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MARTIN BOUNDARIES OF CARTESIAN PRODUCTS OF MARKOV CHAINS

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1. Introduction

Let P and Q be the stochastic transition operators of two time-homogeneous, irreducible Markov chains with countable, discrete state spaces X and Y, respectively. On the Cartesian product $Z = X \times Y$, define a transition operator of the form $R_a = a \cdot P + (1 - a) \cdot Q$, $0 < a < 1$, where P is considered to act on the first variable and Q on the second. The principal purpose of this paper is to describe the minimal Martin boundary of R_a (consisting of the minimal positive eigenfunctions of R_a with respect to some eigenvalue t, also called t-harmonic functions) in terms of the minimal Martin boundaries of P and Q.

The necessary preliminaries are provided in §2. Our main goal is achieved in §3: Theorem 3.2 shows that every minimal t-harmonic function for R_a on Z splits as a product of a minimal r-harmonic function for P on X and a minimal s-harmonic function for Q on Y, where $a \cdot r + (1 - a) \cdot s = t$. This description is completed in Theorem 3.3, where we prove that all such products give rise to minimal t-harmonic functions for R_a on Z.

Theorem 3.2 is related with the work of Molchanov [M1], [M2], who proved an analogous statement for the tensor product $P \otimes Q$ instead of R_a. Molchanov's method (which uses the associated space-time chains) cannot be adapted to our situation, while the converse adaptation in rather easy. Theorem 3.3 (whose analogue is not given in [M1], [M2]) also applies to $P \otimes Q$, thus completing the theorem of Molchanov. For further comments and variations see §3.

We are then interested in describing the topology of the minimal (or even the full) Martin boundary of R_a in terms of the corresponding topologies for P and Q. In general, this leads to a difficulty: we have to use the Martin boundaries of P (and of Q) for different eigenvalues, but we do not know how these boundaries are related to each other. This has led us to introduce the notion of stability of the

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Martin boundary (Definition 2.4): the Martin boundary is *stable* if the Martin compactification does not depend on the eigenvalue (with a possible exception at the critical eigenvalue) and that the Martin kernels are jointly continuous with respect to space variable and eigenvalue. No example is known where the contrary holds. For a detailed discussion, see [PW]. If both P and Q have finite range and stable boundaries, then the topology of the minimal Martin boundary of R_a can be simply described as a product topology (Theorem 4.1).

Finally, we attempt to obtain some evidence about the structure of the complete Martin compactification for R_a. In particular, we show that in a large class of cases (random walks on Cartesian products of nonamenable groups), there have to be nonminimal t-harmonic functions in the Martin compactification for R_a with respect to every eigenvalue (Theorem 4.3), even if the minimal Martin boundary is closed in the Martin compactification.

After finishing the first version of this paper, we were informed that in the continuous setting of Potential Theory on Riemannian manifolds, an analogue of our Theorem 3.3 (the product of minimal harmonic functions is minimal harmonic on the Cartesian product) has been proved by Freire [Fr] and Taylor [Ta].

2. Notation, preliminaries

Let X be a countable, discrete *state space*. A *stochastic transition operator* on X is given by a matrix $P = (p(x, x'))_{x,x' \in X}$ with nonnegative entries and row sums one. It gives rise to a time-homogeneous Markov chain $\mathscr{X}_n$, $n = 0,1,2,\dots$, on X, such that the entries of P are the one-step transition probabilities. If $x, x' \in X$ and $n \geq 0$, then the (x, x')-entry of the matrix power P^n is

$$p^{(n)}(x, x') = \Pr[\mathscr{X}_n = x' \mid \mathscr{X}_0 = x].$$

($P^0 = I$, the identity matrix). We shall always assume that P is *irreducible:* for every $x, x' \in X$, we have $p^{(n)}(x, x') > 0$ for some $n \geq 0$. On real valued functions f on X, P acts by

$$Pf(x) = \sum_{x'} p(x, x') f(x')$$

whenever this sum converges for every $x \in X$. A *t-harmonic function* is an eigenfunction h of P with respect to eigenvalue t: $Ph = t \cdot h$ ($t \in \boldsymbol{R}$). The linear space of t-harmonic functions is denoted by $\mathscr{H}(P, t)$. In particular, we shall be interested in the cone $\mathscr{H}^+(P, t)$ of positive t-harmonic functions. The cone of positive *t-superharmonic* functions is

$$\mathcal{S}^+(P, t) = \{f : X \to R^+ \mid Pf \leq t \cdot f\}.$$

Consider the *convergence norm* (sometimes also called spectral radius)

$$\rho(P) = \limsup_{n\to\infty} p^{(n)}(x, x')^{1/n}.$$

By irreducibility, $\rho(P)$ is independent of x and x'. We say that P has *finite range* if $\{x' \mid p(x, x') > 0\}$ is finite for every $x \in X$.

THEOREM 2.1([Pr]). *$\mathcal{S}^+(P, t)$ is nonvoid if and only if $t \geq \rho(P)$. In particular, $\mathcal{H}^+(P, t)$ is nonvoid only if $t \geq \rho(P)$. If X is infinite and P has finite range then $\mathcal{H}^+(P, t)$ is nonvoid if and only if $t \geq \rho(P)$.*

Let $x_0 \in X$ be a reference point. A function $h \in \mathcal{H}^+(P, t)$ is called *minimal* or *extremal* if $h(x_0) = 1$ and, whenever $0 \leq h_1 \leq h$ and $h_1 \in \mathcal{H}(P, t)$, then h_1/h is constant. The set of minimal t-harmonic is denoted by $\mathcal{E}(P, t)$. The positive t-harmonic functions are determined by the minimal ones. To state this more precisely, we shall recall the construction of the *Martin boundary*.

In addition to the n-step transition probabilities $p^{(n)}(x, x')$, define

$$f^{(n)}(x, x') = \Pr[\mathcal{X}_n = x' ; \mathcal{X}_i \neq x' \text{ for } 0 \leq i < n \mid \mathcal{X}_0 = x],$$
$$f^{(0)}(x, x') = 0$$

and

$$F(x, x' \mid t) = \sum_{n=0}^{\infty} f^{(n)}(x, x') \frac{1}{t^n}, \quad t \geq \rho(P).$$

The *Green kernel* is

$$G(x, x' \mid t) = \sum_{n=0}^{\infty} p^{(n)}(x, x') \frac{1}{t^{n+1}}.$$

It converges for real $t > \rho(P)$. Furthermore, by [Ve], for $t = \rho(P)$

$$\begin{aligned} &\text{either} \quad G(x, x' \mid \rho(P)) < \infty \;\; \forall\; x, x' \in X \\ &\text{or} \quad\;\;\, G(x, x' \mid \rho(P)) = \infty \;\; \forall\; x, x' \in X. \end{aligned}$$

P is called *$\rho(P)$-transient* in the first case and *$\rho(P)$-recurrent* in the second. The following is essentially due to [Pr], see also [PW].

THEOREM 2.2. *Assume that P is $\rho(P)$-recurrent.*

(a) *There is precisely one function h in $\mathcal{E}(P, \rho(P))$, and every function in $\mathcal{S}^+(P, \rho(P))$ is a constant multiple of h.*

(b) *$F(x, x' \mid \rho(P))$ is finite for all $x, x' \in X$, and*

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The *Martin kernel* is

$$K(x, x' \mid t) = \frac{F(x, x' \mid t)}{F(x_0, x' \mid t)}, \quad t \geq \rho(P).$$

Recall the formula $G(x, x' \mid t) = F(x, x' \mid t) G(x', x' \mid t)$. Hence, if $t > \rho(P)$ or if $t = \rho(P)$ and P is $\rho(P)$-transient, then $K(x, x' \mid t) = G(x, x' \mid t) / G(x_0, x' \mid t)$.

The *Martin compactification* $\hat{X}(P, t)$ of X with respect to P and $t \geq \rho(P)$ is uniquely determined up to homomorphism by

(I) $\hat{X}(P, t)$ is compact, and X is discrete, dense and open,

(II) $K(\cdot, \cdot \mid t)$ extends continuously to $X \times \hat{X}(P, t)$, and

(III) the extended kernels (also denoted by $K(\cdot, \cdot \mid t)$) separate the points of the *Martin boundary* $\mathcal{M}(P, t) = \hat{X}(P, t) \backslash X$.

It is known [Do], [Hu], [KSK] that every minimal t-harmonic function is of the form $K(\cdot, \xi \mid t)$, where $\xi \in \mathcal{M}(P, t)$. We identify $\mathcal{E}(P, t)$ with the corresponding (Borel) set of points in the Martin boundary.

THEOREM 2.3 ([Do], [Hu]). *Every $h \in \mathcal{H}^+(P, t)$ has a unique integral representation*

$$h = \int_{\mathcal{M}(P,t)} K(\cdot, \xi \mid t)\, \nu^h(d\xi),$$

where ν^h is a nonnegative Borel measure on $\mathcal{M}(P, t)$ satisfying

$$\nu^h(\mathcal{M}(P, t) \backslash \mathcal{E}(P, t)) = 0.$$

We remark that in our definition of the Martin compactification, $\hat{X}(P, \rho(P))$ is the one point-compactification of X in the $\rho(P)$-recurrent case. This is not the recurrent Martin boundary, as defined in [KSK, Ch. 11]. In the $\rho(P)$-recurrent case we have in particular by Theorem 2.2 (b)

$$K(x, x' \mid \rho(P)) = F(x, x_0 \mid \rho(P)) = h(x) \ \forall\, x, x' \in X,$$

where h is the unique function in $\mathcal{E}(P, \rho(P))$.

In studying the topology of the Martin boundary of a Cartesian product, the following notion of boundary stability will be useful. It has been introduced by the authors in [PW].

DEFINITION 2.4. We say that (X, P) has *stable boundary* if the following conditions are satisfied

(1) For $t_1, t_2 > \rho(P)$, id_X extends to a homeomorphism $\hat{X}(P, t_1) \to \hat{X}(P, t_2)$, (That is, $K(\cdot, x_n \mid t_1)$ converges pointwise on X as $n \to \infty$ if and only if $K(\cdot, x_n \mid t_2)$ converges). In this case, we write $\mathcal{M}(P) = \mathcal{M}(P, t)$, $t > \rho(P)$.

(2) For $t > \rho(P)$, id_X extends to a continuous surjection $\tau : \hat{X}(P, t) \to \hat{X}(P, \rho(P))$. We write $K(\cdot, \xi \mid \rho(P)) = K(\cdot, \tau(\xi) \mid \rho(P))$, if $\xi \in \mathcal{M}(P)$.

(3) The map $(\xi, t) \mapsto K(x, \xi \mid t)$ is jointly continuous on $\mathcal{M}(P) \times [\rho(P), \infty)$.

In addition, we say that (X, P) has *strictly stable boundary* if (1) holds for all $t_1, t_2 \geq \rho(P)$ (or, in other words, τ in (2) is a homeomorphism).

For all irreducible Markov chains with finite range and infinite state space whose boundaries are known, there is at least strong evidence of stability. For a more detailed discussion and examples, see [PW].

3. Cartesian products

We now consider two state spaces X and Y, equipped with irreducible stochastic transition operators P and Q, respectively. We consider the direct product $Z = X \times Y$. If $x \in X$ and $y \in Y$ then we write xy for the resulting pair in Z. P and Q act on functions $f : Z \to \mathbf{R}$ by

$$Pf(xy) = \sum_{x'} p(x, x') f(x'y)$$

$$Qf(xy) = \sum_{y'} q(y, y') f(xy').$$

Now we choose and fix a, $0 < a < 1$, and define the transition operator

$$R = R_a = a \cdot P + (1 - a) \cdot Q$$

on Z. (More precisely, $R = a \cdot P \otimes J + (1 - a) \cdot I \otimes Q$, where I and J are the identity operators on X and Y, respectively, and $\otimes$ denotes tensor product.) Thus R is an irreducible stochastic transition operator on Z.

We want to determine the minimal t-harmonic functions for R on Z in terms of those for P and Q on X and Y, respectively. Molchanov [M1], [M2] has studied (a part of) this problem for the operator $P \otimes Q$ on Z, assuming conditions of "uniform aperiodicity". We remark that, in view of structural considerations concerning X, Y and Z, it is usually more natural to consider an operator of the form R_a on Z rather than $P \otimes Q$. (For example, the simple random walk on $\mathbf{Z}^{d_1+d_2}$ can be written in this way in terms of the simple random walks on $\mathbf{Z}^{d_1}$ and $\mathbf{Z}^{d_2}$.)

If f and g are real valued functions on X and Y, respectively, then we define

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$f \otimes g : Z \to \mathbf{R}$ by

$$f \otimes g(xy) = f(x)g(y).$$

The following is obvious.

LEMMA 3.1.

(a) *As transition operators on Z, P and Q commute.*

(b) *If $f \in \mathcal{H}(P, r)$ and $g \in \mathcal{H}(Q, s)$ then $f \in \mathcal{H}(R, t)$, where $t = a \cdot r + (1 - a) \cdot s$.*

(c) $\rho(R) = a \cdot \rho(P) + (1 - a) \cdot \rho(Q)$.

Now let x_0 and y_0 be the reference points in X and Y, respectively, and choose $z_0 = x_0 y_0$ as the reference point in Z. The Martin kernel for P is denoted by $K_P(\cdot, \cdot \mid r)$, $r \geq \rho(P)$. For $t \geq \rho(R)$, consider the segment

$$I(t) = \{(r, s) \in \mathbf{R}^2 \mid r \geq \rho(P),\ s \geq \rho(Q),\ a \cdot r + (1 - a) \cdot s = t\}.$$

THEOREM 3.2. *If $h \in \mathcal{E}(R, t)$, $t \geq \rho(R)$, then there are $(r, s) \in I(t)$, $f \in \mathcal{E}(P, r)$ and $g \in \mathcal{E}(Q, s)$ such that $h = f \otimes g$.*

Proof. We proceed in several steps.

CLAIM 1. There is $(r, s) \in I(t)$ such that, on Z, $Ph = r \cdot h$ and $Qh = s \cdot h$.

Proof of Claim 1. First, by Lemma 3.1 (a),

$$RPh = PRh = P(t \cdot h) = t \cdot Ph,$$

and $Ph \in \mathcal{H}(R, t)$. Second,

$$h = \frac{1}{t} \cdot Rh \geq \frac{a}{t} \cdot Ph.$$

As h is minimal, we must have $Ph = r \cdot h$ for some $r > 0$. In the same way, $Qh = s \cdot h$ for some $s > 0$. By Theorem 2.1, $(r, s) \in I(t)$. This proves Claim 1.

Therefore on X one has

$$h(\cdot y) \in \mathcal{H}^+(P, r) \ \forall\, y \in Y,$$

and by Theorem 2.3 there is a unique Borel measure ν^y on $\mathcal{M}(P, r)$ such that

$$\nu^y(\mathcal{M}(P, r) \backslash \mathcal{E}(P, r)) = 0$$

for all $y \in Y$ and

$$h(xy) = \int_{\mathcal{M}(P,r)} K_p(x, \xi \mid r) \nu^y(d\xi) \ \ \forall \ xy \in Z.$$

By Claim 1,

$$\int_{\mathcal{M}(P,r)} K_p(x, \xi \mid r) s \cdot \nu^y(d\xi) = s \cdot h(xy) = Qh(xy)$$

$$= \int_{\mathcal{M}(P,r)} K_p(x, \xi \mid r) \sum_{y' \in Y} q(y, y') \nu^{y'}(d\xi).$$

By uniqueness of the representing measure,

$$\sum_{y' \in y} q(y, y') \nu^{y'} = s \cdot \nu^y \ \ \forall y \in Y.$$

If $y, y' \in Y$, then, by irreducibility, $q^{(k)}(y, y') > 0$ for some k. But

$$q^{(k)}(y, y') \nu^{y'} \leq s^k \cdot \nu^y.$$

In particular, all the ν^y are mutually absolutely continuous and have the same support

$$S = \operatorname{supp}(\nu^y) \ \ \forall \ y \in Y.$$

CLAIM 2. S has only one point.

Proof of Claim 2. Suppose the contrary. Then there are two closed disjoint subsets $A, B \subset S$ such that

$$\nu^y(A) > 0 \text{ and } \nu^y(B) > 0$$

for some and hence all $y \in Y$. Consider

$$h_A(xy) = \int_A K_P(x, \xi \mid r) \nu^y(d\xi) \text{ and } h_B(xy) = \int_B K_P(x, \xi \mid r) \nu^y(d\xi).$$

Then $h_A, h_B \in \mathcal{H}(R, t)$, $h_A > 0$, $h_B > 0$ and $h_A + b_B \leq h$. By minimality of h there are constants $c_A, c_B > 0$ such that $h = c_A \cdot h_A = c_B \cdot h_B$. Again by uniqueness of the representing measure

$$\nu^y = c_A \cdot \nu^y \big|_A = c_B \cdot \nu^y \big|_B,$$

which contradicts the fact that A and B are disjoint. This proves Claim 2.

Thus $S = \{\xi\}$ with $\xi \in \mathcal{M}(P, r)$, and

$$\nu^y = g(y) \cdot \delta_\xi, \text{ where } g(y) > 0.$$

Furthermore it must be

$$f = K_P(\cdot, \xi \mid r) \in \mathscr{E}(P, r)$$

Therefore

$$h(xy) = f(x)g(y) \quad \forall\ xy \in Z,$$

and it follows that $g \in \mathscr{H}^+(Q,s)$, $g(y_0) = 1$. Suppose that $g \notin \mathscr{E}(Q, s)$. Then $g = \lambda \cdot g_1 + (1-\lambda) \cdot g_2$ with $0 < \lambda < 1$, $g_1 \in \mathscr{H}(Q,s)$, $g_i(y_0) = 1$ and $g_1 \neq g_2$. But then

$$h = \lambda \cdot f \otimes g_1 + (1-\lambda) \cdot f \otimes g_2,$$

is a proper convex combination of two functions in $\mathscr{H}(R, t)$ by Lemma 3.1 (b). This contradicts the minimality of h. □

We now state the main theorem of this section. It involves a few technical points arising from the fact that a priori we do not know whether certain sets–over which we need to perform an integration–are Borel sets in $\mathscr{M}(R, t)$. (It seems that [M1] does not discuss an analogous problem in a somewhat different situation.)

Theorem 3.3.

$$\mathscr{E}(R,t) = \bigcup_{(r,s) \in I(t)} \mathscr{E}(P, r) \otimes \mathscr{E}(Q,s),$$

that is, products of minimal functions for P *and* Q *respectively are minimal for* R, *and conversely.*

Proof. Theorem 3.2 says that $\mathscr{E}(R,t)$ is contained in the set on the right. Using this fact, we prove the converse inclusion. Let

$$h = f \otimes g, \text{ where } f \in \mathscr{E}(P, r),\ g \in \mathscr{E}(Q, s) \text{ and } (r, s) \in I(t).$$

Then

$$h(xy) = \int_{\mathscr{M}(R,t)} K_R(xy, \zeta \mid t)\, \nu(d\zeta)$$

for a unique probability measure ν with $\nu(\mathscr{M}(R, t) \backslash \mathscr{E}(R, t)) = 0$. Recall that the topology of $\mathscr{M}(R, t)$ is the one induced by pointwise convergence of t-superharmonic functions. By $(\mathscr{E}(P, r) \otimes \mathscr{E}(Q, s))^-$ we denote the closure in this topology.

Claim 3. $\operatorname{supp}(\nu) \subset \mathscr{M}(R, t) \cap (\mathscr{E}(P, r) \otimes \mathscr{E}(Q, s))^-$.

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Proof of Claim 3. For $\varepsilon > 0$, the sets

$$A_\varepsilon = \{u : Z \to \mathbf{R}^+ \mid Ru \le t \cdot u,\ Pu \le (r - \varepsilon) \cdot u\} \cap \mathscr{E}(R, t)$$

and

$$B_\varepsilon = \{u : Z \to \mathbf{R}^+ \mid Ru \le t \cdot u,\ Qu \le (s - \varepsilon) \cdot u\} \cap \mathscr{E}(R, t)$$

are Borel sets with respect to pointwise convergence. Indeed, $\mathscr{E}(R, t)$ is a Borel set by [KSK, Proposition 10.38], and A_ε, B_ε are formed by intersecting $\mathscr{E}(R, t)$ with closed sets. Again, we identify A_ε and B_ε with the corresponding (Borel) subsets of $\mathscr{M}(R, t)$. Let

$$h'(xy) = \int_{A_\varepsilon} K_R(xy, \zeta \mid t) \nu(d\zeta).$$

By Theorem 3.2, in A_ε one has

$$K_R(\cdot, \zeta \mid t) \in \mathscr{E}(P, r') \otimes \mathscr{E}(Q, s'),$$

where

$$s' = \frac{t - ar'}{1 - a} \ge s_\varepsilon \overset{\text{def}}{=} \frac{t - a(r - \varepsilon)}{1 - a} > s.$$

Hence

$$s^n h = Q^n h \ge Q^n h' \ge s_\varepsilon^n h' \quad \forall\ n \in \mathbf{N},$$

and it must be $h' \equiv 0$. Thus $\nu(A_\varepsilon) = 0$ and similarly $\nu(B_\varepsilon) = 0$. Now the set $(\mathscr{E}(P, r) \otimes \mathscr{E}(Q, s))^-$ has empty intersection with $\mathscr{E}(P, r') \otimes \mathscr{E}(Q, s')$ if $(r, s) \ne (r', s')) \in I(t)$. Again by Theorem 3.2,

$$\mathscr{E}(R,t) \setminus (\mathscr{E}(P, r) \otimes \mathscr{E}(Q, s))^- = \bigcup_{n=1}^{\infty} A_{1/n} \cup B_{1/n},$$

and

$$\nu(\mathscr{M}(R, t) \setminus (\mathscr{E}(P, r) \otimes \mathscr{E}(Q, s))^-) = 0.$$

This proves Claim 3.

Note that we do not yet know if $\mathscr{E}(P, r) \otimes \mathscr{E}(Q, s)$ itself is a Borel set in $\mathscr{M}(R, t)$. However,

$$S = \mathscr{E}(R, t) \cap (\mathscr{E}(P, r) \otimes \mathscr{E}(Q, s))^-$$

is a Borel set, and $S \subset \mathscr{E}(P, r) \otimes \mathscr{E}(Q, s)$ by Theorem 3.2. Furthermore, by Claim 3, $\operatorname{supp}(\nu) \subset S^-$ and $\nu(\mathscr{M}(R, t) \setminus S) = 0$. Hence

$$h(xy) = \int_S K_R(xy, \zeta \mid t)\nu(d\zeta).$$

Suppose that $\operatorname{supp}(\nu)$ has more than one element. Then we can decompose S into two disjoint Borel sets B_1 and B_2 with positive ν-mass, such that $S \backslash B_i^- \neq \emptyset$. Setting

$$h_i(xy) = \frac{1}{\nu(B_i)} \int_{B_i} K_R(xy, \zeta \mid t)\nu(d\zeta),\ i = 1,2,$$

one gets

$$Ph_i = r \cdot h_i,\ i = 1,2, \text{ and } h = \nu(B_1)\cdot h_1 + \nu(B_2)\cdot h_2,$$

a proper convex combination because of uniqueness of the representing measure (Theorem 2.3). Then $h_i(\cdot y)/g(y) \in \mathscr{H}^+(P, r)$ and

$$f(x) = \nu(B_1)\,\frac{h_1(xy)}{g(y)} + \nu(B_2)\,\frac{h_2(xy)}{g(y)}.$$

By minimality of f it must be

$$\frac{h_i(xy)}{g(y)} = c_i(y) f(x) \quad \forall\ xy \in Z,$$

where $c_i(y) > 0$, $i = 1,2$. In the same way, by minimality of g,

$$\frac{h_i(xy)}{f(x)} = d_i(x) g(y) \quad \forall\ xy \in Z,$$

where $d_i(x) > 0$, $i = 1,2$. Comparing the last two formulas we get

$$c_i(y) = d_i(x) = d_i(x_0) = 1 \quad \forall\ xy \in Z.$$

Thsu $h_1 = h_2$, a contradiction, and S has only one point. □

A posteriori we get the following from Theorem 3.3 and its proof.

COROLLARY 3.4. (a) *If $(r, s) \in I(t)$, then $\mathscr{E}(P, r) \otimes \mathscr{E}(Q, s)$ is a Borel set in $\mathscr{M}(R, t)$.*

(b) *If $h \in \mathscr{H}^+(P, r) \otimes \mathscr{H}^+(Q, s)$, then its unique representing measure on $\mathscr{M}(R,t)$ (in the sense of Theorem 2.3) gives zero mass to the complement of $\mathscr{E}(P, r) \otimes \mathscr{E}(Q, s)$.*

Note that $I(t)$ collapses to one point if $t = \rho(R)$. Thus $\mathscr{E}(R, \rho(R)) =$

$\mathcal{E}(P, \rho(P)) \otimes \mathcal{E}(Q, \rho(Q))$.

We finish this section by discussing the relation between these theorems and Molchanov's result and by outlining some alternative statements.

1.) The method of [M1], [M2] does not apply to R_a because, contrary to $P \otimes Q$, the Green kernel of the associated space-time chain does not split as a product of the Green kernels associated with the space-time chains of the factors.

2.) Theorems 3.2 and 3.3 apply immediately to the case when R is a transition operator on Z of the form

$$R = a \cdot P + b \cdot Q + \varphi(P, Q),$$

where $a, b > 0$ and φ is a polynomial with nonnegative coefficients. Indeed, if $I(t)$ is replaced by the set $\{(r, s) \mid r, s > 0, a \cdot r + b \cdot s + \varphi(r, s) = t\}$, all the steps of the proofs remain unchanged (s_ε in the proof of Claim 3 has to be modified).

3.) Here is how our proof of Theorem 3.2 can be adapted to deal with $P \otimes Q$. Molchanov's [M2] "uniform aperiodicity" condition for a transition operator P says the following:

(A) The set $M = \{d \in \mathbf{N} \mid P^{m+d} \geq \lambda \cdot P^m$ for some $m \in \mathbf{N}, \lambda > 0\}$ has greatest common divisor one.

The crucial point is the proof of Claim 1; the rest goes through as it stands with the obvious modifications.

Claim 1′. If P and Q satisfy (A) and $h \in \mathcal{E}(R,t)$, where $R = P \otimes Q$, then there are $r, s > 0$ such that $rs = t$ and on Z, $Ph = r \cdot h$ and $Qh = s \cdot h$.

Proof of Claim 1′. Let $h \in \mathcal{E}(R, t)$. Choose $d \in M$, with m and λ associated as above. Then by Lemma 3.1

$$h = \frac{1}{t^{m+d}} \cdot R^{m+d} h \geq \frac{\lambda}{t^{m+d}} \cdot R^m Q^d h = \frac{\lambda}{t^d} Q^d h.$$

As $Q^d h \in \mathcal{H}^+(R,t)$, it must be $Q^d h = s(d) \cdot h$ for some $s(d) > 0$. This holds for every $d \in M$. Now we can find $d_1, \ldots, d_k \in M$, $l_1, \ldots, l_k \in \mathbf{N}$ and j, $1 \leq j \leq k$ such that $n_1 - n_2 = 1$, where $n_1 = \sum_{i=1}^{j} l_i d_i$ and $n_2 = \sum_{i=j+1}^{k} \ell_i d_i$. Thus $Q^{n_1} h = s_1 \cdot h$ and $Q^{n_2} h = s_2 \cdot h$ for some $s_1, s_2 > 0$, and

$$s_1 \cdot h = Q^{n_1} h = Q Q^{n_2} h = s_2 \cdot Qh,$$

so that $Qh = s \cdot h$ for $s = s_1/s_2$. In the same way, $Ph = r \cdot h$ for some $r > 0$, and it must be $rs = t$.

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4.) V.A. Kaimanovich has kindly pointed out to us that condition (A) implies the following weaker condition for the underlying Markov chain (compare with [Ka]):

(B) The stationary sigma-algebra coincides with the tail sigma-algebra.

In order to prove Molchanov's result for the operator $P \otimes Q$, it is sufficient to assume that (B) holds for P and Q.

5.) Finally, we remark that irreducibility is used in our proof of Theorem 3.2 at the point where we show that the measures ν^y all have the same support. With some additional effort, one can adapt the proof in order to drop the irreducibility hypothesis. However, we believe that "connectedness" (irreducibility) is the natural setting for our considerations, so that we have restricted ourselves to this assumption in order to keep the presentation more compact.

4. Cartesian products and the Martin topology

In the preceding section we have determined the minimal t-harmonic functions for the operator $R = a \cdot P + (1-a) \cdot Q$ on the Cartesian product $Z = X \times Y$, so that we know the "essential" part of $\mathcal{M}(R, t)$ in terms of P and Q. However, it is not clear if or when this gives us the whole Martin boundary, and how we can describe the Martin topology of $\mathcal{M}(R, t)$, $t \geq \rho(R)$. This turns out to be a very difficult task, and in this section we obtain only a few indications aiming at a better understanding of this question.

Throughout this section, we assume that (X, P) and (Y, Q) have stable boundaries and that R has finite range. For $t \geq \rho(R)$, we define compact spaces $\mathcal{N}(R, t)$ as follows.

(I) $\mathcal{N}(R, \rho(R)) = \mathcal{M}(R, \rho(R)) \times \mathcal{M}(Q, \rho(Q))$ with the Cartesian product topology.

(II) If $t > \rho(R)$ then set $r_0(t) = (t-(1-a)\rho(Q))/a$, $s_0(t) = (t-a\rho(P))/(1-a)$ and $I^0(t) = I(t) \setminus \{(\rho(P), s_0(t)), (r_0(t), \rho(Q))\}$. Define

$$\begin{aligned}\mathcal{N}(R, t) = (\mathcal{M}(P, \rho(P)) \times \mathcal{M}(Q)) &\cup (\mathcal{M}(P) \times \mathcal{M}(Q) \times I^0(t)) \\ &\cup (\mathcal{M}(P) \times \mathcal{M}(Q, \rho(P))),\end{aligned}$$

a disjoint union.

The topology on $\mathcal{N}(R, t)$ is induced by the product topology on $\mathcal{M}(P) \times \mathcal{M}(Q) \times I(t)$ via the map

$$\pi : \mathcal{M}(P) \times \mathcal{M}(Q) \times I(t) \to \mathcal{N}(R, t)$$

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$$\pi(\xi, \eta, r, s) = \begin{cases} (\xi, \eta, r, s), & \text{if } (r, s) \in I^{\alpha}(t); \\ (\tau_x(\xi), \eta, r, s), & \text{if } r = \rho(P) \text{ and } s = s_0(t); \\ (\xi, \tau_y(\eta), r, s), & \text{if } r = r_0(t) \text{ and } s = \rho(Q). \end{cases}$$

Here, τ_x and τ_y denote the surjections of Definition 2.4 (2) associated with (X, P) and (Y, Q), respectively. If in particular (X, P) and (Y, Q) have strictly stable boundaries, then $\mathcal{N}(R, t) = \mathcal{M}(P) \times \mathcal{M}(Q) \times I(t)$ for $t \geq \rho(R)$. Observe that with $\mathcal{N}(R, t)$ in the place of $\mathcal{M}(R, t)$, all requirements of Definition 2.4 are met. In particular $\mathcal{N}(R, t) \cong \mathcal{N}(R, t')$ for $t, t' > \rho(R)$. By Theorem 3.3, $\mathcal{E}(R, t)$ embeds naturally as a subset of $\mathcal{N}(R, t)$ for $t \geq \rho(R)$.

THEOREM 4.1. *Suppose that (X, P) and (Y, Q) have stable boundaries and finite range. Then we have the following.*

(a) *The Martin topology on $\mathcal{E}(R, t)$ is the relative topology induced by $\mathcal{N}(R, t)$.*

(b) *If all boundary functions of (X, P) and (Y, Q) are minimal, then $\mathcal{E}(R, t) \cong \mathcal{N}(R, t)$ is closed in the Martin topology.*

(c) *If (X, P) and (Y, Q) have strictly stable boundaries and all corresponding boundary functions are minimal, then $\mathcal{E}(R, t) \cong \mathcal{M}(P) \times \mathcal{M}(Q) \times I(t)$ topologically.*

By boundary functions we mean of course the Martin kernels $K_P(\cdot, \xi \mid r)$ and $K_Q(\cdot, \eta \mid s)$, where $\xi \in (P, r)$, $\eta \in \mathcal{M}(Q, s)$, $r \geq \rho(P)$ and $s \geq \rho(Q)$. Recall once more that the Martin topology is the one induced by pointwise convergence of t-superharmonic functions. In view of this fact, of Definition 2.4 and of Theorem 3.3, the proof of Theorem 4.1 is a straightforward topological exercise.

CONJECTURE. *Under our hypotheses, $\mathcal{N}(R, t)$ always embeds naturally into $\mathcal{M}(R, t)$.*

In other words, we believe that every function on Z of the form $K_P(\cdot, \xi \mid r) \otimes K_Q(\cdot, \eta \mid s)$ can be written as a Martin kernel $K_R(\cdot\ \zeta \mid t)$ on Z (even if it is nonminimal).

Vice versa, one might also be tempted to believe that $\mathcal{N}(R, t)$ will give the whole Martin boundary $\mathcal{M}(R, t)$. Our next aim is to disprove this: we exhibit a class of Cartesian products where $\mathcal{E}(R, t) \cong \mathcal{N}(R, t)$ is closed and $\mathcal{M}(R, t)$ contains nonminimal elements for every $t \geq \rho(P)$. Thus $\mathcal{M}(R, t) \setminus \mathcal{E}(R, t)$ is nonvoid and open in $\mathcal{M}(R, t)$. In particular, this disproves the suggestion of [KV] that the Martin boundary can always be obtained as the closure of the minimal (t-)harmonic functions with respect pointwise convergence of (t-)superharmonic functions. The points in $\mathcal{M}(R, t)\backslash\mathcal{E}(R, t)$ arise as limits in the Martin topology of sequences (x_1y_n) or (x_ny_1) in Z, where $n \to \infty$, while x_1 and y_1, respectively, are fixed.

LEMMA 4.2. *Let P and Q have finite range. Suppose that $x_1 \in X$ and that (y_n) is a sequence in Y such that the pointwise limit of*

$$K_R(\cdot, x_1y_n \mid t), \quad n \to \infty,$$

exists on Z and is in $\mathcal{E}(R, t)$, $t \geq \rho(R)$. Then the limit exists for every $x_2 \in X$ in the place of x_1 and is independent of x_1.

Proof. By irreducibility of P, for $x_1, x_2 \in X$ we have $p^{(k)}(x_1, x_2) > 0$ for some $k = k(x_1, x_2) > 0$. Set $C(x_1, x_2) = (a\cdot t)^k p^{(k)}(x_1, x_2)$. If $G = G_R$ denotes the Green kernel for $R = R_a$, then

$$G(xy, x_1y' \mid t) \geq C(x_1, x_2)G(xy, x_2y' \mid t)$$

for all $xy, x_1y', x_2y' \in Z$. Set $D(x_1, x_2) = C(x_1, x_2)C(x_2, x_1)$. Then we obtain for the Martin kernel $K = K_R$

$$D(x_1, x_2) \leq \frac{K(xy, x_1y_n \mid t)}{K(xy, x_2y_n \mid t)} \leq \frac{1}{D(x_1, x_2)} \quad \forall\ xy \in Z.$$

By assumption

$$\lim_{n\to\infty} K(\cdot, x_1y_n \mid t) = f \otimes g \in \mathcal{E}(R, t).$$

Now let $(x_2y_{n_k})$ be a subsequence which converges in the Martin topology. As R has finite range,

$$\lim_{k\to\infty} K(\cdot, x_2y_{n_k} \mid t) = h$$

is a t-harmonic function on Z. Then

$$D(x_1, x_2) \leq \frac{f \otimes g}{h} \leq \frac{1}{D(x_1, x_2)}.$$

By minimality of $f \otimes g$, and using the fact that $h(x_0y_0) = 1 = f(x_0)g(y_0)$, we obtain $h = f \otimes g$.

This holds for every convergent subsequence $(x_2y_{n_k})$ of (x_2y_n). The statement follows by compactness. □

Given a countable group Γ and a probability measure μ on Γ, the *random walk* n Γ with *law* μ is the Markov chain with state space $X = \Gamma$ and transition probbilities $p(x, y) = \mu(x^{-1}y)$, $x, y \in \Gamma$. "Irreducible" then means that the support f μ generates Γ as a semigroup.

Now consider two infinite, finitely generated discrete groups Ξ and Υ carying finitely supported probability measures μ and ν, respectively, each one givng rise to an irreducible random walk. On the Cartesian product $\Gamma = \Xi \times \Upsilon$, conider $\sigma = a \cdot \mu + (1 - a) \cdot \nu$ $(0 < a < 1)$. As the reference points for the corresonding Martin kernels, we choose the respective group identities x_0, y_0 and x_0y_0.)bserve that the Martin kernels satisfy a cocycle identity:

$$K_\mu(x_1, x_2^{-1} x_3 \mid t) = \frac{K_\mu(x_2x_1, x_3 \mid t)}{K_\mu(x_2x_1, x_3 \mid t)} \quad \forall x_1, x_2, x_3 \in \Xi$$

n this situation we have the following.

THEOREM 4.3. *Suppose that* (y_n) *is a sequence of distinct elements in* Υ *such hat for some* $t \geq \rho(\sigma)$,

$$\lim_{n \to \infty} K_\sigma(\cdot, x_0y_n \mid t) = f \otimes g \in \mathscr{E}(\sigma, t)$$

xists. Then

(a) $f(xx') = f(x)f(x') \quad \forall x, x' \in \Xi$;

(b) Ξ *carries an irreducible, finitely supported probability measure* $\bar{\mu}$ *which admits* *o nonconstant bounded* (1-) *harmonic functions;*

(c) *in particular,* Ξ *is an amenable group.*

Proof. By Lemma 4.2 we have

$$\lim_{n \to \infty} K_\sigma(xy, x_1y_n \mid t) = f(x)g(y) \quad \forall x, x_1 \in \Xi, y \in \Upsilon.$$

\pplying the cocycle identity to K_σ yields

$$K_\sigma(xy, x_1^{-1} y_n \mid t) = \frac{K_\sigma((x_1x)y, x_0y_n \mid t)}{K_\sigma(x_1y_0, x_0y_n \mid t)}.$$

Recall that x_0 and y_0 are the respective group identities.) In the limit,

$$f(x)g(y) = f(x_1x)g(y)/f(x_1).$$

:his proves (a).

We have $f \in \mathscr{E}(\mu, r)$ for some $r \geq \rho(\mu)$. Define a new probability measure on Ξ by

$$\bar{\mu}(x) = \mu(x) f(x)/r.$$

(In other words, for $p(x, x') = \mu(x^{-1}x')$ and $\bar{p}(x, x') = \bar{\mu}(x^{-1}x')$ we have $\bar{p}(x, x') = p(x, x') f(x')/rf(x)$.) As $f \in \mathscr{E}(\mu, r)$, a standard argument (see e.g. [KSK, Lemma 10.32]) shows that the constant function 1 is in $\mathscr{E}(\bar{\mu}, 1)$. This means that all bounded (1-) harmonic functions for $\bar{\mu}$ are constant on Ξ, and (b) is proven.

Statenent (c) now follows from [KV]. $\square$

Corollary 4.4. *If in the above situation Ξ or Υ are nonamenable, then $\mathscr{M}(\sigma, t) \setminus \mathscr{E}(\sigma, t)$ is nonvoid for every $t \geq \rho(\sigma)$.*

In particular, in view of the examples of [PW], one can choose Ξ and Υ such that one of the two is nonamenable and equip them with irreducible, finitely supported probability measures μ and ν, respectively, such that

- μ and ν have stable boundaries, and
- the boundaries have no nonminimal elements.

Then, for $\sigma = a \cdot \mu + (1-a) \cdot \nu$ on $\Gamma = \Xi \times \Upsilon$, $\mathscr{E}(\sigma, t) = \mathscr{N}(\sigma, t)$ is closed and $\mathscr{M}(\sigma, t) \setminus \mathscr{E}(\sigma, t)$ is nonvoid for every $t \geq \rho(\sigma)$.

This holds, for example, for the simple random walk on the Cartesian product of two free groups $\mathbf{F}_k$ and $\mathbf{F}_l$ with respective number of free generators $k, l \geq 1$ and not $k = l = 1$. For this case, Fatou-type boundary theory has been studied extensively by Picardello and Sjögren [PS]. Guivarc'h and Taylor [GT] have determined the Martin boundary of the Cartesian product of two or more hyperbolic with respect to the smallest positive eigenvalue. This is in some sense analogous with studying $\mathscr{M}(\sigma, p(\sigma))$ for the simple random walk on $\Gamma = \mathbf{F}_k \times \mathbf{F}_l$ and the methods of [GT] can be adapted to this situation, see also [PS]. For arbitrary $t > \rho(\sigma)$, it is possible to calculate the limits of $K_\sigma(\cdot, x_1 y_n | t)$, $n \to \infty$, on Γ, but the determination of the whole Martin boundary is a more difficult task which stands for further work.

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