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A THEOREM ON COMMUTATIVITY OF SEMI-PRIME RINGS

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The following Theorem is proved: Let R be a semi-prime ring in which either $(xy)^n - x^n y^n$ or $(xy)^n - y^n x^n$ is central, for all x,y in R where $n > 1$ is a fixed integer. Then R is commutative.

1. Introduction.

A theorem of Herstein [6] states that a ring R satisfying the identity $(xy)^n = x^n y^n$, where $n > 1$ is a fixed positive integer, must have nil commutator ideal. Later Awtar [3] and Abu-Khuzam [1] established commutativity of the rings satisfying the above identity imposing the torsion conditions on the additive group R^+. In this direction Bell [5] proved that if R is an n-torsion free ring with identity 1 and satisfies the two identities $(xy)^n = x^n y^n$ and $(xy)^{n+1} = x^{n+1} y^{n+1}$, then R is commutative. Recently Abu-Khuzam [2] extended the mentioned results as follows: "If R is a semi-prime ring in which, for each x in R, there exists an integer $n = n(x) > 1$ such that $(xy)^n = x^n y^n$, for all y in R, then R is commutative."

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Our present aim is to generalize the above result. In fact we prove the following:

THEOREM: *Let* $n > 1$ *be a fixed positive integer and* R *be a semi-prime ring which satisfies one of the following polynomial identities:*

(Z_1) *For all* x,y,z *in* R, $[(xy)^n - x^n y^n, z] = 0$,

(Z_2) *For all* x,y,z *in* R, $[(xy)^n - y^n x^n, z] = 0$.

Then R *is commutative.*

Throughout the paper R denotes an associative ring R and for all x,y in R, $[x,y] = xy - yx$.

2. The following lemma is due to Bell [4]:

LEMMA 2.1. *Let* R *be a ring satisfying an identity* $q(X) = 0$, *where* $q(X)$ *is a polynomial in a finite number of non-commuting indeterminates, its coefficients being integers with highest common factor* 1 . *If there exists no prime* p *for which the ring of* 2×2 *matrices over* $GF(p)$ *satisfies* $q(X) = 0$, *then* R *has a nilcommutator ideal and the nilpotent elements of* R *form an ideal.*

LEMMA 2.2. *Let* R *be a prime ring satisfying the hypothesis of the theorem. Then* R *has no nonzero nilpotent element.*

Proof. Let x be an element of R such that $x^2 = 0$. Using the hypothesis (Z_1) or (Z_2) of the Theorem, we get $(xy)^n z = z(xy)^n$, for all $y, z \in R$. With $z = x$, we get $(xy)^n x = 0$, that is $(xy)^{n+1} = 0$, for all y in R . Whence it follows that xR is a right ideal of R in which $t^{n+1} = 0$, for each $t \in xR$. Thus $xR = (0)$, by lemma 1.1 of [7] . This implies that $x = 0$, since R is prime.

Proof of the Theorem. We shall prove the result for rings satisfying (Z_1) . In the other case one can get the result by proceeding on the same lines. Since R is semi-prime satisfying the identity $q(x,y,z) = (xy)^n z - x^n y^n z - z(xy)^n + zx^n y^n = 0$, then it is isomorphic to a subdirect-sum of prime rings R_α each of which as a homomorphic image of R

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satisfies the hypothesis placed on R . Hence we can assume that R is a prime ring satisfying $q(x,y,z) = (xy)^n z - x^n y^n z - z(xy)^n + zx^n y^n = 0$, which is a polynomial identity with co-prime integral coefficients. Now if we consider $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we find that no 2×2 matrix ring over $GF(p)$, p a prime, satisfies the identity. Hence by Lemma 2.1, R has a nilcommutator ideal. But by Lemma 2.2, R has no non-zero nilpotent elements. Thus the commutator ideal is zero and R is commutative.

The following example shows that the above theorem is not true for arbitrary rings:

EXAMPLE. Let D be a division ring and

$$A_K = \{(a_{ij}) \in D_K | a_{ij} = 0 (i \geq j)\} \qquad K > 2 .$$

Then A_3 is a noncommutative nilpotent ring of index 3, which is not semi-prime, satisfies the identities in the hypothesis of the theorem.

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