \mathfrak{M}_{2d} of projective irreducible holomorphic symplectic varieties of $K3^{[n]}$ -type polarized by a line bundle of degree $2d$. For all $(X, H) \in \mathfrak{M}$, outside at most a finite number of connected components, the following holds:*

- (1) *there exists a ruling genus 0 stable curve in X with cohomology class proportional to the Poincaré-dual of the class of H ;*
- (2) *there exists a positive integer m such that the linear system $|mH|$ contains a uniruled divisor.*

Remark 1.3. The ruling curve in (1) may be chosen to have primitive cohomology class, see the comments below. However, we are not able to control the integer m in (2).

Some comments are in order. The Beauville–Bogomolov form induces an embedding $H^2(X, \mathbb{Z}) \hookrightarrow H_2(X, \mathbb{Z})$, $H \mapsto H^\vee$. By abuse of notation, we again denote by q the quadratic form on $H_2(X, \mathbb{Z})$. We can make explicit the components of \mathfrak{M} for which the existence is obtained.

Remark 1.4. The previous statement may be split into two parts. On the one hand, Theorem 4.5 ensures the existence of uniruled divisors covered by primitive rational curves if there exist integers p, g and ϵ such that $p \geq g$ and $\epsilon = 0$ or 1 such that the following two conditions hold:

- (i) the class $\alpha := H^\vee / \text{div}(H) \in H_2(X, \mathbb{Z})$ can be written as $\gamma + (2g - \epsilon)\eta$, with η in the monodromy orbit of the class of the exceptional curve on a $K3^{[n]}$ and $\gamma \in \eta^\perp$;
- (ii) $q(\gamma) = 2p - 2$ (hence, $q(\alpha) = 2p - 2 - (2g - \epsilon)^2 / (2n - 2)$).

On the other hand, thanks to Proposition 4.6, we can show that the two conditions above are satisfied outside at most a finite number of connected components. Observe furthermore that conditions (i) and (ii) determine the monodromy orbit of the polarization H (cf. Corollary 2.8 for details).

We list some relevant cases in which the conditions (i) and (ii) of Remark 1.4 are easily seen to be satisfied in the following.

Remark 1.5. (i) If $q(\alpha) \geq n - 1$, then a multiple of H is uniruled by primitive rational curves of class α (see Proposition 4.6).

(ii) If $\rho(X) \geq 2$, then X always contains an ample uniruled divisor covered by primitive rational curves (cf. Corollary 4.7).

(iii) If $n \leq 7$, then the conclusion of the theorem holds for *all* the connected components of \mathfrak{M} (cf. Remark 4.8).

(iv) If $n - 1$ is a power of a prime number, then by [Mar11, Lemma 9.2 and subsequent comment], the monodromy group is maximal. Therefore, it suffices to check that the square $q(\alpha)$ is of the form $2p - 2 - (2g - \epsilon)^2 / (2n - 2)$, with $p \geq g$.

The existence of uniruled divisors ruled by primitive rational curves on *any* projective holomorphic symplectic variety of $K3^{[n]}$ -type was wrongly claimed in [CP14]. Counterexamples were recently provided by Oberdieck, Shen and Yin in [OSY19, Corollary A.3]. The proof presented in [CP14] was based upon the following three ingredients: (a) the existence of a ‘controlled’ polarized deformation of a polarized holomorphic symplectic variety (X, H) to a $((K3)^{[n]}, H')$, as a consequence of Verbitsky’s global Torelli theorem and Markman’s study of the monodromy group (see § 2); (b) a geometric criterion to deform a rational curve on a holomorphic symplectic variety X along its Hodge locus inside the moduli space of X , which we derive from Mumford’s theorem on 0-cycles and deformation theoretic arguments (cf. § 3); (c) the existence of uniruled divisors on a $(K3)^{[n]}$, via points on nodal curves in the hyperplane linear system (see § 4). These three parts of the proof are correct and are presented here essentially as in [CP14]. The main difference is that, after the appearance of [OSY19], we realized that the examples we provided in ingredient (c) did not (and actually could not, because of [OSY19, Corollary A.3]) cover all the connected components of \mathfrak{M} . The same type of considerations and results hold in the generalized Kummer case, treated in [MP18] and amended in [MP21].

In the present paper we also show that conditions (i) and (ii) in Remark 1.4 are precisely satisfied in *all cases* where the obstruction discovered by [OSY19] does not prevent uniruled divisors covered by primitive rational curves to exist. In other words, we show that our result is sharp (see § 5.1 for details). In all cases where the theorem fails, using [KLM19], we can check the existence of a family of the expected dimension of rational curves covering a coisotropic subvariety of codimension $c \geq 2$ (cf. Proposition 5.13. Moreover, in some of these cases we can actually prove that $c = 2$ (see Theorem 5.11 in § 5.3).

In § 5.2 we discuss the extent to which it might be possible to find uniruled divisors covered by possibly non-primitive rational curves. We find an explicit condition (cf. Proposition 5.6) for the existence of such divisors, which can be easily checked on examples. For instance, this allows us to check the existence of ample uniruled divisors covered by non-primitive rational curves *for all the connected components* of the moduli space \mathfrak{M} up to dimension 26.

It should be noted that applications of the existence of uniruled divisors to the study of Chow groups of 0-cycles does not make use of the primitivity of the relevant rational curves. These applications are presented in § 6. To state our results recall that if Y is a variety, $CH_0(Y)_{\text{hom}}$ is the subgroup of $CH_0(Y)$ consisting of 0-cycles of degree 0.

DEFINITION 1.6. Let D be an irreducible divisor on X . We denote by $S_1CH_0(X)_{D,\text{hom}}$ the subgroup

$$S_1CH_0(X)_{D,\text{hom}} := \text{Im}(CH_0(D)_{\text{hom}} \rightarrow CH_0(X)),$$

of $CH_0(X)$. We denote by $S_1CH_0(X)_D$ the subgroup

$$S_1CH_0(X)_D := \text{Im}(CH_0(D) \rightarrow CH_0(X)),$$

of $CH_0(X)$.

We prove the following.

THEOREM 1.7. *Let X be a projective holomorphic symplectic variety. Suppose that X possesses an ample ruling curve. Then the subgroups $S_1CH_0(X)_{D,\text{hom}}$ and $S_1CH_0(X)_D$ are independent of the irreducible uniruled divisor D .*

In light of the previous result, we set

$$S_1CH_0(X)_{\mathbb{Q}} := \text{Im}(j_* : CH_0(D)_{\mathbb{Q}} \rightarrow CH_0(X)_{\mathbb{Q}}),$$

where $j : D \hookrightarrow X$ is any irreducible uniruled divisor. It is natural to ask whether such a subgroup of $CH_0(X)_{\mathbb{Q}}$ has an intersection-theoretic interpretation, as for $K3$ surfaces. This is indeed the case.

THEOREM 1.8. *Let X be a projective holomorphic symplectic variety. Suppose that X possesses an ample ruling curve and that the group of cohomology classes of curves on X is generated over \mathbb{Q} by classes of ruling curves. Then, for any non-trivial $L \in \text{Pic}(X)$, we have*

$$S_1CH_0(X)_{\text{hom}} = L \cdot CH_1(X)_{\text{hom}} \quad \text{and} \quad S_1CH_0(X) = L \cdot CH_1(X).$$

In particular, the conclusions of the results given previously hold for all projective holomorphic symplectic variety of $K3^{[n]}$ -type, for $n \leq 13$, and, for higher n , for all but finitely many components of \mathfrak{M} , as specified in Remark 1.4. The hypothesis on the Picard group is verified in applications by showing the existence of uniruled divisors linearly equivalent to certain multiple of each ample divisor. There is no evidence this could not hold in general.

The theorems given previously may be regarded as a higher-dimensional analogue of the Beauville–Voisin result. It is important to note that, for holomorphic symplectic varieties of higher dimension, Beauville has stated in [Bea07] a far-reaching conjectural generalization of their result, called the ‘weak splitting property’, which can be deduced by a (conjectural) splitting of the (conjectural) Bloch–Beilinson filtration on the Chow group. The conjecture was further refined by Voisin in [Voi08]. These conjectures have been studied intensively in recent years. After the appearance of a first version of this paper, Voisin [Voi16] has unveiled a surprising conjectural connection between the weak splitting property conjecture and the existence of subvarieties whose zero cycles are supported in lower dimension. A first instance of such an existence result is provided by Theorem 1.2. For a (non-exhaustive) list of works in this research direction, see [Voi08, Fer12, Fu13, Fu15, Huy14, Rie16, Voi15, Lat18, Lin20, Lin16, LP19, Yin15, SV16, MP18, Via17, SYZ20, SY20, FLVS19, OSY19, Voi22].

We end the paper with some speculations on possible generalizations of our results.

After the appearance of [CP14], other researchers studied the existence of rational curves on projective holomorphic symplectic varieties. We already mentioned [MP18, MP21] where the analogous existence results of ample uniruled divisors are established for deformations of generalized Kummer varieties. The existence of primitive rational curves moving in a family of the expected dimension on projective holomorphic symplectic varieties deformations of punctual

Hilbert schemes on a $K3$ surface or of generalized Kummer varieties is shown in [MO20, Theorem 3.2] and [MP18, Theorem 5.1]. While finishing the first version of the paper [CP14], independent work of Amerik and Verbitsky [AV15] has appeared and § 4 of [AV15] has some overlap with the results and arguments presented here in § 3 concerning the deformations of rational curves on holomorphic symplectic varieties. Amerik and Verbitsky are more concerned with negative rational curves, whereas we focus on positive ones (i.e. dual to an ample class). As the goals and the results of the two papers are quite different, for the sake of completeness we did not try to eliminate similar discussions. We refer the reader to § 3 here for precise references to the similar results appearing in [AV15].

We always work over the field \mathbb{C} of complex numbers.

2. Varieties of $K3^{[n]}$ -type and their polarizations

In this section we collect a number of known results on Hilbert schemes of points on a $K3$ surface and then move on to study the *polarized* deformations of irreducible holomorphic symplectic varieties of $K3^{[n]}$ -type.

2.1 Lattices

The general theory of irreducible holomorphic symplectic varieties as in [Bea83] shows that the group $H^2(X, \mathbb{Z})$ is endowed with a natural symmetric bilinear form q_X of signature $(3, 20)$, the *Beauville–Bogomolov form*. When no confusion is possible, we denote the square $q_X(h)$ of an element $h \in H^2(X, \mathbb{Z})$ simply by $q(h)$ or by h^2 . Therefore, $(H^2(X, \mathbb{Z}), q_X)$ is naturally a lattice. We try to use the notation h for elements of the cohomology lattice and the capital letter H for divisors/line bundles (but no confusion should hopefully arise if we do otherwise somewhere).

DEFINITION 2.1. Let Λ be a free \mathbb{Z} -module of finite rank endowed with a symmetric bilinear form. If h is an element of Λ , the *divisibility* of h is the non-negative integer t such that

$$h \cdot \Lambda = t\mathbb{Z}.$$

It will be denoted by $\text{div}(h)$. If (X, h) is a polarized irreducible holomorphic symplectic variety, then the divisibility of h is its divisibility as an element of the lattice $H^2(X, \mathbb{Z})$ endowed with the Beauville–Bogomolov form.

We use the following maps relating a lattice (Λ, q) and its dual Λ^\vee .

There is a natural map

$$\Lambda \hookrightarrow \Lambda^\vee, \quad \lambda \mapsto q(\lambda, \cdot). \tag{2.1}$$

Given a class $c \in \Lambda^\vee$ there exists a unique class $\lambda_c \in \Lambda \otimes \mathbb{Q}$ such that for all $\lambda' \in \Lambda$ we have

$$q(\lambda_c, \lambda') = c(\lambda').$$

This induces a natural map

$$\Lambda^\vee \hookrightarrow \Lambda \otimes \mathbb{Q}, \quad c \mapsto \lambda_c. \tag{2.2}$$

Using the previous results we can endow Λ^\vee with a quadratic form taking *rational* values. By abuse of notation, we still denote it by q .

We often make use of the following set-theoretic map:

$$\Lambda \hookrightarrow \Lambda \otimes \mathbb{Q}, \quad \lambda \mapsto \frac{\lambda}{\text{div}(\lambda)}. \tag{2.3}$$

One may easily check that the image is contained in the image of Λ^\vee under the map (2.2) and gives all primitive elements in it.

Finally, the map given previously will be composed with the projection onto the *discriminant group* Λ^\vee/Λ :

$$\Lambda \hookrightarrow \Lambda^\vee \rightarrow \Lambda^\vee/\Lambda, \quad \lambda \mapsto \left[\frac{\lambda}{\text{div}(\lambda)} \right]. \tag{2.4}$$

2.2 Hilbert schemes of points on a K3 surface

Let S be a compact complex projective surface. If n is a positive integer, denote by $S^{[n]}$ the Hilbert scheme (or the Douady space in the non-projective case) of length n subschemes of S . By [Fog68], $S^{[n]}$ is a smooth complex variety. The general theory of the Hilbert scheme shows that $S^{[n]}$ is projective if S is.

Assume that S is a K3 surface. By [Bea83], $S^{[n]}$ is an irreducible holomorphic symplectic variety: it is simply connected, and the space $H^0(S^{[n]}, \Omega_{S^{[n]}}^2)$ is one-dimensional, generated by a holomorphic symplectic form. Let X be an irreducible holomorphic symplectic variety. We say that X is of $K3^{[n]}$ -type if X is deformation equivalent, as a complex variety, to $S^{[n]}$, where S is a complex K3 surface.

Let S be a K3 surface, and let $n > 1$ be an integer. We briefly recall the description of the group $H^2(S^{[n]}, \mathbb{Z})$ as in [Bea83, Proposition 6].

Let $S^{(n)}$ be the n th symmetric product of S , and let $\epsilon : S^{[n]} \rightarrow S^{(n)}$ be the Hilbert–Chow morphism. The map

$$\epsilon^* : H^2(S^{(n)}, \mathbb{Z}) \rightarrow H^2(S^{[n]}, \mathbb{Z}),$$

is injective. Furthermore, let $\pi : S^n \rightarrow S^{(n)}$ be the canonical quotient map, and let p_1, \dots, p_n be the projections from S^n to S . There exists a unique map

$$i : H^2(S, \mathbb{Z}) \rightarrow H^2(S^{[n]}, \mathbb{Z}),$$

such that for any $\alpha \in H^2(S, \mathbb{Z})$, $i(\alpha) = \epsilon^*(\beta)$, where $\pi^*(\beta) = p_1^*(\alpha) + \dots + p_n^*(\alpha)$. The map i is an injection.

Let E_n be the exceptional divisor in $S^{[n]}$, that is, the divisor that parametrizes non-reduced subschemes (we will drop the index n and simply write $E = E_n$ when no confusion is possible). The cohomology class of E in $H^2(S^{[n]}, \mathbb{Z})$ is uniquely divisible by two; see [Bea83, Remarque after Proposition 6]. Let $\delta := \delta_n$ be the element of $H^2(S^{[n]}, \mathbb{Z})$ such that $2\delta_n = [E]$. Then we have

$$H^2(S^{[n]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus_{\perp} \mathbb{Z}\delta_n, \tag{2.5}$$

where the embedding of $H^2(S, \mathbb{Z})$ into $H^2(S^{[n]}, \mathbb{Z})$ is that given by the map i .

The decomposition (2.5) is orthogonal with respect to the Beauville–Bogomolov form q , and the restriction of q to $H^2(S, \mathbb{Z})$ is the canonical quadratic form on the second cohomology group of a surface induced by cup product. We have

$$\delta^2 = -2(n - 1).$$

By Poincaré duality, $H_2(S^{[n]}, \mathbb{Z})$ may be identified to the dual lattice of $H^2(S^{[n]}, \mathbb{Z})$. Therefore, using (2.2), to a class $Z \in H_2(S^{[n]}, \mathbb{Z})$ we can associate a unique class $D_Z \in H^2(S^{[n]}, \mathbb{Q})$ such that for all $D' \in H^2(S^{[n]}, \mathbb{Z})$ we have

$$q(D_Z, D') = Z \cdot D'.$$

The class D_Z is called the *dual* of Z with respect to the Beauville–Bogomolov quadratic form. In this way, we obtain a quadratic form on the homology group $H_2(S^{[n]}, \mathbb{Z})$ taking rational values

and such that

$$H_2(S^{[n]}, \mathbb{Z}) = H_2(S, \mathbb{Z}) \oplus^\perp \mathbb{Z}r_n,$$

where r_n is the homology class orthogonal to $H_2(S, \mathbb{Z})$ and such that

$$r_n \cdot \delta_n = -1. \tag{2.6}$$

In particular, we have

$$r_n = \frac{1}{2(n-1)}\delta_n,$$

which implies

$$q(r_n) = -\frac{1}{2(n-1)}.$$

Geometrically, r_n is the class of an exceptional rational curve that is the general fiber of the Hilbert–Chow morphism (see, e.g., [HT10]).

By abuse of notation, if $h \in H^2(S, \mathbb{Z})$, we again denote by h the induced class in $H^2(S^{[n]}, \mathbb{Z})$ as well as that in $H_2(S^{[n]}, \mathbb{Z})$, using the embedding (2.1).

2.3 Polarized deformations of varieties of $K3^{[n]}$ -type

In this paper, we are interested in the possible deformation types for primitively polarized varieties (X, h) , where X is a variety of $K3^{[n]}$ -type and h is a primitive polarization of X , that is, the numerical equivalence class of a primitive and ample line bundle on X .

DEFINITION 2.2. Let X and X' be two compact complex manifolds, and let h, h' be numerical equivalence classes of line bundles on X and X' , respectively. We say that the pairs (X, h) and (X', h') are deformation equivalent if there exists a connected complex variety S , a smooth, proper morphism $\pi : \mathcal{X} \rightarrow S$, a line bundle \mathcal{L} on \mathcal{X} and two points s, s' of S such $(\mathcal{X}_s, c_1(\mathcal{L}_s))$ is isomorphic to (X, h) and $(\mathcal{X}_{s'}, c_1(\mathcal{L}_{s'}))$ is isomorphic to (X', h') .

Remark 2.3. Let X be a variety of $K3^{[n]}$ -type, and let h be a polarization on X . Then Markman shows in [Mar11, Proposition 7.1] that there exists a $K3$ surface S and a polarization h' on $S^{[n]}$ such that (X, h) is deformation equivalent to $(S^{[n]}, h')$.

In the surface case, that is, when $n = 1$, the global Torelli theorem implies that two primitively polarized $K3$ surfaces (X, h) and (X', h') are deformation equivalent if and only $h^2 = h'^2$. The situation is different in higher dimension.

Let X be a variety of $K3^{[n]}$ -type. If $n = 1$, the lattice $H^2(X, \mathbb{Z})$ is unimodular, so that the divisibility of any non-zero primitive element of $H^2(X, \mathbb{Z})$ is 1. This is no longer the case as soon as $n > 1$.

By (2.3) we have the following map of sets

$$H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Q}), \quad h \mapsto \frac{1}{\text{div}(h)}h. \tag{2.7}$$

Let (X, h) be a primitively polarized irreducible holomorphic symplectic variety. Both h^2 and the divisibility of h are constant along deformations of (X, h) . However, as shown in [Apo14], it is not true that these two invariants determine the deformation type of (X, h) . In this section, we give explicit representatives for all the deformation-equivalence classes of primitively polarized varieties of $K3^{[n]}$ -type.

We start by describing results due to Markman on deformation-equivalence of polarized varieties of $K3^{[n]}$ -type. These results rely both on the global Torelli theorem [Ver13] and the

computation of the monodromy group of varieties of $K3^{[n]}$ -type in [Mar10]. Recently, Kreck and Su [KS21] provided a counterexample to some of the statements contained in Verbitsky’s global Torelli theorem. However, this does not affect the results we are using, as they rely on Markman’s formulation of the global Torelli theorem using marked moduli spaces instead of the Teichmüller space used by Verbitsky. We refer the reader to [Loo21, Theorem 3.1 and Remark 3.3] for the correct statement and a comment about the difference between the Teichmüller space and marked moduli spaces with respect to the global Torelli theorem (see also [Ver20]).

Let S be a $K3$ surface, and $n > 1$ an integer. Let $\tilde{\Lambda}$ be the Mukai lattice of S

$$\tilde{\Lambda} = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}),$$

endowed with the quadratic form defined by

$$\langle (a, b, c), (a', b', c') \rangle = bb' - ac' - a'c.$$

Let $v_n = (1, 0, 1 - n)$. We identify $H^2(S^{[n]}, \mathbb{Z})$ endowed with the Beauville–Bogomolov quadratic form with the orthogonal of v_n in $\tilde{\Lambda}$. The inclusion

$$H^2(S, \mathbb{Z}) \hookrightarrow (v_n)^\perp,$$

is compatible with the decomposition (2.5).

If h is any class in $H^2(S^{[n]}, \mathbb{Z}) \subset \tilde{\Lambda}$, let $T_S(h)$ be the saturation in $\tilde{\Lambda}$ of the lattice spanned by h and v_n .

PROPOSITION 2.4. *Let S and S' be two $K3$ surfaces, and let $n > 1$ be an integer. Let h (respectively, h') be the numerical equivalence class of a big line bundle on $S^{[n]}$ (respectively, $S'^{[n]}$). The pairs $(S^{[n]}, h)$ and $(S'^{[n]}, h')$ are deformation equivalent if and only if there exists an isometry*

$$T_S(h) \rightarrow T_{S'}(h'),$$

mapping h to h' .

Proof. In case h and h' are ample, this is the result of Markman written up in [Apo14, Proposition 1.6].

In the general case, choose small deformations (X, h) and (X', h') of $(S^{[n]}, h)$ and $(S'^{[n]}, h')$, respectively, such that both X and X' have Picard number 1. By a theorem of Huybrechts [Huy99], h and h' are ample classes on X and X' , respectively.

The construction of the rank 2 lattices $T_S(h)$ and $T_{S'}(h')$ generalizes to X and X' to provide rank 2 lattices $T_X(h)$ and $T_{X'}(h')$. This follows from the work of Markman as in [Mar11], Corollary 9.5. We refer to [Mar11] and the discussion in [Apo14], § 1 for the precise construction.

The formation of $T_X(h)$ is compatible with parallel transport. As a consequence, the isomorphism $T_S(h) \rightarrow T_{S'}(h')$ mapping h to h' induces an isomorphism $T_X(h) \rightarrow T_{X'}(h')$ mapping h to h' . It follows once again from [Apo14, Proposition 1.6], that (X, h) and (X', h') are deformation equivalent. □

We now state the main result of this section.

THEOREM 2.5. *Let $n > 1$ be an integer and let (X, h) be a primitively polarized irreducible holomorphic symplectic variety of $K3^{[n]}$ -type. Let t be the divisibility of h , and let $I \subset \mathbb{Z}$ be a system of representatives of $\mathbb{Z}/t\mathbb{Z}$, up to the action of -1 on $\mathbb{Z}/t\mathbb{Z}$. Then t divides $2n - 2$ and there exists a $K3$ surface S , a primitive polarization h_S on S and an integer $\mu \in I$ such that the pair (X, h) is deformation equivalent to $(S^{[n]}, th_S - \mu\delta_n)$.*

Remark 2.6. The class $th_S - \mu\delta_n$ is not ample in general. However, the argument of the proof of Proposition 2.4 shows that it is big.

Proof. Remark 2.3 allows us to assume that $X = S^{[n]}$ for some $K3$ surface S . Let $\tilde{\Lambda} = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$ be the Mukai lattice of S and let $v_n = (1, 0, 1 - n)$.

Write $2d$ and t for the Beauville–Bogomolov square and the divisibility of h , respectively. We start by describing the structure of the lattice $T_S(h)$.

Write

$$h = (\mu, \lambda h_S, \mu(n - 1)) = \lambda h_S - \mu \delta_n,$$

where λ and μ are two integers and h_S is primitive. As h is primitive, λ and μ are relatively prime. It is readily checked that the divisibility of h is

$$t = \gcd(\lambda, 2n - 2),$$

and that μ and t are relatively prime.

The element

$$w = \frac{1}{t}h - \frac{\mu}{t}v_n = \left(0, \frac{\lambda}{t}h_S, \frac{\mu(2n - 2)}{t}\right) \in \tilde{\Lambda}, \tag{2.8}$$

belongs to $T_S(h)$, and the computation of [Apo14, Proposition 2.2] show that w generates the group $T_S(h)/(\mathbb{Z}h \oplus \mathbb{Z}v_n) \simeq \mathbb{Z}/t\mathbb{Z}$.

The lattice N spanned by h and v_n in $\tilde{\Lambda}$ is isomorphic to $\langle 2d \rangle \oplus \langle 2n - 2 \rangle$. Its discriminant group N^\vee/N is $\mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/(2n - 2)\mathbb{Z}$.

As in paragraph 4 of [Nik79] and under the identifications given previously, the inclusion

$$\langle 2d \rangle \oplus \langle 2n - 2 \rangle \simeq \mathbb{Z}h \oplus \mathbb{Z}v_n \subset T_S(h),$$

induces an injective morphism

$$\varphi : \mathbb{Z}/t\mathbb{Z} \simeq T_S(h)/(\mathbb{Z}h \oplus \mathbb{Z}v_n) \hookrightarrow \mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/(2n - 2)\mathbb{Z}.$$

By (2.8), φ sends 1 to $(2d/t, \mu(2n - 2)/t)$.

Let S' be a $K3$ surface with a primitive polarization $h_{S'}$. Let μ' be an arbitrary integer, and define

$$h' = th_{S'} - \mu'\delta_n \in H^2(S'^{[n]}, \mathbb{Z}) = H^2(S', \mathbb{Z}) \oplus \mathbb{Z}\delta_n.$$

By [Nik79, Proposition 1.4.2] and the discussion given previously, a sufficient condition ensuring that there exists an isomorphism of lattices $T_S(h) \rightarrow T_{S'}(h')$ sending h to h' is that

$$t^2 h_{S'}^2 - \mu'^2(2n - 2) = 2d,$$

and

$$\frac{\mu'(2n - 2)}{t} = \pm \frac{\mu(2n - 2)}{t} \pmod{2n - 2},$$

that is,

$$\mu' = \pm \mu \pmod{t}.$$

Now let μ' be an integer in I such that

$$\mu' = \pm \mu \pmod{t}.$$

Then μ' is prime to t because μ is. Furthermore, λ is divisible by t and h_S^2 is even, so that $\lambda^2 h_S^2$ is divisible by $2t^2$. In addition, $(\mu^2 - \mu'^2)(2n - 2)$ is divisible by $2t^2$: both $(\mu^2 - \mu'^2)$ and $2n - 2$ are divisible by t , and at least one of these terms is divisible by $2t$, depending on the parity of t .

As a consequence, the integer

$$2d + \mu'^2(2n - 2) = \lambda^2 h_S^2 - (\mu^2 - \mu'^2)(2n - 2),$$

is divisible by $2t^2$. Write

$$2d = t^2 2d' - \mu'^2(2n - 2),$$

and let S' be a $K3$ surface with a primitive polarization $h_{S'}$ of degree $2d'$.

By construction, there exists an isomorphism of lattices $T_S(h) \rightarrow T_{S'}(h')$ sending h to h' . Proposition 2.4 shows that $(S^{[n]}, h)$ and $(S'^{[n]}, th_{S'} - \mu'\delta_n)$ are deformation equivalent. \square

The following is an immediate consequence of the theorem.

COROLLARY 2.7. *Let $n > 1$ be an integer and let (X, h) be a primitively polarized irreducible holomorphic symplectic variety of $K3^{[n]}$ -type. Let $I \subset \mathbb{Z}$ be a system of representatives of $\mathbb{Z}/(2n - 2)\mathbb{Z}$, up to the action of -1 on $\mathbb{Z}/(2n - 2)\mathbb{Z}$. Then there exists a positive integer m , a $K3$ surface S , a primitive polarization h_S on S and an integer $\mu \in I$ such that the pair (X, mh) is deformation equivalent to $(S^{[n]}, (2n - 2)h_S - \mu\delta_n)$.*

We use the dual statement on curve classes.

COROLLARY 2.8. *Let $n > 1$ be an integer and let X be a primitively polarized irreducible holomorphic symplectic variety of $K3^{[n]}$ -type. Let $C \in H_2(X, \mathbb{Z})$ be a primitive curve class with positive square.*

Then there exists a $K3$ surface S , a primitive polarization h_S on S and an integer $\mu \in [0, n - 1)$ such that there exists a polarized deformation from X to $S^{[n]}$, carrying C to $h_S - \mu r_n$.

Proof. Following (2.2) and (2.3) we write $C = h/\text{div}(h)$, for some primitive and positive $h \in H^2(X, \mathbb{Z})$. As in the proof of Proposition 2.4 we may assume that h is ample. By Corollary 2.7 there exists a polarized $K3$ surface (S, h_S) and an integer $\mu \in [0, n - 1)$ such that the pair (X, mh) is deformation equivalent to $(S^{[n]}, (2n - 2)h_S - \mu\delta_n)$, for some integer $m > 0$. The divisibility of $(2n - 2)h_S - \mu\delta_n$ equals $2n - 2$. Therefore, via the map (2.7), we obtain

$$(2n - 2)h_S - \mu\delta_n \mapsto \frac{1}{2n - 2}((2n - 2)h_S - \mu\delta_n) = h_S - \mu r_n.$$

This class yields the desired parallel transport of C . \square

Remark 2.9. The proofs of the previous two corollaries work for any choice of a system of representatives of $\mathbb{Z}/(2n - 2)\mathbb{Z}$, up to the action of -1 on $\mathbb{Z}/(2n - 2)\mathbb{Z}$.

3. Deforming rational curves

Let $\pi : \mathcal{X} \rightarrow B$ be a smooth projective morphism of complex quasi-projective varieties of relative dimension d , and let α be a global section of Hodge type $(d - 1, d - 1)$ of the local system $R^{2d-2}\pi_*\mathbb{Z}$. Fixing such a section α , we can consider the relative Kontsevich moduli stack of genus zero stable curves $\overline{\mathcal{M}}_0(\mathcal{X}/B, \alpha)$. We refer the reader to [BM96, FP97, AV02] for details and constructions.

The space $\overline{\mathcal{M}}_0(\mathcal{X}/B, \alpha)$ parametrizes maps $f : C \rightarrow X$ from genus zero stable curves to fibers $X = \mathcal{X}_b$ of π such that $f_*[C] = \alpha_b$. The map $\overline{\mathcal{M}}_0(\mathcal{X}/B, \alpha) \rightarrow B$ is proper. If f is a stable map, we denote by $[f]$ the corresponding point of the Kontsevich moduli stack.

For the remainder of this section, let X be a smooth projective irreducible holomorphic symplectic variety of dimension $2n$ and let $f : C \rightarrow X$ be a map from a stable curve C of genus zero to X . We assume furthermore f is unramified at the generic point of each irreducible

component of C . Let $\mathcal{X} \rightarrow B$ be a smooth projective morphism of smooth connected quasi-projective varieties and let 0 be a point of B such that $\mathcal{X}_0 = X$. Let α be a global section of Hodge type $(2n - 1, 2n - 1)$ of $R^{4n-2}\pi_*\mathbb{Z}$ such that $\alpha_0 = f_*[C]$ in $H^{4n-2}(X, \mathbb{Z})$.

PROPOSITION 3.1. *Let M be an irreducible component of $\overline{M}_0(X, f_*[C])$ containing $[f]$. Then the following hold:*

- (1) *the stack M has dimension at least $2n - 2$;*
- (2) *if M has dimension $2n - 2$, then any irreducible component of the Kontsevich moduli stack $\overline{\mathcal{M}}_0(\mathcal{X}/B, \alpha)$ that contains M dominates B .*

In other words, when assumption (2) holds, the stable map $f : C \rightarrow X$ deforms over a finite cover of B . Related results have been obtained by Ran, see, for instance, Example 5.2 of [Ran95]. See also [AV15, Theorem 4.1] for an alternative proof.

Proof. Let $\tilde{\mathcal{X}} \rightarrow S$ be a local universal family of deformations of X such that B is the Noether–Lefschetz locus associated to $f_*[C]$ in S . In particular, B is a smooth divisor in S .

Lemma 11 of [BHT11] and our hypothesis on f show that we have an isomorphism

$$\mathbb{R}\mathcal{H}om(\Omega_f^\bullet, \mathcal{O}_C) \simeq N_f[-1],$$

in the derived category of coherent sheaves on C , for some coherent sheaf N_f . As a consequence, standard deformation theory shows that any component of the deformation space of the stable map f over S has dimension at least

$$\dim S + H^1(\mathbb{R}\mathcal{H}om(\Omega_f^\bullet, \mathcal{O}_C)) - H^2(\mathbb{R}\mathcal{H}om(\Omega_f^\bullet, \mathcal{O}_C)) = \dim S + \chi(N_f) = \dim B + 2n - 2,$$

the latter equality following from the Riemann–Roch theorem on C and the triviality of the canonical bundle of X .

As the image in S of any component of the deformation space of the stable map f is contained in B , the fibers of such a component all have dimension at least $2n - 2$. If any fiber has dimension $2n - 2$, it also follows that the corresponding component has to dominate B , which shows the result. □

This result holds for any smooth projective variety X with trivial canonical bundle. In order to use it, we need to study the locus spanned by a family of rational curves. The following gives a strong restriction on this locus, and makes crucial use of the symplectic form on X .

PROPOSITION 3.2. *Let X be a projective manifold of dimension $2n$ endowed with a symplectic form, and let Y be a closed subvariety of codimension k of X . Let $W \subset X$ be a subvariety such that any point of Y is rationally equivalent to a point of W . Then the codimension of W is at most $2k$.*

For a similar result, see [AV15, Theorem 4.4].

Before proving the proposition, we first record the following fact from linear algebra.

LEMMA 3.3. *Let (F, ω) be a symplectic vector space of dimension $2n$ and V a subspace of codimension k of F . Then V contains a subspace V' of codimension at most $2k$ in F such that the restriction $\omega|_{V'}$ of the 2-form is symplectic on V' . In particular, $\omega|_V^{2n-2k} \neq 0$.*

Proof of Lemma 3.3. Let V^\perp be the orthogonal to V with respect to the symplectic form ω . As ω is non-degenerate, we have

$$\dim(V^\perp) = k,$$

which implies that $\dim(V \cap V^\perp) \leq k$.

Let V' be a subspace of V such that V' and $V \cap V^\perp$ are in direct sum in V . Then $\dim(V') \geq 2n - 2k$. Furthermore, any $v \in V$ is orthogonal to $V \cap V^\perp$, so that $(V')^\perp = V^\perp$. As a consequence, $V' \cap (V')^\perp = V' \cap V^\perp = 0$, and the restriction of ω to V' is non-degenerate. \square

Let us recall the following result, proved by Voisin [Voi16, Lemma 1.1] as an application of Mumford’s theorem on 0-cycles.

LEMMA 3.4. *Let $f : Z \rightarrow X$ be a morphism between smooth projective varieties. Assume that there exists a surjective morphism $p : Z \rightarrow B$ to a smooth projective variety with the property that any two points of Z with the same image under p are mapped by f to rationally equivalent points of X .*

Then for any holomorphic form η on X there exists a holomorphic form η_B on B such that

$$f^*\eta = p^*\eta_B.$$

Proof of Proposition 3.2. Assume by contradiction that W has dimension at most $2n - 2k - 1$. We argue as in [Voi16, Proof of Theorem 1.3]. If w is any point of W , define

$$O_w := \{x \in X : x \equiv_{\text{rat}} w\},$$

where \equiv_{rat} denotes rational equivalence. Then O_w is a countable union of subvarieties of X , and a dimension count shows that for any w , O_w contains a component of dimension at least $k + 1$.

By the countability of Hilbert schemes, there exists a generically finite cover $\varphi : B \rightarrow W$, a family $f : Z \rightarrow B$ of varieties of dimension $k + 1$ and a morphism $f : Z \rightarrow X$ mapping every fiber of p generically finitely onto points that are all rationally equivalent in X . We can assume that B and Z are smooth and projective. Lemma 3.4 shows that for any holomorphic form η on X there exists a holomorphic form η_B on B such that

$$f^*\eta = p^*\eta_B. \tag{3.1}$$

On the other hand, if ω is the symplectic form on X , then, by Lemma 3.3, the $(2n - 2k)$ -holomorphic form $\eta = \omega^{n-k}$ verifies

$$f^*(\omega^{n-k}) \neq 0.$$

As $\dim(B) = \dim(W) < 2n - 2k$, the latter contradicts (3.1). \square

These results allow us to give a simple criterion for the existence of uniruled divisors on polarized deformations of a given holomorphic symplectic variety X . For similar results, see [AV15, Corollaries 4.5 and 4.8].

COROLLARY 3.5. *Let $f : C \rightarrow X$ be a genus zero stable curve in X , and let $\alpha = f_*[C]$. Assume that f is ruling. Then the following holds.*

- (1) *There exists an irreducible component of $\overline{\mathcal{M}}_0(\mathcal{X}/B, \alpha)$ containing $[f]$ that dominates B . In particular, the stable map $[f]$ deforms over a finite cover of B .*
- (2) *For any point b of B , the fiber \mathcal{X}_b contains a uniruled codimension 1 subscheme D whose codimension 1 component of its cohomology class is a positive multiple of the Poincaré dual of α , and such that D is ruled by a curve of class α .*

Proof. Let M be an irreducible component of $\overline{M}_0(X, f_*[\mathbb{P}^1])$ containing $[f]$ such that, denoting by Y the subscheme of X covered by the deformations of f parametrized by M , Y is a divisor in X .

Let $C \rightarrow M$ be the universal curve. By Proposition 3.1, the dimension of M is at least $2n - 2$. We claim that equality holds, which implies statement (1), again by Proposition 3.1. Assume

by contradiction that $\dim(M) > 2n - 2$. As $\dim(Y) = 2n - 1$, this implies that any fiber of the evaluation map $C \rightarrow Y \subset X$ is at least one-dimensional, which, in turn, shows that there exists a subvariety $W \subset X$ of dimension at most $\dim Y - 2 = 2n - 3$ such that any point of Y is rationally equivalent to a point in W . Proposition 3.2 provides the contradiction.

To show statement (2), it suffices to consider the case where B has dimension 1 and passes through a very general point of the Noether–Lefschetz locus associated to α . Let \mathcal{M} be an irreducible component of $\overline{\mathcal{M}}_0(\mathcal{X}/B, \alpha)$ containing M . Then \mathcal{M} dominates B . Let $\mathcal{Y} \subset \mathcal{X} \rightarrow B$ be the locus in \mathcal{X} covered by the deformations of f parametrized by \mathcal{M} . Since \mathcal{M} dominates B , any irreducible component of \mathcal{Y} dominates B . Since the fiber of $\mathcal{Y} \rightarrow B$ over 0 is a divisor in $\mathcal{X}_0 = X$, the fiber of $\mathcal{Y} \rightarrow B$ at any point b has codimension 1.

By construction, the irreducible codimension 1 part of Y_b is ruled by a curve of class α . Note that we cannot deduce that the class of the divisor Y_b inside \mathcal{X}_b equals that of Y because \mathcal{Y}_0 contains Y and the inclusion could be proper.

Let b be very general in the Noether–Lefschetz locus. The Néron–Severi group of \mathcal{X}_b has rank at most 2, and it is generated over \mathbb{Q} by the Poincaré dual α^\vee of α and the class of the polarization. In particular, the codimension one component of the cohomology class of Y_b is a linear combination of α^\vee and the class of the polarization. As a consequence, the same holds for Y_0 . This holds for any choice of a polarization on X .

First assume that $b_2(X) \geq 5$, so that $h^{1,1}(X) \geq 3$. Assume that α^\vee is not proportional to the codimension one component of the cohomology class of Y_0 . Then the Picard number of X_b is 2 for general b , and we may choose B in such a way that there exists b' in B such that the Néron–Severi group of $NS(X_{b'})$ has rank at least 3. We may replace X with $X_{b'}$ and assume that $NS(X)$ has rank at least 3.

Let h and h' be two ample classes on X such that α^\vee, h and h' span a subspace of rank 3 of $NS(X)$. Then the codimension one component of the cohomology class of Y_0 is both a linear combination of α^\vee and h and of α^\vee and h' . As a consequence, it is proportional to α^\vee .

To give a general argument that covers the case $b_2 = 4$, we may replace the moduli space of stable genus 0 curves – for which classical references require projectivity assumptions – with the (relative) Douady space parametrizing rational curves, and work with non projective manifolds as well. In this framework, we may deform X with the class α and the uniruled codimension one subscheme under consideration to an irreducible holomorphic symplectic variety with Néron–Severi group generated by α^\vee , which proves the proportionality result.

Knowing that the cohomology class of the codimension 1 component Y_b is proportional to the Poincaré dual of α , let λ be the rational number such that

$$[Y_b] = \lambda[\alpha]^\vee.$$

Then λ is independent of b . To prove that λ is positive, we may assume that X_b is projective by choosing a suitable deformation of the pair (X, α) . Let H be an ample class on X_b . Then $q(H, Y_b) > 0$ and $H\alpha > 0$, so that the coefficient of proportionality is positive. □

4. Examples, proof of Theorem 1.2 and further remarks

In this section, we use the notation and the basic facts recalled in §2.1. We start by constructing examples of uniruled divisors using the classical Brill–Noether theory applied to the desingularizations of curves on a $K3$ surface. Then we present the proof of Theorem 1.2. We also compare conditions (i) and (ii) in Remark 1.4 with the condition appearing in [OSY19, Corollary A.3].

Finally, we show the existence of codimension 2 coisotropic subvarieties on projective holomorphic symplectic manifold of $K3^{[n]}$ -type in some of the cases where Theorem 1.2 does not provide uniruled divisors.

4.1 Examples

Let n be a positive integer, and let k be an integer between 1 and n . Write $g = k - 1$. Let (S, H) be a general polarized $K3$ surface, $m > 0$ an integer and $L := mH$ with $p_a(L) := p \geq g$. Consider the Severi variety parametrizing nodal genus g curves inside $|L|$. It is a locally closed subvariety of $|L|$ (see, e.g., [Ser06] for the basic facts on Severi varieties).

Recall that the Severi variety on regular surfaces has the expected codimension (equal to the number of nodes) whenever non-empty (see, e.g., [CS97, Example 1.3]). In the case of multiples of the hyperplane section on a general $K3$ surface non-emptiness has been shown by [Che02] (see also [GK14] for another proof and a generalization). The case $m = 1$ corresponds to the ‘classical’ Bogomolov–Mumford theorem (see, e.g., [BHPV04, Ch. VIII, § 23]).

Consider a g -dimensional family $\mathcal{C}_T \rightarrow T$, $T \subset |L|$, of curves of geometric genus g whose general member is a nodal curve. Hence, T is given by an irreducible component of the Severi variety parametrizing curves with $(p - g)$ -nodes inside $|L|$.

It is well known that the relative symmetric product $\mathcal{C}_T^{(g+1)}$ is uniruled. Indeed, for each $t \in T$ the symmetric product $C_t^{(g+1)}$ is uniruled as the Abel–Jacobi morphism $AJ_{g+1} : \tilde{C}_t^{(g+1)} \rightarrow \text{Pic}^{g+1}(\tilde{C}_t)$ onto the g -dimensional Picard variety of the desingularization $\tilde{C}_t \rightarrow C_t$ has fibers isomorphic to $\mathbb{P}H(\tilde{C}_t, L)$, for all $L \in \text{Pic}^{g+1}(\tilde{C}_t)$.

Its dimension is $2g + 1$. We have the following key result.

PROPOSITION 4.1. *Let S and L be as given previously. Then $S^{[g+1]}$ contains a uniruled divisor that is the image of $\mathcal{C}_T^{(g+1)}$ under the natural rational map*

$$\mathcal{C}_T^{(g+1)} \dashrightarrow S^{[g+1]},$$

for a certain irreducible component T of the Severi variety parametrizing curves with $(p - g)$ -nodes inside $|L|$.

Proof. We may assume that $\text{Pic}(S) = \mathbb{Z}H$. We prove this statement by induction on g . For $g = 0$, the statement is simply the existence of nodal rational curves proved by [Che02].

It is sufficient to show the claim on the symmetric product $S^{(g+1)}$ of S . More precisely, we prove the following statement: there exists an irreducible component V of the Zariski closure of the Severi variety parametrizing nodal genus g curves inside $|L|$ such that, if $\mathcal{C}_V \rightarrow V$ denotes the universal curve and $\mathcal{C}_V^{(g+1)} \rightarrow V$ the relative symmetric product, the natural morphism

$$\mathcal{C}_V^{(g+1)} \rightarrow S^{(g+1)},$$

is generically finite onto its image. Note that this is equivalent to saying that $(g + 1)$ -generic points on a generic curve of the family lie only on a finite number of curves of the family.

Indeed as

$$\dim \mathcal{C}_V^{(g+1)} = \text{reldim}(\mathcal{C}_V^{(g+1)}) + \dim V = (g + 1) + g = 2g + 1,$$

it follows that the image is a divisor inside $S^{(g+1)}$, and such divisor is uniruled as observed previously.

Note also that the positive dimensional fibers of the morphism $\mathcal{C}_V^{(g+1)} \rightarrow S^{(g+1)}$ cannot lie in a fiber of $\mathcal{C}_V^{(g+1)} \rightarrow V$, as $C_t^{(g+1)}$ injects into $S^{(g+1)}$ for every $t \in V$.

By inductive hypothesis, there exists an irreducible component W of the (Zariski closure of the) Severi variety parametrizing nodal genus $g - 1$ curves inside $|L|$ such that, if $\mathcal{C}_W \rightarrow W$ denotes the universal curve and $\mathcal{C}_W^{(g)} \rightarrow W$ the relative symmetric product, the natural morphism

$$\mathcal{C}_W^{(g)} \rightarrow S^{(g)},$$

is generically finite onto its image.

Now let V be the Zariski closure of an irreducible component of the Severi variety of nodal genus g curves in $|H|$ obtained by smoothing one node of the curves in W (which can be done again by the regularity of the Severi variety, [CS97, Example 1.3]). By construction, $W \subset V$. Let $\mathcal{C}_V \rightarrow V$ be the universal curve. Its restriction over W yields a map $\mathcal{C}_W \rightarrow W$. Let D be the image of the morphism

$$\mathcal{C}_V^{(g+1)} \rightarrow S^{(g+1)}.$$

Observe that D contains the image D_W of

$$\mathcal{C}_W^{(g+1)} \rightarrow S^{(g+1)}.$$

We claim that by the inductive hypothesis D_W has codimension 2 or, equivalently, that the morphism $\mathcal{C}_W^{(g+1)} \rightarrow S^{(g+1)}$ is generically finite onto its image. Indeed if $\xi = x_1 + \dots + x_{g+1}$ is a generic point of the image, then, say, $x_1 + \dots + x_g$ is a generic point of the image of the morphism $\mathcal{C}_W^{(g)} \rightarrow S^{(g)}$. By the inductive hypothesis the points x_1, \dots, x_g lie on finitely many curves of the family W , *a fortiori* that will be true for x_1, \dots, x_g, x_{g+1} and the claim follows.

We want to prove that D contains D_W strictly. If this were not the case, by irreducibility, we would have $D = D_W$. Let $U \subset D$ be an open subset over which the morphisms $\mathcal{C}_W^{(g+1)} \rightarrow S^{(g+1)}$ and $\mathcal{C}_V^{(g+1)} \rightarrow S^{(g+1)}$ are smooth and let $p_1 + p_2 + \dots + p_{g+1}$ be a point in U . Let C be a nodal genus g curve in V containing these points. Let us fix the first g points p_1, \dots, p_g . By induction, these points are contained inside a finite number of curves of genus $g - 1$ belonging to W . Let B_1, \dots, B_m be all such curves. Let $U_C \subset C$ be an open subset such that for all $q \in U_C$ we have $p_1 + \dots + p_g + q \in U$. As we have shown previously, p_1, \dots, p_g, q lie on finitely many curves of genus $g - 1$ belonging to W , and these curves must be B_1, \dots, B_m . Therefore, as q varies in U_C , we deduce that U_C is a subset of a finite union of genus $(g - 1)$ curves. As C is irreducible, there is an i such that $C = B_i$, which is clearly a contradiction. Therefore, D must strictly contain D_W and be a divisor, which is necessarily uniruled. \square

Remark 4.2. Let $n \geq 2$ be an integer. Let S be a general projective $K3$ surface, H an ample divisor on it and $L = mH$, $m \geq 1$, a line bundle with $p_a(L) \geq n - 1$. For all $k = 1, \dots, n$, consider a $(k - 1)$ -dimensional family $T \subset |L|$ of nodal curves of geometric genus $k - 1$. Then consider the closure of the image of the rational map

$$\mathcal{C}_T^{(k)} + S^{(n-k)} \dashrightarrow S^{[n]}.$$

By Proposition 4.1 we obtain n distinct uniruled divisors D_1, \dots, D_n inside $S^{[n]}$.

Let us now compute the class of the curve in the ruling of the divisors given previously. Let us denote by $\mathbb{P}_{\mathfrak{g}_k^1}^1 \subset S^{[k]}$ the image of a rational curve associated to one of the \mathfrak{g}_k^1 given, as in the proof of Proposition 4.1, by any k -points on the curve. We remark that the hypotheses imply that for a general choice, these linear series are simple and the nodes of the curve are non-neutral (a node $p \in C$ is said to be non-neutral with respect to a linear series \mathfrak{g}_k^1 on the desingularization $\nu : \tilde{C} \rightarrow C$ if $\mathfrak{g}_k^1(-\nu^{-1}(p)) = \emptyset$, i.e. if the two points above the node do not belong to the same

fiber of the morphism associated to the linear series). Then, the Riemann–Hurwitz formula (see, e.g., [CK14, § 2]) yields

$$[\mathbb{P}^1_{\mathfrak{g}_k^1}] = mh - 2(k - 1)r_k, \tag{4.1}$$

where mh is the class of L in the Néron–Severi group of S .

If we add $(n - k)$ -distinct generic points $\eta = q_1 + \dots + q_{n-k}$ to $\mathbb{P}^1_{\mathfrak{g}_k^1} \subset S^{[k]}$ we obtain a rational curve inside $S^{[n]}$, which we denote by R_k .

PROPOSITION 4.3. *Let k be an integer between 1 and n . Then:*

- (1) *the class of $R_k \subset S^{[n]}$ in $H_2(S^{[n]}, \mathbb{Z})$ is $mh - (2k - 2)r_n$;*
- (2) *the class of $D_k \subset S^{[n]}$ in $H^2(S^{[n]}, \mathbb{Z})$ is proportional to $(2n - 2)mh - (2k - 2)\delta_n$.*

Proof. (1) Write $R_k = ah - br_n$. As $R_k = \mathbb{P}^1_{\mathfrak{g}_k^1} + \eta$, the intersection product $R_k \cdot h$ equals $2p - 2$, from which, using the facts recalled in § 2.1 we deduce $a = m$. Again for the choice of η we have that $R_k \cdot \delta_n = \mathbb{P}^1_{\mathfrak{g}_k^1} \cdot \delta_k$. From this, from (4.1) and from § 2.1 we deduce that $b = 2(k - 1)$. This proves the first statement.

(2) For the second statement we argue similarly as follows. Write $D_k = ah - b\delta_n$. Let $x_1, \dots, x_{n-1} \in S$ be general points and let $C \in |L|$ be a general curve. Set $\xi := x_1 + \dots + x_{n-1}$ and consider the curve in $S^{[n]}$ given by $\xi + C$. Such a curve has class $h \in H_2(S, \mathbb{Z})$. We first describe all the points in the intersection between D_k and $\xi + C$. Let I be a subset of $k - 1$ indices among $\{1, \dots, n - 1\}$ and let ξ_I be the corresponding zero-dimensional subscheme of length $k - 1$. Notice that, as shown in Proposition 4.1, there exists a finite subfamily T_ξ of cardinality M given by curves in T passing through ξ_I . For each curve $C' \in T_\xi$ and each point q of the $2p - 2$ intersection points between C' and C we get an intersection between D_k and $\xi + C$ given by $q + \xi$. All in all we obtain that

$$D_k \cdot h = M \binom{n - 1}{k - 1} (2p - 2),$$

from which we deduce that $a = mM \binom{n-1}{k-1}$.

Consider $p, x_1, \dots, x_{n-2} \in S$ general points. Set $\xi = x_1 + \dots + x_{n-2}$ and consider the curve $\mathbb{P}T_p(S) + \xi$, which has class r_n . We now describe all the points in the intersection between D_k and $\mathbb{P}T_p(S) + \xi$. Let I be a subset of $k - 2$ indices among $\{1, \dots, n - 2\}$ and let ξ_I be the corresponding zero-dimensional subscheme of length $k - 2$. As above, there exists a finite subfamily $T_{p+\xi}$ of cardinality M given by curves in T passing through $p + \xi_I$. For each curve $C' \in T_{p+\xi}$ we get an intersection between D_k and r_n given by $p_{C'} + \xi$, where $p_{C'}$ is the length two zero-dimensional subschemes supported on p and determined by the tangent direction of C' at p . We therefore have

$$D_k \cdot r_n = M \binom{n - 2}{k - 2},$$

from which we deduce that $b = M \binom{n-2}{k-2}$. Therefore

$$D_k = M \binom{n - 2}{k - 2} \frac{1}{(k - 1)} (m(n - 1)h - (k - 1)\delta_n)$$

and we are done. □

To account for other polarization types, we proceed as follows. For $k = 2, \dots, n$ let $\xi' \in S^{[k]}$ be a zero-dimensional non-reduced subscheme corresponding to a ramification point of a \mathfrak{g}_k^1 on

a general curve \tilde{C}_t . Let $\xi \in S^{[n]}$ be a zero-dimensional subscheme obtained by adding $(n - k)$ distinct generic points to ξ' . Let \mathbb{P}_ξ^1 the exceptional rational curve passing through ξ . Consider the curve

$$R'_k := R_k \cup \mathbb{P}_\xi^1,$$

obtained by glueing along ξ the curve R_k corresponding to the \mathfrak{g}_k^1 and the exceptional rational curve \mathbb{P}_ξ^1 . Working with families of rational curves, we obtain as before uniruled divisors D'_2, \dots, D'_n . The divisor D'_k is the union of D_k with the exceptional divisor of $S^{[n]}$. As a direct consequence of Proposition 4.3, we can compute the relevant cohomology classes.

PROPOSITION 4.4. *Let k be an integer between 2 and n . Then:*

- (1) *the class of $R'_k \subset S^{[n]}$ in $H_2(S^{[n]}, \mathbb{Z})$ is $h - (2k - 1)r_n$;*
- (2) *the class of $D'_k \subset S^{[n]}$ in $H^2(S^{[n]}, \mathbb{Z})$ is proportional to $(2n - 2)h - (2k - 1)\delta_n$.*

We now move on to the result we alluded to in Remark 1.4 the introduction.

THEOREM 4.5. *Let (X, H) be a polarized holomorphic symplectic variety of $K3^{[n]}$ -type. Suppose there exist integers p, g and ϵ such that $p \geq g$ and $\epsilon = 0$ or 1 such that the following two conditions hold:*

- (i) *the class $\alpha := H^\vee / \text{div}(H) \in H_2(X, \mathbb{Z})$ can be written as $\gamma + (2g - \epsilon)\eta$, with η in the monodromy orbit of the class of the exceptional curve on a $K3^{[n]}$ and $\gamma \in \eta^\perp$;*
- (ii) *$q(\gamma) = 2p - 2$ (hence, $q(\alpha) = 2p - 2 - (2g - \epsilon)^2 / (2n - 2)$).*

Then there exists an integer $m > 0$ such that the linear system $|mH|$ contains a uniruled divisor covered by rational curves of primitive class equal to α .

Proof. Let $\alpha \in H_2(X, \mathbb{Z})$ as in the statement of the theorem. By Corollary 2.8, there exists a $K3$ surface S as given previously such that the pair (X, α) is deformation equivalent to $(S^{[n]}, [R_k])$ for some k between 1 and n , or to $(S^{[n]}, [R'_k])$ for some k between 2 and n (with R_k and R'_k as in Propositions 4.3 and 4.4). Note that being deformation equivalent means that (X, α) and $(S^{[n]}, R_k)$ (or $(S^{[n]}, [R'_k])$) are connected by a family $f : \mathcal{X} \rightarrow B$, where B is an irreducible curve, so that the parallel transport brings α to $[R_k]$ (respectively, $[R'_k]$).

Corollary 3.5 shows indeed that X contains a uniruled divisor with class a multiple of h and the theorem now follows. □

4.2 Finiteness of the exceptions and proof of the main theorem

In this section we prove that, for every dimension, there is at most a finite number of components of the moduli space of polarized irreducible holomorphic symplectic manifolds (X, H) of $K3^{[n]}$ -type where the strategy of the previous sections does not work. Together with Theorem 4.5 this will conclude the proof of Theorem 1.2.

The uniruled divisors we constructed in the previous paragraph have class $h_S - (2g)r_n$ (or $h_S - (2g - 1)r_n$), with $2p - 2 = h_S^2$ and h_S the primitive polarization on the $K3$ surface. We have the following result.

PROPOSITION 4.6. *Let C be a primitive class of a curve on a manifold of $K3^{[n]}$ -type such that its square C^2 with respect to the Beauville–Bogomolov form is > 0 . If $q(C) \geq n - 1$, then C is deformation equivalent to the class of one of the curves constructed in the previous section.*

Proof. We know by Corollary 2.8 that C is deformation equivalent to either $h_S - 2gr_n$ or $h_S - (2g - 1)r_n$, with $2g \leq n - 1$ (respectively, $2g \leq n$). If p were $< g$, then, in both cases, we would

have

$$\begin{aligned} n - 1 \leq q(C) &= q(h_S) - 4g^2 \frac{1}{2(n - 1)} \\ &= 2(p - 1) - 4g^2 \frac{1}{2(n - 1)} < 2(g - 1) - 4g^2 \frac{1}{2(n - 1)} \leq n - 2, \end{aligned}$$

which is a contradiction. □

Proof of Theorem 1.2. The components of \mathfrak{M} are in bijective correspondence with the monodromy orbits of a given class of positive square in $H^2(X, \mathbb{Z})$, see [Apo14, Corollary 2.4]. For a fixed square of h , there is a finite number of orbits (cf. [GHS10, Proposition 1.2]), so it follows that if X has a uniruled divisor when $q(h)$ is big enough, our claim will hold. Let C be a curve class in $H_2(X, \mathbb{Z})$ such that $C = h/\text{div}(h)$ under the map (2.7). The divisibility of h is at most $2n - 2$, therefore if $q(h) \geq (2n - 2)^2(n - 1)$ the curve class C has square at least $n - 1$, so that Proposition 4.6 applies and both items of the theorem follow from Theorem 4.5. □

COROLLARY 4.7. *Let X be a projective irreducible holomorphic symplectic variety of $K3^{[n]}$ -type with Picard rank at least two. Then X has an ample divisor ruled by primitive rational curves.*

Proof. As X is projective and has Picard rank at least two, its Picard lattice is indefinite and contains primitive elements of positive arbitrary Beauville–Bogomolov square. The same holds for ample classes. Let h be an ample divisor such that $q(h) \geq (2n - 2)^2(n - 1)$. Let C be a curve class in $H_2(X, \mathbb{Z})$ such that $C = h/\text{div}(h)$ under the map (2.7). As in the proof of Theorem 1.2 it follows that $q(C) \geq n - 1$ and Proposition 4.6 yields our claim. □

Remark 4.8. The estimate of Proposition 4.6 is definitely not sharp, indeed all primitive curves of positive square on irreducible holomorphic symplectic manifolds of $K3^{[n]}$ -type with $n \leq 7$ are deformation equivalent to the curves we construct. Indeed, by Corollary 2.8 we can suppose that our pair is $(S^{[n]}, h_S - \mu r_n)$ with $0 \leq \mu \leq n - 1$ and S is a $K3$ of genus p . This curve is constructed from a curve of class $h_S - 2gr_n$ with the eventual addition of a tail of class r_n , so that $2g \leq n$. Let us suppose that $g > p$ and $n \leq 7$. We have

$$q(h_S - 2gr_n) = 2p - 2 - 2\frac{g^2}{n - 1} \leq 2p - 2 - 2\frac{(p + 1)^2}{n - 1} \leq 2p - 2 - 2\frac{(p + 1)^2}{6}.$$

However, the last value is never positive, hence $q(h_S - 2gr_n)$ can only be negative. Analogously, for $C = h_S - (2g - 1)r_n$, we have $q(C) \leq (20p - 25 - 4p^2)/12$ with $g \geq p + 1$ and $2g \leq n$, which is again not positive.

5. Comparison with the work of Oberdieck, Shen and Yin and further results

In this section we compare our results with those of Oberdieck, Shen and Yin [OSY19, Corollary A.3]. We show that conditions (i) and (ii) in Theorem 4.5 are precisely satisfied in *all cases* where the obstruction discovered by [OSY19] does not prevent uniruled divisors covered by primitive rational curves to exist. In other words, we show that our result is sharp. We discuss how the counterexamples to the existence of uniruled divisors covered by rational curves of primitive class propagate in higher dimensions. Moreover, up to a relatively high value of the dimension ($2n = 26$), we show that the existence of uniruled divisors can nevertheless be obtained via non-primitive rational curves. Finally in some of the ‘exceptional cases’ (those where existence of uniruled divisors covered by rational curves of primitive class is excluded) we show that the codimension of the locus covered by the primitive rational curves is two.

5.1 Comparison with the work of Oberdieck, Shen and Yin

To conclude this section let us recall the condition in [OSY19, Corollary A.3].

PROPOSITION 5.1 [OSY19]. *Let β be a curve class on a manifold X of $K3^{[n]}$ -type. Then there is a uniruled divisor swept out by β if*

$$\beta^2 = -2 + \sum_i^{n-1} 2d_i - \frac{1}{2n-2} \left(\sum_i^{n-1} r_i \right)^2, \tag{5.1}$$

$$[\beta] = \pm \left[\sum r_i \right], \tag{5.2}$$

$$4d_i - r_i^2 \geq 0. \tag{5.3}$$

Here $[\beta]$ denotes the class of β seen as an element of the discriminant group $H^2(X, \mathbb{Z})^\vee / H^2(X, \mathbb{Z})$ (with a generator of square $-1/(2n-2)$, which is in the same monodromy orbit of the class r_n of exceptional lines on $S^{[n]}$ for any $K3$ surface S). The converse holds if β is irreducible.

PROPOSITION 5.2. *Let $n, g > 0$ be integers such that $2g \leq n$. Let p be an integer number. The condition $p \geq g$ for a curve of class $h_S - 2gr_n$ with $h_S^2 = 2p - 2$, on an Hilbert scheme $S^{[n]}$ is equivalent to the conditions in Proposition 5.1.*

Proof. Let us call $\beta = h_S - 2gr_n$. We have $\beta^2 = -2 + 2p - (1/(2n-2))4g^2$ and $[\beta] = [2gr_n]$ in the discriminant group. Therefore, we must have

$$\sum d_i = p, \tag{5.4}$$

$$\sum r_i = 2g. \tag{5.5}$$

If $p \geq g$, we can set $r_i = 2$ for g indices i such that $d_i \neq 0$ and set $r_i = 0$ for all the others, so that the conditions in Proposition 5.1 are satisfied. On the other hand, if $g > p$, there is at least one $r_i > 2d_i$, so that $4d_i - r_i^2 < 0$, contradicting the third item in Proposition 5.1. \square

Remark 5.3. Observe that if p, g and n are as in Proposition 5.2, then, for all $n' \geq n$, the integers p, g and n' again provide examples of primitive classes $h_S - 2gr_{n'} \in H_2(S^{[n']}, \mathbb{Z})$ which cannot rule a divisor.

Remark 5.4. Condition (ii) in Remark 1.4 is not sufficient to ensure the existence of a uniruled divisor covered by primitive rational curves. Indeed, let $n = 11$ and consider two general polarized $K3$ surfaces (S_1, h_1) and (S_2, h_2) of genus 2 and 4, respectively. One checks that the classes $C_1 := h_{S_1} - r_{11}$ on $S_1^{[11]}$ and $C_2 := h_{S_2} - 9r_{11}$ on $S_2^{[11]}$ have the same square $= 2 - 1/20$. The divisors $h_i, i = 1, 2$, such that $C_i = h_i/\text{div}(h_i)$ are $h_1 = 20h_{S_1} - \delta_{11}$ and $h_2 = 20h_{S_2} - 9\delta_{11}$. Nevertheless, they are not in the same orbit under the monodromy action. This can be seen as follows: by Markman, two classes are monodromy equivalent if and only if they have the same square and their images in the discriminant group are equal up to sign. Now the image of the first class is $[r_{11}]$ whereas that of the second is $[9r_{11}]$. Moreover, by Proposition 5.1, only the first can be the class of rational curves covering a divisor.

5.2 Uniruledness via non-primitive rational curves

Let ϵ be equal to 0 or 1. Suppose n, p and g are positive integers such that

$$\frac{(n-1+\epsilon)}{2} \geq g \geq p+1; \tag{5.6}$$

$$(2p-2)(2n-2) > (2g-\epsilon)^2. \tag{5.7}$$

Let (S, h_S) be a general polarized $K3$ surface of genus $p = p_a(h_S)$. Let $C \in H_2(S^{[n]}, \mathbb{Z})$ be a primitive curve class of the form

$$h_S - (2g - \epsilon)r_n.$$

Note that condition (5.7) is equivalent to $q(C) > 0$. By condition (5.6) we cannot apply our main result to C (and by [OSY19, Corollary A.3] there is no way to obtain a uniruled divisor covered by rational curves of class C). Nevertheless, it makes sense to ask the following.

Question 5.5. Does there exist an integer $m > 0$ such that mC is represented by rational curves covering a divisor?

In particular, as the Severi varieties of $|mh_S|$ are known to be non-empty by [Che02], it is natural to try and extend our approach to the multiple hyperplane linear system. Precisely, we can look for an integral nodal curve $C' \in |mh_S|$ of genus $g' = \lceil (2mg - m\epsilon)/2 \rceil$ and take $g' + 1$ points on C' to obtain a rational curve in $S^{[n]}$ of class mC (possibly after the union of an exceptional tail, depending on the parity of m).

The obvious necessary numerical conditions to be satisfied are

$$g' + 1 \leq n; \tag{5.8}$$

$$g' \leq p_a(mh_S) = m^2(p - 1) + 1. \tag{5.9}$$

If such an integer m exists, by applying Proposition 4.1 and the same strategy of Theorem 1.2 we would obtain the existence of uniruled divisors in the components of \mathfrak{M} (which is the union $\cup_{d>0} \mathfrak{M}_{2d}$ of the moduli spaces \mathfrak{M}_{2d} of projective irreducible holomorphic symplectic varieties of $K3^{[n]}$ -type polarized by a line bundle of degree $2d$) left out from Theorem 1.2.

Let us define the following quantities (coming from conditions (5.8) and (5.9) by distinguishing according to the parity of m):

$$m_{\max}^{\text{even}} := \frac{2(n - 1)}{2g - \epsilon}; \tag{5.10}$$

$$m_{\max}^{\text{odd}} := \frac{2n - 3}{2g - \epsilon}; \tag{5.11}$$

$$m_{\min}^{\text{even}} := \frac{g - \epsilon/2 + \sqrt{(g - \epsilon/2)^2 - 4(p - 1)}}{2(p - 1)}; \tag{5.12}$$

$$m_{\min}^{\text{odd}} := \frac{g - \epsilon/2 + \sqrt{(g - \epsilon/2)^2 - 2(p - 1)}}{2(p - 1)}. \tag{5.13}$$

From the discussion given previously, we deduce the following result.

PROPOSITION 5.6. *If there exists an integer $m > 0$ such that*

$$m_{\min}^\bullet \leq m \leq m_{\max}^\bullet, \tag{5.14}$$

then Question 5.5 has a positive answer.

To apply Proposition 5.6, one must first check that $m_{\min}^\bullet \leq m_{\max}^\bullet$. This easily follows from (5.7) when m is even or $\epsilon = 0$.

We can now show that the apparent persistence of the pathologies, observed in Remark 5.3, can be avoided by taking non-primitive curves.

PROPOSITION 5.7. *Let n, p and g as given previously. Then for all $n' \geq g + 1 + n$ there exists an integer $m > 0$ satisfying (5.14).*

TABLE 1. Primitive curve classes not ruling a divisor.

Class	p	g	ϵ	m	n
$h_S - 5r_n$	2	3	1	2	8
$h_S - 7r_n$	3	4	1	2	8
$h_S - 8r_n$	3	4	0	2	10
$h_S - 9r_n$	4	5	1	2	10
$h_S - 10r_n$	4	5	0	2	10
$h_S - 6r_n$	2	3	0	3	11
$h_S - 9r_n$	3	5	1	2	12
$h_S - 11r_n$	4	6	1	2	12
$h_S - 11r_n$	5	6	1	2	12
$h_S - 12r_n$	5	6	0	2	13

Proof. If $n' \geq n + 1$, we always have $m_{\min}^\bullet \leq m_{\max}^\bullet$. Moreover, for $n' \geq n + 1 + g$ the value of m_{\max}^\bullet increases by at least one. \square

We have shown in Remark 4.8 that we do have existence of uniruled divisors covered by primitive rational curves in sufficiently ample linear systems on irreducible holomorphic symplectic manifolds of $K3^{[n]}$ -type with $n \leq 7$.

There are cases in which condition (5.14) is easily checked to hold: for instance, for all p, g and n verifying (5.6) and (5.7) and such that $(2p - 2)|(2g - \epsilon)$. In particular, this yields the following.

THEOREM 5.8. *Let $8 \leq n \leq 13$ be an integer. Let $\mathfrak{M} = \cup_{d>0} \mathfrak{M}_{2d}$ be the union of the moduli spaces \mathfrak{M}_{2d} of projective irreducible holomorphic symplectic varieties of $K3^{[n]}$ -type polarized by a line bundle of degree $2d$. For all $(X, H) \in \mathfrak{M}$, there exists a positive integer a such that the linear system $|aH|$ contains a uniruled divisor.*

Proof. We follow the strategy of taking a curve in the multiple hyperplane system $|mH_S|$ outlined previously. In Table 1, we list the curve classes arising as exceptions to Theorem 1.2, together with the genus p of the $K3$ surface S , the values of g and ϵ , the minimal m which satisfies (5.14) and the smallest n such that $h_S - (2g - \epsilon)r_n$ does not satisfy Theorem 1.2.

The conclusion immediately follows from Table 1. \square

The first case when this strategy does not work appears for $n = 14$, by taking

$$C := h_S - 10r_{14},$$

where h_S is a polarization of genus $p_a(h_S) = 3$ and one checks that $2 < m_{\min} < m_{\max} < 3$.

The bad news is that, even asymptotically in n , there is no hope that condition (5.14) can hold as shown by the following.

Example 5.9. Take $g = \lceil (n - 1)/3 \rceil$, $p - 1 = \lceil (n - 1)/9 \rceil + 1$ and n large enough. Conditions (5.6) and (5.7) are satisfied. However, both m_{\min}^\bullet and m_{\max}^\bullet are < 3 , but $m_{\min}^\bullet \rightarrow_{n \rightarrow +\infty} 3$.

Remark 5.10. One may wonder whether a minor modification of this strategy might still lead to the existence of uniruled divisors in all cases. One possibility is to construct *different* rational curves coming from the Brill–Noether theory of nodal curves in the multiple hyperplane linear system of a general $K3$. This approach presents two difficulties. One has first to control the Brill–Noether theory of such nodal curves (which does not seem to be an easy task, knowing that already smooth curves in multiples of the hyperplane section are not Brill–Noether general).

Second, even if one disposed of a family of g_n^1 on nodal curves in $|mh_S|$ of the right dimension, the analogue of Proposition 4.1 should still hold for such a family.

5.3 Some codimension 2 coisotropic subvarieties

In this section, we look at the cases where, by [OSY19], there are no uniruled divisors ruled by primitive rational curves, and try to study the codimension of the ruled locus in this case. In particular, we have the following.

THEOREM 5.11. *Let X be a polarized irreducible holomorphic symplectic manifold of $K3^{[n]}$ -type. Let C be a curve class such that the pair (X, C) is deformation equivalent to $(S^{[n]}, h_S - (2g - 1)r_n)$, where $q(h_S) = 2g - 4$, for a certain integer $n/2 \geq g > 2$. Then X has a codimension 2 locus covered by rational surfaces ruled by a primitive curve class.*

This will follow from the following result, proven in [KLM19, Theorem 6.1]. We refer the reader to [KLM19] for the notation.

THEOREM 5.12 [KLM19, Theorem 6.1]. *Let (S, h_S) be a very general primitively polarized $K3$ surface of genus $p := p_a(h_S) \geq 2$. Let $0 \leq g \leq p$ and $n \geq 2$ be integers satisfying*

$$2(p - g) + 2 \leq \chi := g - n + 3 \leq p - g + n + 1. \tag{5.15}$$

Then on $S^{[n]}$ there exists a $(2n - 2)$ -dimensional family of rational curves of class

$$h_S - (g + n - 1)r_n, \tag{5.16}$$

which covers a subvariety birational to a $\mathbb{P}^{\chi-2(p-g)-1}$ -bundle on a holomorphic symplectic manifold of dimension $2(n + 1 + 2(p - g) - \chi)$.

This theorem applies only to finitely many deformation types of (X, h) for any dimension, but it turns out that it can be used to produce coisotropic subvarieties of codimension at least 2 in the exceptions to Theorem 1.2.

PROPOSITION 5.13. *Let (X, h) be a polarized irreducible holomorphic symplectic manifold of $K3^{[n]}$ -type and let C be the primitive curve class equal to $h/\text{div}(h)$. Suppose that the conditions in Proposition 5.1 are not satisfied. Then there exist a polarized $K3$ surface (S, h_S) of genus $p = p_a(h_S)$ and an integer $g \leq p$ such that:*

- (i) p and g satisfy conditions (5.15);
- (ii) X is deformation equivalent to $S^{[n]}$ and the class C is sent to $h_S - (g + n - 1)r_n$ by the parallel transport.

Proof. By Corollary 2.8 the pair (X, C) is deformation equivalent to $(\Sigma^{[n]}, h_\Sigma - (2\gamma - \epsilon)r_n)$, where Σ is a $K3$ surface of genus $\pi := p_a(h_\Sigma)$, γ a positive integer and $\epsilon = 0, 1$. As the conditions in Proposition 5.1 are not satisfied, by Proposition 5.2 we have $p_a(h_\Sigma) < \gamma$. By Remark 2.9 we can choose as system of representatives of $\mathbb{Z}/(2n - 2)\mathbb{Z}$, up to the action of -1 , the set $[2n - 2, 3n - 3] \cap \mathbb{N}$. Hence, we can deform to a different punctual Hilbert scheme on a polarized $K3$ surface (S, h_S) such that our curve has class $h_S - (2n - 2 + 2\gamma - \epsilon)r_n$ and the genus p of S is $p = \pi + n - 1 + 2\gamma - \epsilon$. Set $g := n - 1 + 2\gamma - \epsilon$.

It follows that $\chi = 2\gamma + 2 - \epsilon$. The conditions we must check are

$$2(p - g) + 2 \leq \chi \leq (p - g) + n + 1.$$

As $p - g = \pi < \gamma$ and $2\gamma - \epsilon \leq n - 1$, they are always satisfied and Theorem 5.12 applies, giving the desired locus of codimension $\chi - 2(p - g) - 1 \geq 2$. □

Proof of Theorem 5.11. Let C be as in the statement. The conditions of Proposition 5.1 are not satisfied, as $p_a(h_S) = g - 1$. Hence, we can apply Proposition 5.13 and obtain a new pair $(S_1^{[n]}, h_{S_1} - ((2g - 1 + n - 1) + n - 1)r_n)$ satisfying conditions (5.15). Therefore, we can apply Theorem 5.12 to $(S_1^{[n]}, h_{S_1} - ((2g - 1 + n - 1) + n - 1)r_n)$ to deduce the existence of a $(2n - 2)$ -dimensional family of rational curves covering a coisotropic subvariety. By Proposition 3.1 these rational curves deform to the initial variety X and by construction they have class equal to C . By [OSY19], the codimension of the locus covered by their deformations on X cannot be less than 2. Thus, if it is 2 on $S_1^{[n]}$ that must be the case also on a general point X' in the component of the moduli space \mathfrak{M} containing (X, C) . Observe that, in this case, we have $\chi = 2g + 1$ and the locus covered by these rational curves has therefore codimension $2g + 1 - 2g + 1 = 2$. Hence, we have a coisotropic subvariety $Z' \subset X'$ covered by rationally chain connected surfaces F' . The flat limit $Z \subset X$ of Z' is covered by the flat limits F of the RCC surfaces F' and of course F is RCC. The theorem follows. \square

Remark 5.14. For $n = 8, 9$, where the first exceptions discovered by [OSY19] appear, Theorem 5.11 applies. This is not the case for the exceptions of classes $h_S - 8r_{10}$ and $h_S - 10r_{10}$ in dimension 20 (cf. Theorem 5.8).

6. Application to 0-cycles

In what follows, we always consider Chow groups with rational coefficients. Throughout this section, let X be an irreducible holomorphic symplectic variety. If Y is a variety, let $CH_0(Y)_{\text{hom}}$ be the subgroup of $CH_0(Y)$ consisting of 0-cycles of degree 0.

DEFINITION 6.1. Let D be an irreducible divisor on X . We denote by $S_1CH_0(X)_{D,\text{hom}}$ the subgroup

$$S_1CH_0(X)_{D,\text{hom}} := \text{Im}(CH_0(D)_{\text{hom}} \rightarrow CH_0(X)),$$

of $CH_0(X)$. We denote by $S_1CH_0(X)_D$ the subgroup

$$S_1CH_0(X)_D := \text{Im}(CH_0(D) \rightarrow CH_0(X))$$

of $CH_0(X)$.

LEMMA 6.2. Let D and D' be two irreducible uniruled divisors on X and let R and R' be the general curves in the respective rulings. If $DR' \neq 0$ and $D'R \neq 0$, then $S_1CH_0(X)_D = S_1CH_0(X)_{D'}$ and $S_1CH_0(X)_{D,\text{hom}} = S_1CH_0(X)_{D',\text{hom}}$.

Proof. Let $\pi : \tilde{D} \rightarrow T$ and $\pi' : \tilde{D}' \rightarrow T'$ be rulings of varieties \tilde{D} and \tilde{D}' mapping finitely to D and D' , respectively. The curves R and R' are the images of the general fibers of π and π' respectively. As a consequence of the hypothesis, both projections in the diagram

$$\begin{array}{ccc} \pi^{-1}(\Sigma) & \longrightarrow & \Sigma := D \cap D' \longleftarrow (\pi')^{-1}(\Sigma) \\ \pi_{|\Sigma} \downarrow & & \downarrow \pi'_{|\Sigma} \\ T & & T' \end{array} \tag{6.1}$$

are surjective, which implies that

$$S_1CH_0(X)_D = \text{Im}(CH_0(\Sigma) \rightarrow CH_0(X)) = S_1CH_0(X)_{D'}$$

and

$$S_1CH_0(X)_{D,\text{hom}} = \text{Im}(CH_0(\Sigma)_{\text{hom}} \rightarrow CH_0(X)) = S_1CH_0(X)_{D',\text{hom}}. \quad \square$$

Proof of Theorem 1.7. We give the proof for $S_1CH_0(X)_D$. The proof for $S_1CH_0(X)_{D,\text{hom}}$ is exactly the same.

Applying Corollary 3.5, we may find an ample divisor H , ruled by a rational curve of class α , Poincaré dual to that of H . First note that, since H is ample, for any index i we have

$$H_i \cdot \alpha = q(H_i, H) > 0.$$

Then, by Lemma 6.2 we conclude that the groups $S_1CH_0(X)_{H_i}$ are independent of i .

Let D be an irreducible uniruled divisor on X and let R be a general curve of its ruling. Since H is ample, we may find an integer i such that $H_i R \neq 0$. Furthermore, as above, we have that $R_i D \neq 0$. By Lemma 6.2, we obtain the equality

$$S_1CH_0(X)_D = S_1CH_0(X)_{H_i},$$

which concludes the proof. □

Thanks to Theorem 1.7 we can drop the dependence on D from the notation and, in what follows, under the same hypotheses, we will simply write $S_1CH_0(X)$ and $S_1CH_0(X)_{\text{hom}}$ for the groups $S_1CH_0(X)_D$ and $S_1CH_0(X)_{D,\text{hom}}$.

PROPOSITION 6.3. *Let X be a projective holomorphic symplectic variety, and let D be an irreducible uniruled divisor on X . Suppose that X possesses an ample ruling curve R . Then*

$$S_1CH_0(X)_{\text{hom}} = D \cdot CH_1(X)_{\text{hom}} \quad \text{and} \quad S_1CH_0(X) = D \cdot CH_1(X).$$

Proof. We give the proof for $S_1CH_0(X)_{\text{hom}}$. Basic intersection theory [Ful98, chapter 6] guarantees the inclusion

$$D \cdot CH_1(X)_{\text{hom}} \subset \text{Im}(CH_0(D)_{\text{hom}} \rightarrow CH_0(X)_{\text{hom}}) = S_1CH_0(X)_{\text{hom}}.$$

To prove the other inclusion consider any irreducible uniruled component H_i of the ample divisor H ruled by R . By Theorem 1.7 we have $S_1CH_0(X)_{D,\text{hom}} = S_1CH_0(X)_{H_{i_0},\text{hom}}$. Notice, moreover, that by the hypothesis,

$$D \cdot R = D \cdot \lambda H^\vee = \lambda q(D, H) \neq 0 \text{ for some } \lambda \neq 0. \tag{6.2}$$

Consider

$$Z := \sum n_k x_k \in S_1CH_0(X)_{H_{i_0},\text{hom}},$$

where the x_k lie in H_{i_0} . For each x_k , let R_{x_k} be a curve in the ruling of H_i containing it. Then, by (6.2), there exists a rational number $\mu > 0$, independent of k , such that

$$x_k = \mu D \cdot R_{x_k}.$$

Hence

$$Z = \sum n_i x_i = \mu D \cdot \left(\sum n_k R_{x_k} \right)$$

holds in $CH_0(D)$, from which we see that

$$S_1CH_0(X)_{D,\text{hom}} = S_1CH_0(X)_{H_{i_0},\text{hom}} \subset D \cdot CH_1(X)_{\text{hom}}. \tag{□}$$

Proof of Theorem 1.8. Write $L = \sum m_i D_i$, where D_i is irreducible and uniruled for all i . We have

$$L \cdot CH_1(X)_{\text{hom}} \subset \sum D_i \cdot CH_1(X)_{\text{hom}} = S_1 CH_0(X)_{\text{hom}},$$

where the last equality holds thanks to Proposition 6.3 (note that we do not automatically have the equality, because some of the m_i may be negative).

To prove the other inclusion we argue as in Proposition 6.3. We take an irreducible uniruled divisor D such that $L \cdot R_D \neq 0$, where R_D is a curve in the ruling of D . Such a divisor exists by the hypothesis, as we can take any irreducible component of the divisor H ruled by the ample curve R . Let $Z := \sum n_i x_i \in \text{Im}(CH_0(D)_{\text{hom}} \rightarrow CH_0(X)) = S_1 CH_0(X)$. Then the equality

$$\sum n_i x_i = \lambda L \cdot \left(\sum n_i D_{x_i} \right),$$

holds in $CH_0(X)$ for some rational number λ , hence

$$S_1 CH_0(X)_D \subset L \cdot CH_1(X)_{\text{hom}}. \quad \square$$

7. Some open questions

We briefly discuss some questions raised by our results. Theorem 1.2 suggests a natural extension to general projective holomorphic symplectic varieties in the following way.

Question 7.1. Let X be a projective holomorphic symplectic variety of dimension $2n$, and let k be an integer between 0 and n . Does there exist a subscheme Y_k of X of pure dimension $2n - k$ such that its 0-cycles are supported in dimension $2n - 2k$?

This question has been put into a larger perspective by Voisin, in [Voil6], as a key step towards the construction of a multiplicative splitting in the Chow group. In view of our constructions, it seems natural to hope for a positive answer for Question 7.1. Note that this is the case if X is of the form $S^{[n]}$ for some K3 surface S as follows by taking Y to be the closure in $S^{[n]}$ of the locus of points $s_1 + \dots + s_n$, where the s_i are distinct points of S , k of which lie on a given rational curve of S .

It would be interesting to refine Question 7.1 to specify the expected cohomology classes of the subschemes Y_k .

The particular case of middle-dimensional subschemes seems of special interest in view of the study of rational equivalence on holomorphic symplectic varieties.

Question 7.2. Let X be a projective holomorphic symplectic variety of dimension $2n$. Does there exist a rationally connected subvariety Y of X such that Y has dimension n and non-zero self-intersection ?

A positive answer to Question 7.2 would lead to the existence of a canonical 0-cycle of degree 1 on X , as in the case of K3 surfaces. This raises the following question.

Question 7.3. Assume that Question 7.2 has a positive answer for X and let y be any point of Y . Let H_1, \dots, H_r be divisors on X , and let k_1, \dots, k_n be non-negative integers such that $r + \sum_i 2ik_i = 2n$.

Do we have

$$H_1 \cdots H_r \cdot c_2(X)^{k_1} \cdots c_{2n}(X)^{k_n} = \text{deg}(H_1 \cdots H_r \cdot c_2(X)^{k_1} \cdots c_{2n}(X)^{k_n}) \cdot y \quad (7.1)$$

inside $CH_0(X)$?

The existence of a degree 1 0-cycle c_X verifying the equality (7.1) is a consequence of the Beauville conjecture. We wonder whether such a 0-cycle can be realized in a geometrically meaningful way as a point on a rationally connected half-dimensional subvariety.

Even in the case of a general polarized fourfold of $K3^{[2]}$ -type, we do not know the answer to the preceding questions.

Finally, Question 7.1 raises a counting problem as in the case of the Yau–Zaslow conjecture for rational curves on $K3$ surfaces [YZ96], which was solved in [KMPS10]. We do not know of a precise formulation for this question.

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CONFLICTS OF INTEREST

None.

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