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Abstract

Let k be an infinite field. Let R be the semi-local ring of a finite family of closed points on a k-smooth affine irreducible variety, let K be the fraction field of R, and let G be a reductive simple simply connected R-group scheme isotropic over R. Our Theorem 1.1 states that for any Noetherian k-algebra A the kernel of the map

$$H^1_{\text{ét}}(R \otimes_k A, G) \to H^1_{\text{ét}}(K \otimes_k A, G)$$

induced by the inclusion of R into K is trivial. Theorem 1.2 for $A = k$ and some other results of the present paper are used significantly in Fedorov and Panin [*A proof of Grothendieck–Serre conjecture on principal bundles over a semilocal regular ring containing an infinite field*, Preprint (2013), arXiv:1211.2678v2] to prove the Grothendieck–Serre's conjecture for regular semi-local rings R containing an infinite field.

1. Introduction

Recall that an R-group scheme G is called reductive (respectively, semi-simple or simple), if it is affine and smooth as an R-scheme and if, moreover, for each ring homomorphism $s : R \to \Omega(s)$ to an algebraically closed field $\Omega(s)$, its scalar extension $G_{\Omega(s)}$ is a connected reductive (respectively, semi-simple or simple) algebraic group over $\Omega(s)$. The class of reductive group schemes contains the class of semi-simple group schemes which in turn contains the class of simple group schemes. This notion of a simple R-group scheme coincides with the notion of a simple semi-simple R-group scheme from Demazure and Grothendieck [SGA3, Exp. XIX, Definition 2.7 and Exp. XXIV, 5.3]. *Throughout the present paper, R denotes an integral domain and G denotes a semi-simple R-group scheme, unless explicitly stated otherwise. All commutative rings that we consider are assumed to be Noetherian.*

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A semi-simple R-group scheme G is called *simply connected* (respectively, *adjoint*), provided that for an inclusion $s : R \hookrightarrow \Omega(s)$ of R into an algebraically closed field $\Omega(s)$ the scalar extension $G_{\Omega(s)}$ is a simply connected (respectively, adjoint) $\Omega(s)$-group scheme. This definition coincides with the one from [SGA3, Exp. XXII. Definition 4.3.3].

A well-known conjecture due to Serre and Grothendieck [Ser58, Remarque, p. 31], [Gro58, Remarque 3, pp. 26–27], and [Gro68, Remarque 1.11.a] asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme G over R the map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion of R into K, has trivial kernel. The following theorem, which is one of the main result of the present paper, asserts that for simple and simply connected isotropic group schemes over certain rings R this is indeed the case (*recall that a simple R-group scheme is called isotropic if it contains a split torus $\mathbb{G}_{m,R}$*). Actually, we prove something significantly stronger, namely the following theorem.

Theorem 1.1. *Let k be an infinite field. Let $\mathcal{O}$ be the semi-local ring of finitely many closed points on a k-smooth irreducible affine k-variety X and let K be its field of fractions. Let G be an isotropic simple simply connected group scheme over $\mathcal{O}$. Then for any Noetherian k-algebra A the map*

$$H^1_{\text{ét}}(\mathcal{O} \otimes_k A, G) \to H^1_{\text{ét}}(K \otimes_k A, G),$$

induced by the inclusion $\mathcal{O}$ into K, has trivial kernel.

In other words, under the above assumptions on $\mathcal{O}$ and G each principal G-bundle P over $\mathcal{O} \otimes_k A$ which is trivial over $K \otimes_k A$ is itself trivial. In the case $A = k$ the main result of [FP13] is much stronger, since there are no anisotropy assumptions on G there. This theorem is wrong without the isotropy assumption, as is proved in Lemma 10.1.

Theorem 1.1 extends easily to the case of sufficiently isotropic simply connected semi-simple group schemes, using the Faddeev–Shapiro lemma. However, in this generality its statement is a bit more technical and we postpone it till § 11 (see Theorem 11.1). All other results stated below extend to semi-simple simply connected group schemes as well.

Theorem 1.1 is a consequence of the following two theorems as explained in Remark 1.4 below.

Theorem 1.2. *Let k, $\mathcal{O}$, K, A be the same as in Theorem 1.1. Let G be a not necessarily isotropic simple simply connected group scheme over $\mathcal{O}$. Let $\mathcal{G}$ be a principal G-bundle over $\mathcal{O} \otimes_k A$ which is trivial over $K \otimes_k A$. Then there exists a principal G-bundle $\mathcal{G}_t$ over $\mathcal{O}[t] \otimes_k A$ and a monic polynomial $f(t) \in \mathcal{O}[t]$ such that the following hold.*

(i) *The G-bundle $\mathcal{G}_t$ is trivial over $(\mathcal{O}[t]_f) \otimes_k A$.*
(ii) *The evaluation of $\mathcal{G}_t$ at $t = 0$ coincides with the original G-bundle $\mathcal{G}$.*
(iii) *$f(1) \in \mathcal{O}$ is invertible in $\mathcal{O}$.*

Theorem 1.3. *Let k be a not necessarily infinite field. Let B be a Noetherian k-algebra. Let G be an isotropic simple simply connected group scheme over B; that is, G contains a torus $\mathbb{G}_{m,B}$. Let P_t be a principal G-bundle over $B[t]$ and let $h(t) \in B[t]$ a monic polynomial such that the following hold.*

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(i) *The G-bundle P_t is trivial over $B[t]_h$.*
(ii) *$h(1) \in B$ is invertible in B.*

Then the principal G-bundle P_t is trivial.

Remark 1.4. To prove Theorem 1.1 one needs to substitute in Theorem 1.3 $B = \mathcal{O} \otimes_k A$, $P_t := \mathcal{G}_t$, $h(t) = f(t) \otimes 1$ from Theorem 1.2. By Theorem 1.3 the G-bundle $\mathcal{G}_t$ is trivial. Now, by Theorem 1.2(ii), the original G-bundle $\mathcal{G}$ is trivial.

Remark 1.5. Theorem 1.3 does not hold in the case of anisotropic G even for $B = \mathcal{O}$ with $\mathcal{O}$ as in Theorem 1.2. There are various counterexamples. Only a weaker form of Theorem 1.3 holds in the case of anisotropic G and $B = \mathcal{O}$, as is proved in [FP13, Theorem 2]. This is why we are skeptical that Theorem 1.1 holds in the anisotropic case.

Theorem 1.6. *Let R be a regular semi-local domain containing an infinite field and let K be the fraction field of R. Let G be an isotropic simple simply connected group scheme over R, containing a split rank-1 torus $\mathbb{G}_{m,R}$. Then for any Noetherian commutative ring A the map*

$$H^1_{\text{ét}}(R \otimes_{\mathbb{Z}} A, G) \to H^1_{\text{ét}}(K \otimes_{\mathbb{Z}} A, G),$$

induced by the inclusion R into K, has trivial kernel.

Combining Theorem 1.6 with the well-known result of Raghunathan and Ramanathan [RR84], we obtain the following corollary.

Corollary 1.7. *Let R be a regular domain containing $\mathbb{Q}$. Let G be an isotropic simple simply connected group scheme over R, containing a split rank-1 torus $\mathbb{G}_{m,R}$. Then the map*

$$H^1_{\text{ét}}(R[t_1, t_2, \dots, t_n], G) \to H^1_{\text{ét}}(R, G),$$

induced by evaluation at $t_1 = t_2 = \cdots = t_n = 0$, has trivial kernel.

The proof of this corollary is postponed till § 10. Using principal bundles constructed in [OS71, Par86, Rag89] one can show that this Corollary is wrong without the isotropy condition (see Lemma 10.1).

To put all these statements into context, let us recall other known results on the Serre–Grothendieck conjecture.

- The case where the group scheme G comes from the ground field k is completely solved by Colliot-Thélène, Ojanguren, Raghunathan and Gabber: in [CO92] and [Rag94, Rag95] when k is infinite, while Gabber announced a proof for an arbitrary ground field k.
- The case of an arbitrary reductive group scheme over a discrete valuation ring is completely solved by Nisnevich in [Nis84].
- The case where G is an arbitrary torus over a regular local ring was settled by Colliot-Thélène and Sansuc in [CS87].
- For most simple group schemes of classical series the Serre–Grothendieck conjecture was solved in works of the first author, Suslin, Ojanguren and Zainoulline [PS98, OP01, Zai01, OPZ04]. In fact, unlike our Theorem 1.6, *no isotropy hypotheses* was imposed there.
- The first author, the second author, and Petrov proved the Serre–Grothendieck conjecture for strongly inner adjoint groups of type E_6 or E_7 [PPS09], and for groups of type F_4 with trivial f_3 invariant [PS09], under the same assumptions on R. The results of [PPS09, PS09] use Theorem 1.1 of the present paper.

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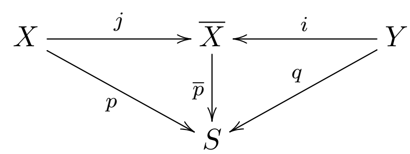
- Chernousov [Che10] established the Serre–Grothendieck conjecture for groups of type F_4 with trivial g_3 invariant, under the assumption that R is a regular local ring containing a field of characteristic 0.
- The case $A = k$ of Theorem 1.1 and [Pan13, Theorem 1.0.1] yields the following: if k is an infinite field and $\mathcal{O}$ is a semi-local regular ring as in Theorem 11.1 and G is a reductive group $\mathcal{O}$-scheme such that the simply connected cover G^{sc} of the derived group DG satisfies the assumption of Theorem 11.1, then Grothendieck–Serre's conjecture holds for this $\mathcal{O}$ and this G.
- In [FP13], Fedorov and the first author prove the Serre–Grothendieck conjecture for an arbitrary reductive group scheme over a semi-local regular domain containing an infinite field. Their work relies heavily on the results of the present paper and of [Pan13].

2. Almost elementary fibrations

In this section we modify a result of Artin from [Art73] concerning existence of nice neighborhoods. The following notion is a modification of the one introduced by Artin in [Art73, Exp. XI, Définition 3.1].

Definition 2.1. An *almost elementary fibration* over a scheme S is a morphism of schemes $p : X \to S$ which can be included in a commutative diagram



(1)

of morphisms satisfying the following conditions.

(i) j is an open immersion dense at each fibre of $\overline{p}$, and $X = \overline{X} - Y$.
(ii) $\overline{p}$ is smooth projective all of whose fibres are geometrically irreducible of dimension one.
(iii) q is a finite flat morphism all of whose fibres are non-empty.
(iv) The morphism i is a closed embedding and the ideal sheaf $I_Y \subset \mathcal{O}_{\overline{X}}$ defining the closed sub-scheme Y in $\overline{X}$ is locally principal.

Remark 2.2. This definition is motivated by the following example. Take a field k and $S = \mathrm{Spec}(k)$, take $\overline{X} = \mathbf{P}^1_k$, take a closed point $y \in \mathbf{P}^1_k$, and set $X = \mathbf{P}^1_k - \{y\}$, $Y = y$. Then the structure morphism $X \to S$ is an almost elementary fibration. If the field extension $k(y)/k$ is purely inseparable, then $X \to S$ is not an elementary fibration in the sense of Artin [Art73, Exp. XI, Définition 3.1].

We prove the following result, which is a slight modification of Artin's result [Art73, Exp. XI, Proposition 3.3].

Proposition 2.3. *Let k be an infinite field, and X be a smooth geometrically irreducible affine variety over k, $x_1, x_2, \ldots, x_n \in X$ be closed points. Then there exists a Zariski open neighborhood X^0 of the family $\{x_1, x_2, \ldots, x_n\}$ and an almost elementary fibration $p : X^0 \to S$, where S is an open sub-scheme of the projective space $\mathbf{P}^{\dim X - 1}$.*

If, moreover, Z is a closed co-dimension one subvariety in X, then one can choose X^0 and p in such a way that $p|_{Z \bigcap X^0} : Z \bigcap X^0 \to S$ is finite surjective.

The proofs of the above Proposition and of the following one are provided in Appendix A.1.

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Proposition 2.4. *Let $p : X \to S$ be an almost elementary fibration. If S is a regular semi-local irreducible scheme, then there exists a commutative diagram of S-schemes*

$$\begin{array}{ccccc} X & \xrightarrow{\ j\ } & \overline{X} & \xleftarrow{\ i\ } & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\overline{\pi}} & & \downarrow \\ \mathbf{A}^1 \times S & \xrightarrow{\ \text{in}\ } & \mathbf{P}^1 \times S & \xleftarrow{\ i\ } & \{\infty\} \times S \end{array} \tag{2}$$

such that the left-hand side square is Cartesian. Here j and i are the same as in Definition 2.1, while $\mathrm{pr}_S \circ \pi = p$, where pr_S is the projection $\mathbf{A}^1 \times S \to S$.

In particular, $\pi : X \to \mathbf{A}^1 \times S$ is a finite surjective morphism of S-schemes, where X and $\mathbf{A}^1 \times S$ are regarded as S-schemes via the morphism p and the projection pr_S, respectively.

3. Nice triples

In the present section we introduce and study certain collections of geometric data and their morphisms. The concept of a *nice triple* is very similar to that of a *standard triple* introduced by Voevodsky [Voe00, Definition 4.1], and was in fact inspired by the latter notion. Let k be an infinite field, let X/k be a smooth geometrically irreducible affine variety, and let $x_1, x_2, \ldots, x_n \in X$ be a family of closed points. Further, let $\mathcal{O} = \mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}}$ be the corresponding geometric semi-local ring.

Definition 3.1. Let $U := \mathrm{Spec}(\mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}})$. A *nice triple* over U consists of the following data:

(i) a smooth morphism $q_U : \mathcal{X} \to U$, where $\mathcal{X}$ is an irreducible scheme;
(ii) an element $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$;
(iii) a section Δ of the morphism q_U,

subject to the following conditions.

(a) Each irreducible component of each fibre of the morphism q_U has dimension one.
(b) The module $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finite as a $\Gamma(U, \mathcal{O}_U) = \mathcal{O}$-module.
(c) There exists a finite surjective U-morphism $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$.
(d) The following holds: $\Delta^*(f) \neq 0 \in \Gamma(U, \mathcal{O}_U)$.

A motivation to require the condition $\Delta^*(f) \neq 0 \in \Gamma(U, \mathcal{O}_U)$ rather than $\Delta^*(f) \in \Gamma(U, \mathcal{O}_U)^{\times}$ is this. If for the triple $(q_U : \mathcal{X} \to U, f, \Delta)$ from § 7 one has $\Delta^*(f) \in \Gamma(U, \mathcal{O}_U)^{\times}$, then $\mathrm{f} \in \mathcal{O}^{\times}$. So, in this case the principal G-bundle P is trivial already over $\mathcal{O} \otimes_k A = \mathcal{O}_{\mathrm{f}} \otimes_k A$. And there is nothing to prove.

Definition 3.2. *A morphism* between two nice triples

$$(q' : \mathcal{X}' \to U, f', \Delta') \to (q : \mathcal{X} \to U, f, \Delta)$$

is an étale morphism of U-schemes $\theta : \mathcal{X}' \to \mathcal{X}$ such that:

(1) $q'_U = q_U \circ \theta$;
(2) $f' = \theta^*(f) \cdot h'$ for an element $h' \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$;
(3) $\Delta = \theta \circ \Delta'$.

Two observations are in order here.

- Condition (2) implies in particular that $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\theta^*(f) \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a finite $\mathcal{O}$-module.
- It should be emphasized that no conditions are imposed on the interrelation of Π' and Π.

Let U be as in Definition 3.1. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. Then for each finite surjective U-morphism $\sigma : \mathcal{X} \to \mathbf{A}^1 \times U$ and the corresponding $\mathcal{O}$-algebra inclusion $\mathcal{O}[t] \hookrightarrow \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ the algebra $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finitely generated as an $\mathcal{O}[t]$-module. Since both rings $\mathcal{O}[t]$ and $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ are regular, the algebra $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finitely generated and projective as an $\mathcal{O}[t]$-module by theorem [Eis95, Corollary 18.17]. Let $T^r - a_{n-1}T^{r-1} + \cdots \pm N(f)$ be the characteristic polynomial of the $\mathcal{O}[t]$-module endomorphism $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{f} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and set

$$g_{f,\sigma} := f^{r-1} - a_{n-1}f^{r-2} + \cdots \pm a_1 \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}). \tag{3}$$

Lemma 3.3. *One has $f \cdot g_{f,\sigma} = \pm\sigma^*(N(f)) \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. The scheme $\{N(f) = 0\}$ is finite over U and the top coefficient of the polynomial $N(f)$ is a unit in $\mathcal{O}$.*

Proof. Indeed, the characteristic polynomial of the operator $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{f} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ vanishes on f. Since the scheme $\{f = 0\}$ is finite over U and the morphism σ is a finite morphism of U-schemes, the scheme $\{N(f) = 0\}$ is finite over U too. Thus the top coefficient of $N(f)$ is a unit in $\mathcal{O}$. □

Let us state two crucial results which will be used in our main construction. Their proofs are given in §§ 4 and 5 respectively.

Theorem 3.4. *Let U be as in Definition 3.1. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over U, such that f vanishes at every closed point of $\Delta(U)$. There exists a distinguished finite surjective morphism*

$$\sigma : \mathcal{X} \to \mathbf{A}^1 \times U$$

of U-schemes which enjoys the following properties.

(1) *σ is étale along the closed subset $\{f = 0\} \cup \Delta(U)$.*

(2) *For $g_{f,\sigma}$ and $N(f)$ defined by the distinguished σ, one has equalities of closed subsets*

$$\sigma^{-1}(\sigma(\{f = 0\})) = \{\sigma^*(N(f)) = 0\} = \{f = 0\} \sqcup \{g_{f,\sigma} = 0\}.$$

(3) *Denote by $\mathcal{X}^0 \hookrightarrow \mathcal{X}$ the largest open subscheme where the morphism σ is étale. Write g for $g_{f,\sigma}$ in this item. Then the square*

$$\begin{array}{ccc} \mathcal{X}^0_{N(f)} = \mathcal{X}^0_{fg} & \xrightarrow{\text{inc}} & \mathcal{X}^0_g \\ \Big\downarrow{\scriptstyle \sigma^0_{fg}} & & \Big\downarrow{\scriptstyle \sigma^0_g} \\ (\mathbf{A}^1 \times U)_{N(f)} & \xrightarrow{\text{inc}} & \mathbf{A}^1 \times U \end{array} \tag{4}$$

is an elementary distinguished square in the category of smooth U-schemes in the sense of [MV99, Definition 3.1.3]. More precisely, this square is Cartesian, the horizontal arrows are open embedding and the induced morphism of the closed sub-schemes

$$\sigma^0_g|_{\{(\sigma^0_g)^*(N(f))=0\}} : \{(\sigma^0_g)^*(N(f)) = 0\} = \{f = 0\} \to \{N(f) = 0\}$$

of the schemes $\mathcal{X}^0_g$ and $\mathbf{A}^1 \times U$ is an isomorphism.

(4) *One has $\Delta(U) \subset \mathcal{X}^0_g$.*

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Remark 3.5. One readily sees that if in Theorem 3.4 we let $\mathcal{X}^0$ be any open sub-scheme of $\mathcal{X}$ such that σ is étale on $\mathcal{X}^0$ and $\mathcal{X}^0$ contains the closed subset $\{f=0\} \cup \Delta(U)$, then all the claims of this theorem are still valid. In particular, if needed, one can assume that $\mathcal{X}^0$ is an affine scheme and $\mathcal{X}^0_g$ is an affine scheme too.

Theorem 3.6. *Let U be as in Definition 3.1. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. Let $G_{\mathcal{X}}$ be a semi-simple $\mathcal{X}$-group scheme, and let $G_U := \Delta^*(G_{\mathcal{X}})$. Finally, let G_{const} be the pull-back of G_U to $\mathcal{X}$. Then there exists a morphism $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ of nice triples and an isomorphism*

$$\Phi : \theta^*(G_{\mathrm{const}}) \to \theta^*(G_{\mathcal{X}})$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = \mathrm{id}_{G_U}$.

4. Proof of Theorem 3.4

The nearest aim is to prove Theorem 3.4. We will use analogues of three lemmas from [Pan05] making them characteristic free. Lemma 4.3 is a refinement of [OP99, Lemma 5.2].

Lemma 4.1. *Let k be an infinite field and let S be an k-smooth equidimensional k-algebra of dimension one. Let $f \in S$ be a non-zero divisor.*

Let $\mathfrak{m}_0$ be a maximal ideal with $S/\mathfrak{m}_0 = k$. Let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ be pairwise distinct maximal ideals of S (possibly $\mathfrak{m}_0 = \mathfrak{m}_i$ for some i). Then there exists a non-zero divisor $\bar{s} \in S$ such that S is finite over $k[\bar{s}]$ and the following hold.

(1) *The ideals $\mathfrak{n}_i := \mathfrak{m}_i \cap k[\bar{s}]$, $1 \leqslant i \leqslant n$, are pairwise distinct. If $\mathfrak{m}_0$ is distinct from all the $\mathfrak{m}_i$, then $\mathfrak{n}_0 := \mathfrak{m}_0 \cap k[\bar{s}]$ is distinct from all the $\mathfrak{n}_i$.*
(2) *The extension $S/k[\bar{s}]$ is étale at each $\mathfrak{m}_i$, $i = 1, 2, \ldots, n$, and at $\mathfrak{m}_0$.*
(3) *$k[\bar{s}]/\mathfrak{n}_i = S/\mathfrak{m}_i$ for each $i = 1, 2, \ldots, n$.*
(4) *$\mathfrak{n}_0 = \bar{s}k[\bar{s}]$.*

Proof. Let x_i, $0 \leqslant i \leqslant n$, be the points on $\mathrm{Spec}(S)$ corresponding to the ideals $\mathfrak{m}_i$. Consider a closed embedding $\mathrm{Spec}(S) \hookrightarrow \mathbf{A}^n_k$ and find a generic linear projection $p : \mathbf{A}^n_k \to \mathbf{A}^1_k$, defined over k and such that the following hold.

(a) For all $i, j \geqslant 0$ one has $p(x_i) \neq p(x_j)$, provided that $x_i \neq x_j$.
(b) For each index $i \geqslant 0$ the map $p|_{\mathrm{Spec}(S)} : \mathrm{Spec}(S) \to \mathbf{A}^1_k$ is étale at the point x_i.
(c) For each i, the separable degree of the extension $k(x_i)/k(p(x_i))$ is one.

These items imply equalities $k(p(x_i)) = k(x_i)$, for all i. Indeed, the extension $k(x_i)/k(p(x_i))$ is separable (b). By (c) we conclude that $k(p(x_i)) = k(x_i)$. The lemma follows. □

Lemma 4.2. *Under the hypotheses of Lemma 4.1 let $f \in S$ be a non-zero divisor which does not belong to a maximal ideal distinct from $\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Let $\bar{s} \in S$ be an element satisfying conditions (1)–(4) of Lemma 4.1. Let $N(f) = N_{S/k[\bar{s}]}(f)$ be the norm of f. Then one has:*

(a) *$N(f) = fg$ for an element $g \in S$;*
(b) *$fS + gS = S$;*
(c) *the map $k[\bar{s}]/(N(f)) \to S/(f)$ is an isomorphism.*

Proof. The proof is straightforward. □

Lemma 4.3. *Let k be an infinite field, and let R be a domain which is a semi-local essentially smooth k-algebra with maximal ideals $\mathfrak{p}_i$, $1 \leqslant i \leqslant m$. Let $A \supseteq R[t]$ be another domain, smooth as an R-algebra and finite over $R[t]$. Assume that for each i the $R/\mathfrak{p}_i$-algebra $A_i = A/\mathfrak{p}_i A$ is equidimensional of dimension one. Let $\epsilon : A \to R$ be an R-augmentation and $I = \operatorname{Ker}(\epsilon)$. Given an $f \in A$ with*

$$0 \neq \epsilon(f) \in \bigcap_{i=1}^{m} \mathfrak{p}_i \subset R$$

and such that the R-module A/fA is finite, one can find an element $u \in A$ satisfying the following conditions:

1. *A is a finite projective module over $R[u]$;*
2. *$A/uA = A/I \times A/J$ for some ideal J;*
3. *$J + fA = A$;*
4. *$(u-1)A + fA = A$;*
5. *set $N(f) = N_{A/R[u]}(f)$, then $N(f) = fg \in A$ for some $g \in A$;*
6. *$fA + gA = A$;*
7. *the composition map $\varphi : R[u]/(N_{A/R[u]}(f)) \to A/(N_{A/R[u]}(f)) \to A/(f)$ is an isomorphism.*

Proof. Replacing t by $t - \epsilon(t)$ we may assume that $\epsilon(t) = 0$. Since A is finite over $R[t]$, it follows from a theorem of Grothendieck [Eis95, Corollary 17.18] that it is a finite projective $R[t]$-module.

Since A is finite over $R[t]$ and A/fA is finite over R we conclude that $R[t]/(N_{A/R[t]}(f))$ is finite over R, and hence $R/(tN_{A/R[t]}(f))$ is finite over R.

Setting $v = tN_{A/R[t]}(f)$, we get an integral extension $R[t]$ over $R[v]$. Thus A is finite over $R[v]$. By the theorem of Grothendieck [Eis95, Corollary 17.18] A is a finite projective $R[v]$-module.

Applying Lemma 3.3 to A over $R[t]$ (not over $R[v]$) one gets an equality $N_{A/R[t]}(f) = f \cdot g_{f,t} \in A$ for an element $g_{f,t} \in A$. Thus

$$v = t \cdot N_{A/R[t]}(f) = t \cdot f \cdot g_{f,t} \in fA \quad \text{and} \quad \epsilon(v) = \epsilon(t) \cdot \epsilon(N_{A/R[t]}(f)) = 0.$$

Below, we use the bar $^{-}$ to denote reduction modulo an ideal, and the subscript i to indicate that reduction is modulo $\mathfrak{p}_i A$, $1 \leqslant i \leqslant m$. Let $l_i = \bar{R}_i = R/\mathfrak{p}_i$. By the assumption of the lemma, the l_i-algebra $\bar{A}_i$ is l_i-smooth equidimensional of dimension 1. The element $\bar{f}_i \in \bar{A}_i$ is a non-zero divisor since $\bar{A}_i/\bar{f}_i\bar{A}_i = \overline{(A/fA)}_i$ is a finite l_i-module. Let $\mathfrak{m}_1^{(i)}, \mathfrak{m}_2^{(i)}, \ldots, \mathfrak{m}_{n_i}^{(i)}$ be distinct maximal ideals of $\bar{A}_i$ dividing $\bar{f}_i$ and let $\mathfrak{m}_0^{(i)} = \operatorname{Ker}(\bar{\epsilon}_i)$. Let $\bar{s}_i \in \bar{A}_i$ be such that the extension $\bar{A}_i/l_i[\bar{s}_i]$ satisfies conditions (1)–(4) of Lemma 4.1.

Let $s \in A$ be a common lifting of the $\bar{s}_i$, in other words, $\overline{s} = \bar{s}_i$ in $\bar{A}_i$ for all $i = 1, \ldots, m$. Replacing s by $s - \epsilon(s)$ we may assume that $\epsilon(s) = 0$ and, as above, $\overline{s} = \bar{s}_i$ for all $i = 1, \ldots, m$.

Let $s^n + p_1(v)s^{n-1} + \cdots + p_n(v) = 0$ be an integral dependence relation for s. Let N be an integer larger than $\max\{2, \deg(p_j(t))\}$, where $j = 1, 2, \ldots, n$. Then for any $r \in k^\times$ the element $u = s - rv^N$ has the following property: v is integral over $R[u]$. Thus, for any $r \in k^\times$ the ring A is integral over $R[u]$.

On the other hand, one has $\bar{v}_i \in \mathfrak{m}_j^{(i)}$ for all $1 \leqslant i \leqslant m$ and all $0 \leqslant j \leqslant n_i$, since $v \in fA$ and $\epsilon(v) = 0$. It implies that each element $\bar{u}_i = \bar{s}_i - r\bar{v}_i{}^N$ still satisfies conditions (1)–(4) of Lemma 4.1.

We claim that the element $u \in R$ has all the properties listed in the statement of the present lemma, for almost all $r \in k^\times$.

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Indeed, for almost all $r \in k^{\times}$ the element u satisfies conditions (1)–(4) of Lemma 4.3. It remains to show that conditions (5)–(7) hold for all $r \in k^{\times}$.

Since A is finite over $R[u]$, the same theorem of Grothendieck [Eis95, Corollary 17.18] implies that it is a finite projective $R[u]$-module. To prove condition (5) of Lemma 4.3, consider the characteristic polynomial of the operator $A \xrightarrow{f} A$ as an $R[u]$-module operator. This polynomial vanishes on f and its free term equals $\pm N_{A/R[u]}(f)$, the norm of f. Thus, $f^n - a_1 f^{n-1} + \cdots \pm N_{A/R[u]}(f) = 0$ and $N_{A/R[u]}(f) = f \cdot g_{f,u}$ for some $g_{f,u} \in A$.

To prove condition (6), one has to verify that the above g is a unit modulo the ideal fA. It suffices to check that for each index i the element $\bar{g}_i \in \bar{A}_i$ is a unit modulo the ideal $\bar{f}_i \bar{A}_i$. To that end observe that the field $l_i = R/\mathfrak{p}_i$, the l_i-algebra $S_i = \bar{A}_i$, its maximal ideals $\mathfrak{m}_0^{(i)}, \mathfrak{m}_1^{(i)}, \ldots, \mathfrak{m}_{n_i}^{(i)}$ and the element $\bar{u}_i$ satisfy the hypotheses of Lemma 4.2, with u replaced by $\bar{u}_i$. Now, by item (b) of Lemma 4.2 the reduction $\bar{g}_i$ is a unit modulo the ideal $\bar{f}_i \bar{R}_i$.

To prove condition (7), observe that $R[u]/(N_{A/R[u]}(f))$ and A/fA are finite R-modules. Thus, it remains to check that the map $\varphi : R[u]/(N_{A/R[u]}(f)) \to A/fA$ is an isomorphism modulo each maximal ideal $\mathfrak{p}_i$. To that end it suffices to verify that the map $\bar{\varphi}_i : l_i[\bar{u}_i]/(N(\bar{f}_i)) \to \bar{A}_i/\bar{f}_i \bar{A}_i$ is an isomorphism for each index i, where $N(\bar{f}_i) := N_{\bar{A}_i/l_i[\bar{u}]}(\bar{f}_i)$. Now, by item (c) of Lemma 4.2 the map $\bar{\varphi}_i$ is an isomorphism. This finishes the proof. □

Proof of Theorem 3.4. Let $U = \operatorname{Spec}(\mathcal{O}_{X,\{x_1,x_2,\ldots,x_r\}})$ be as in Definition 3.1. Write R for $\mathcal{O}_{X,\{x_1,x_2,\ldots,x_r\}}$. It is a domain which is a semi-local essentially smooth k-algebra with maximal ideals $\mathfrak{p}_i$, $1 \leqslant i \leqslant r$. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. We show that it gives rise to certain data subject to the hypotheses of Lemma 4.3.

Let $A = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. It is a domain, since $\mathcal{X}$ is irreducible. It is an R-algebra via the ring homomorphism $q_U^* : R \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Furthermore, it is smooth as an R-algebra. The triple $(\mathcal{X}, f, \Delta)$ is a nice triple. Thus, there exists a finite surjective U-morphism $\Pi : \mathcal{X} \to \mathbf{A}^1_U$. It induces an R-algebra inclusion $R[t] \hookrightarrow \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = A$ such that A is finitely generated as an $R[t]$-module. Also, for all $i = 1, \ldots, r$, the $R/\mathfrak{p}_i$-algebra $A/\mathfrak{p}_i A$ is equidimensional of dimension one. Let

$$\epsilon = \Delta^* : A \to R$$

be an R-algebra homomorphism induced by the section Δ of the morphism q_U. Clearly, this ϵ is an augmentation; set $I = \operatorname{Ker}(\epsilon)$. Further, since $(\mathcal{X}, f, \Delta)$ is a nice triple, $\epsilon(f) \neq 0 \in R$ and A/fA is finite as an R-module. Finally, f vanishes at every closed point of $\Delta(U)$ by the assumption of the Theorem. Summarizing the above, we conclude that we are in the setting of Lemma 4.3, and may use the conclusion of that Lemma.

Thus, there exists an element $u \in A$ subject to conditions (1)–(7) of Lemma 4.3. This u induces an R-algebra inclusion $R[u] \hookrightarrow A$ such that A is finite as an $R[u]$-module. Let

$$\sigma : \mathcal{X} \to \mathbf{A}^1 \times U$$

be the U-scheme morphism induced by the above inclusion $R[u] \hookrightarrow A$. Clearly, σ is finite and surjective. In the rest of the proof we write t instead of u, and consider A as an $R[t]$-module via σ. Let $N(f) := N_{A/R[t]}(f) \in R[t] \subseteq A$ and $g_{f,\sigma} \in A$ be the elements defined just above Lemma 3.3.

We claim that this morphism σ and the chosen elements $N(f)$ and $g_{f,\sigma}$ satisfy conclusions (1)–(4) of Theorem 3.4. Let us verify this claim. Since A is finite as an $R[t]$-module and both rings $R[t]$ and A are regular, the $R[t]$-module A is finitely generated and projective, see [Eis95, Corollary 18.17]. Thus, σ is étale at a point $x \in \mathcal{X}$ if and only if the $k(\sigma(x))$-algebra $k(\sigma(x)) \otimes_{R[t]} A$

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is étale. If the point x belongs to the closed sub-scheme $\mathrm{Spec}(A/\mathfrak{p}_iA)$ for some maximal ideal $\mathfrak{p}_i$ of R, then

$$k(\sigma(x)) \otimes_{R[t]} A = k(\sigma(x)) \otimes_{(R/\mathfrak{p}_i)[t]} A/\mathfrak{p}_iA.$$

We can conclude that σ is étale at a specific point x if and only if the $(R/\mathfrak{p}_i)[t]$-algebra $A/\mathfrak{p}_iA$ is étale at the point x. It follows from the proof of Lemma 4.3 that the morphism σ induces a morphism $\mathrm{Spec}(A/\mathfrak{p}_iA) \xrightarrow{\sigma_i} \mathbf{A}^1_{l_i}$ on the closed fibre $\mathrm{Spec}(A/\mathfrak{p}_iA)$ for each i. This induced morphism is étale along the vanishing locus of the function $\bar{f}_i$ and along each point $\overline{\Delta}_i(\mathrm{Spec}\, l_i)$. Indeed, for the vanishing locus of the function $\bar{f}_i$ this follows from items (6) and (7) of Lemma 4.3. It follows from the hypotheses of Lemma 4.3 that the function f vanishes at each maximal ideal containing I. Thus σ is étale along the closed sub-scheme $\mathcal{X}$ defined by the ideal I, that is along $\Delta(U)$. This settles item (1) of Theorem 3.4.

Consider item (2). Write g for $g_{f,\sigma}$. The first of the following equalities

$$\sigma^{-1}(\sigma(\{f=0\})) = \{\sigma^*(N(f))=0\} = \{f=0\} \sqcup \{g=0\}$$

is a commonplace. The second one follows from the equality $\sigma^*(N(f)) = \pm f \cdot g$, proved in Lemma 3.3 and item (6) of Lemma 4.3.

Clearly, the square (4) is Cartesian and the morphism σ^0_g is étale. The scheme $\mathcal{X}^0_g$ contains a closed sub-scheme $\Delta(U)$, and hence is non-empty. Item (7) of Lemma 4.3 shows that the induced morphism of the closed sub-schemes

$$\sigma^0_g|_{\{(\sigma^0_g)^*(N(f))=0\}} : \{(\sigma^0_g)^*(N(f))=0\} = \{f=0\} \to \{N(f)=0\}$$

of the schemes $\mathcal{X}^0_g$ and $\mathbf{A}^1 \times U$ is an isomorphism. Thus, we have checked item (3) of Theorem 3.4.

It remains only to check item (4). We already know that $\{f=0\} \subset \mathcal{X}^0_g$. Both schemes $\Delta(U)$ and $\{f=0\}$ are semi-local and the set of closed points of $\Delta(U)$ is contained in the set of closed points of the closed set $\{f=0\}$ by the assumptions of the theorem. Thus, $\Delta(U) \subset \mathcal{X}^0_g$. This concludes the proof of item (4) of Theorem 3.4 and thus of the theorem itself. □

5. Proof of Theorem 3.6

The aim of this section is to proof of Theorem 3.6. We begin with the following Proposition which is a straightforward analogue of [OP01, Proposition 7.1].

Proposition 5.1. *Let S be a regular semi-local irreducible scheme and let G_1, G_2 be two semi-simple S-group schemes which are twisted forms of each other. Further, let $T \subset S$ be a closed sub-scheme of S and $\varphi : G_1|_T \to G_2|_T$ be an S-group scheme isomorphism. Then there exists a finite étale morphism $\widetilde{S} \xrightarrow{\pi} S$ together with its section $\delta : T \to \widetilde{S}$ over T and an $\widetilde{S}$-group scheme isomorphism $\Phi : \pi^*G_1 \to \pi^*G_2$ such that $\delta^*(\Phi) = \varphi$.*

Since the proof of the Proposition 5.1 is rather long we first give an outline. Clearly, G_1 and G_2 are of the same type. By [SGA3, Exp. XXIV, Corollary 1.8] there exists an S-scheme $\mathrm{Isom}_S(G_1, G_2)$ representing the functor that sends an S-scheme W to the set of all W-group scheme isomorphisms from $W \times_S G_1$ to $W \times_S G_2$. The isomorphism φ from the hypothesis of Proposition 5.1 determines a section $\delta : T \to \mathrm{Isom}_S(G_1, G_2)$ of the structure map $\mathrm{Isom}_S(G_1, G_2) \to S$. By Lemmas 5.4 and 5.2 below there exists a closed sub-scheme $\widetilde{S}$ of $\mathrm{Isom}_S(G_1, G_2)$

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which is finite étale over S and contains $\delta(T)$. So, we have a commutative diagram of S-schemes

$$\begin{array}{ccccc} T & \xrightarrow{\delta} & \widetilde{S} & \longrightarrow & \mathrm{Isom}_S(G_1, G_2) \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle \pi} & \swarrow & \\ & & S & & \end{array} \tag{5}$$

such that the horizontal arrows are closed embedding. Thus we get an isomorphism $\Phi : \pi^*(G_1) \to \pi^*(G_2)$ such that $\delta^*(\Phi) = \varphi$.

The precise proof of the Proposition requires some auxiliary results and will be given right below Lemma 5.4. Clearly, G_1 and G_2 are of the same type. Let G_0 be a split semi-simple simply connected algebraic group over the ground field k such that G_1 and G_2 are twisted forms of the S-group scheme $S \times_{\mathrm{Spec}(k)} G_0$. Let $\mathrm{Aut}_k(G_0)$ be the automorphism scheme of the algebraic k-group G_0. It is known that $\mathrm{Aut}_k(G_0)$ is a semi-direct product of the algebraic k-group G_0^{ad} and a finite constant group, where G_0^{ad} is the adjoint group to G_0 [SGA3, Exp. XXIV, Theorem 1.3 (iii) and Corollary 1.6]. Also, $\mathrm{Aut}_k(G_0)$ is a smooth affine algebraic k-group (for example, by [SGA3, Exp. XXIV, Corollary 1.8]). For brevity, set $\mathrm{Aut} := \mathrm{Aut}_k(G_0)$ and Aut_S for the S-group scheme $S \times_{\mathrm{Spec}(k)} \mathrm{Aut}$.

Consider an S-scheme $\mathrm{Isom}_S(G_{0,S}, G_2)$ constructed in [SGA3, Exp. XXIV, Corollary 1.8] and representing a functor that sends an S-scheme W to the set of all W-group scheme isomorphisms $\varphi_2 : W \times_S G_{0,S} \to W \times_S G_2$. Similarly, consider an S-scheme $\mathrm{Aut}_S(G_2)$ constructed in [SGA3, Exp. XXIV, Corollary 1.8] and representing a functor that sends an S-scheme W to the set of all W-group scheme automorphisms $\alpha : W \times_S G_2 \to W \times_S G_2$.

The functor transformation $(\varphi_2, \alpha_2) \mapsto \varphi_2 \circ \alpha_2^{-1}$ defines an S-scheme morphism

$$\mathrm{Isom}_S(G_{0,S}, G_2) \times_S \mathrm{Aut}_S \to \mathrm{Isom}_S(G_{0,S}, G_2)$$

which makes the S-scheme $\mathrm{Isom}_S(G_{0,S}, G_2)$ a principal right Aut_S-bundle. The functor transformation $(\beta_2, \varphi_2) \mapsto \beta_2 \circ \varphi_2$ defines an S-scheme morphism

$$\mathrm{Aut}_S(G_2) \times_S \mathrm{Isom}_S(G_{0,S}, G_2) \to \mathrm{Isom}_S(G_{0,S}, G_2)$$

which makes the S-scheme $\mathrm{Isom}_S(G_{0,S}, G_2)$ a principal left $\mathrm{Aut}_S(G_2)$-bundle.

Analogously, the functor transformation $(\alpha_1, \varphi_1) \mapsto \alpha_1 \circ \varphi_1$ makes the S-scheme $\mathrm{Isom}_S(G_1, G_{0,S})$ a principal left Aut_S-bundle and the functor transformation $(\varphi_1, \beta_1) \mapsto \varphi_1 \circ \beta_1$ makes the S-scheme $\mathrm{Isom}_S(G_1, G_{0,S})$ a principal right $\mathrm{Aut}_S(G_1)$-bundle.

Let ${}_2P_r$ be a left principal $\mathrm{Aut}_S(G_2)$-bundle and at the same time a right principal Aut_S-bundle such that the two actions commute. Let ${}_lP_1$ be a left principal Aut_S-bundle and at the same time a right principal $\mathrm{Aut}_S(G_1)$-bundle such that the two actions commute. Let Y be a k-variety equipped with a left and a right Aut_k-actions which commute. Then the k-scheme

$$({}_2P_r) \times_S (Y_S) \times_S ({}_lP_1)$$

is equipped with a left $\mathrm{Aut}_k \times \mathrm{Aut}_k$-action given by

$$(\alpha_2, \alpha_1)(p_2, y, p_1) = (p_2\alpha_2^{-1}, \alpha_2 y \alpha_1^{-1}, \alpha_1 p_1).$$

The orbit space does exist (it can be constructed by descent). Denote it by ${}_2Y_1$. We now show that it is an S-scheme. Indeed, the structure morphism $Y \to \mathrm{Spec}(k)$ defines a morphism

$$({}_2P_r) \times_S (Y_S) \times_S ({}_lP_1) \to ({}_2P_r) \times_S ({}_lP_1)$$

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respecting the Aut × Aut-actions on both sides. Thus it defines a morphism of the orbit spaces

$$ {}_2Y_1 \to ({}_2\mathrm{Spec}(k)_1) = S. $$

The latter equality holds since $({}_2P_r) \times_S ({}_lP_1)$ is a principal left Aut × Aut-bundle with respect the left action given by $(\alpha_2, \alpha_1)(p_2, p_1) = (p_2\alpha_2^{-1}, \alpha_1 p_1)$.

The construction $Y \mapsto {}_2Y_1$ has several nice properties, namely, the following.

(i) It is natural with respect to k-morphisms of k-varieties $Y \to Y'$ commuting with the given two-sided Aut × Aut-actions on Y and Y'.

(ii) It takes a closed embedding to a closed embedding.

(iii) It takes an open embedding to an open embedding.

(iv) It takes k-products to S-products.

(v) Locally in the étale topology on S, the S-schemes Y_S and ${}_2Y_1$ are isomorphic.

Set ${}_2P_r = \mathrm{Isom}_S(G_{0,S}, G_2)$ and ${}_lP_1 = \mathrm{Isom}_S(G_1, G_{0,S})$. The functor transformation $(\varphi_2, \alpha, \varphi_1) \mapsto \varphi_2 \circ \alpha \circ \varphi_1$ gives a morphism of representable S-functors

$$ \mathrm{Isom}_S(G_{0,S}, G_2) \times_S (\mathrm{Aut}_S) \times_S \mathrm{Isom}_S(G_1, G_{0,S}) \xrightarrow{\Phi} \mathrm{Isom}_S(G_1, G_2). $$

The equality

$$ \varphi_2 \circ \alpha \circ \varphi_1 = (\varphi_2 \circ \alpha_2^{-1}) \circ (\alpha_2 \circ \alpha \circ \alpha_1^{-1}) \circ (\alpha_1 \circ \varphi_1) $$

shows that the morphism Φ induces a morphism $\bar{\Phi} : {}_2(\mathrm{Aut})_1 \to \mathrm{Isom}_S(G_1, G_2)$.

Lemma 5.2. *The S-morphism*

$$ \bar{\Phi} : {}_2(\mathrm{Aut})_1 \to \mathrm{Isom}_S(G_1, G_2) $$

is an isomorphism.

Proof. It suffices to prove that $\bar{\Phi}$ is an isomorphism locally in the étale topology on S. The latter follows from property (v). □

Now let G_0 and Aut be as above. There is a closed embedding of algebraic groups $\rho : \mathrm{Aut} \hookrightarrow \mathrm{GL}_{V,k}$ for an n-dimensional k-vector space V. Replacing ρ with $\rho \oplus \det^{-1} \circ \rho$ we get a closed embedding of algebraic k-groups $\rho_1 : \mathrm{Aut} \hookrightarrow \mathrm{SL}_{W,k}$, where $W = V \oplus k$. Let $\mathrm{End} := \mathrm{End}_k(W)$. Clearly, the composition $in : \mathrm{Aut} \xrightarrow{\rho_1} \mathrm{SL}_{W,k} \hookrightarrow \mathrm{End}$ is a closed embedding. We will identify Aut with its image in End. Let $\overline{\mathrm{Aut}}$ be the closure of Aut in the projective space $\mathbf{P}(k \oplus \mathrm{End})$. Set $\mathrm{Aut}_\infty := \overline{\mathrm{Aut}} - \mathrm{Aut}$ regarded as a reduced scheme. So, we get a commutative diagram of k-varieties

$$ \begin{array}{ccccc} \mathrm{Aut} & \xrightarrow{j} & \overline{\mathrm{Aut}} & \xleftarrow{i} & \mathrm{Aut}_\infty \\ \downarrow{\scriptstyle in} & & \downarrow{\scriptstyle \overline{in}} & & \downarrow{\scriptstyle in_\infty} \\ \mathrm{End} & \xrightarrow{J} & \mathbf{P}(k \oplus \mathrm{End}) & \xleftarrow{I} & \mathbf{P}(\mathrm{End}) \end{array} \qquad (6) $$

where the left square is Cartesian. All varieties are equipped with the left Aut × Aut-action induced by Aut × Aut-action on the affine space $k \oplus \mathrm{End}$ given by $(g_1, g_2)(c, \alpha) = (c, g_1 \alpha g_2^{-1})$. All

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the arrows in this diagram respect this action. Applying to this diagram the above construction $Y \mapsto {}_2Y_1$, we obtain a commutative diagram of S-schemes

$$\begin{array}{ccccc} {}_2\mathrm{Aut}_1 & \xrightarrow{j} & {}_2(\overline{\mathrm{Aut}})_1 & \xleftarrow{i} & {}_2(\mathrm{Aut}_\infty)_1 \\ \downarrow{\scriptstyle in} & & \downarrow{\scriptstyle \overline{in}} & & \downarrow{\scriptstyle in_\infty} \\ {}_2\mathrm{End}_1 & \xrightarrow{J} & \mathbf{P}(\mathcal{O}_S \oplus {}_2\mathrm{End}_1) & \xleftarrow{I} & \mathbf{P}({}_2\mathrm{End}_1) \end{array} \tag{7}$$

where the square on the left is Cartesian.

From now on we assume that S is a semi-local irreducible scheme. Then the vector bundle ${}_2\mathrm{End}_1$ is trivial. Since it is trivial, we may choose homogeneous coordinates Y_i on $\mathbf{P}(\mathcal{O}_S \oplus {}_2\mathrm{End}_1)$ such that the closed sub-schemes $\{Y_0 = 0\}$ and $\mathbf{P}({}_2\mathrm{End}_1)$ of the scheme $\mathbf{P}(\mathcal{O}_S \oplus {}_2\mathrm{End}_1)$ coincide and the S-scheme $\mathbf{P}(\mathcal{O}_S \oplus {}_2\mathrm{End}_1)$ itself is isomorphic to the projective space $\mathbf{P}^{n^2}_S$. Thus the diagram (7) of S-schemes and of S-scheme morphisms can be rewritten as follows:

$$\begin{array}{ccccc} {}_2\mathrm{Aut}_1 & \xrightarrow{j} & {}_2(\overline{\mathrm{Aut}})_1 & \xleftarrow{i} & {}_2(\mathrm{Aut}_\infty)_1 \\ \downarrow{\scriptstyle in} & & \downarrow{\scriptstyle \overline{in}} & & \downarrow{\scriptstyle in_\infty} \\ \{Y_0 \neq 0\} & \xrightarrow{J} & \mathbf{P}^{n^2}_S & \xleftarrow{I} & \{Y_0 = 0\} \end{array} \tag{8}$$

where the square on the left is Cartesian. Since ${}_2(\mathrm{Aut}_\infty)_1 = {}_2(\overline{\mathrm{Aut}})_1 - {}_2\mathrm{Aut}_1$, the set-theoretic intersection ${}_2(\overline{\mathrm{Aut}})_1 \cap \{Y_0 = 0\}$ in $\mathbf{P}^{n^2}_S$ coincides with ${}_2(\mathrm{Aut}_\infty)_1$.

The following Lemma is the lemma [OP01, Lemma 7.2].

Lemma 5.3. *Let $S = \mathrm{Spec}(R)$ be a regular semi-local scheme and T a closed sub-scheme of S. Let $\bar{X}$ be a closed sub-scheme of $\mathbf{P}^N_S = \mathrm{Proj}(S[Y_0, \dots, Y_N])$ and $X = \bar{X} \cap \mathbf{A}^N_S$, where $\mathbf{A}^N_S$ is the affine space defined by $Y_0 \neq 0$. Let $X_\infty = \bar{X} \backslash X$ be the intersection of $\bar{X}$ with the hyperplane at infinity $Y_0 = 0$. Assume further that the following hold.*

(1) *X is smooth and equidimensional over S, of relative dimension r.*

(2) *For every closed point $s \in S$ the closed fibres of X_∞ and X satisfy*

$$\dim(X_\infty(s)) < \dim(X(s)) = r.$$

(3) *Over T there exists a section $\delta : T \to X$ of the canonical projection $X \to S$.*

Then there exists a closed sub-scheme $\widetilde{S}$ of X which is finite étale over S and contains $\delta(T)$.

Diagram (8) shows that the S-schemes $X = {}_2\mathrm{Aut}_1$, $\bar{X} = {}_2(\overline{\mathrm{Aut}})_1$ and $X_\infty = {}_2(\mathrm{Aut}_\infty)_1$ satisfy all the hypotheses of Lemma 5.3 except possibly the conditions (2) and (3). To check condition (2), observe that the diagram of S-schemes

$${}_2\mathrm{Aut}_1 \xrightarrow{j} {}_2(\overline{\mathrm{Aut}})_1 \xleftarrow{i} {}_2(\mathrm{Aut}_\infty)_1 \tag{9}$$

locally in the étale topology on S is isomorphic to the diagram of S-schemes

$$\mathrm{Aut} \times S \xrightarrow{j} (\overline{\mathrm{Aut}}) \times S \xleftarrow{i} (\mathrm{Aut}_\infty) \times S. \tag{10}$$

This follows from property (v) of the construction $Z \to {}_2Z_1$. Since Aut is equidimensional and $\overline{\mathrm{Aut}}$ is the closure of Aut in $\mathbf{P}(\mathrm{End} \oplus k)$, one has

$$\dim(\mathrm{Aut}_\infty) < \dim(\overline{\mathrm{Aut}}) = \dim \mathrm{Aut}.$$

Thus the assumption (2) of Lemma 5.3 is fulfilled. Whence we have proved the following.

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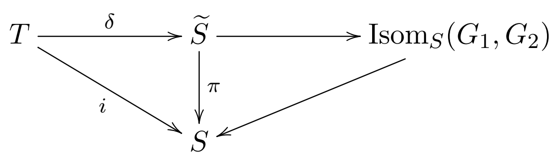
Lemma 5.4. *Assume S is a regular semi-local irreducible scheme and assume we are given with a closed sub-scheme $T \subset S$ equipped with a section $\delta : T \to {}_2\mathrm{Aut}_1$ of the structure map ${}_2\mathrm{Aut}_1 \to S$. Then there exists a closed sub-scheme $\widetilde{S}$ of ${}_2\mathrm{Aut}_1$ which is finite and étale over S and contains $\delta(T)$.*

Proof of Proposition 5.1. By Lemma 5.2 the S-schemes $\mathrm{Isom}_S(G_1, G_2)$ and ${}_2\mathrm{Aut}_1$ are naturally isomorphic as S-schemes. The isomorphism φ from the hypotheses of the Proposition 5.1 determines a section $\delta : T \to \mathrm{Isom}_S(G_1, G_2) = {}_2\mathrm{Aut}_1$ of the structure map $\mathrm{Isom}_S(G_1, G_2) = {}_2\mathrm{Aut}_1 \to S$. By Lemma 5.4 there exists a closed sub-scheme $\widetilde{S}$ of ${}_2\mathrm{Aut}_1 = \mathrm{Isom}_S(G_1, G_2)$ which is finite étale over S and contains $\delta(T)$. So, we have morphisms (even closed inclusions) of S-schemes.



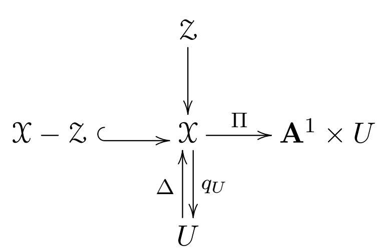
(11)

Thus we get an isomorphism $\Phi : \pi^*(G_1) \to \pi^*(G_2)$ such that $\delta^*(\Phi) = \varphi$. □

Proof of Theorem 3.6. We can start by almost literally repeating arguments from the proof of [OP01, Lemma 8.1], which involve the following purely geometric lemma [OP01, Lemma 8.2]. For the reader's convenience we state below that lemma with notation adapted to that of § 3. Namely, let U be as in Definition 3.1 and let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. Further, let $G_{\mathcal{X}}$ be a simple simply-connected $\mathcal{X}$-group scheme, $G_U := \Delta^*(G_{\mathcal{X}})$, and let G_{const} be the pull-back of G_U to $\mathcal{X}$. Finally, by the definition of a nice triple there exists a finite surjective morphism $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$ of U-schemes.

Lemma 5.5. *Let $\mathcal{Y}$ be a closed nonempty sub-scheme of $\mathcal{X}$, finite over U. Let $\mathcal{V}$ be an open subset of $\mathcal{X}$ containing $\Pi^{-1}(\Pi(\mathcal{Y}))$. There exists an open set $\mathcal{W} \subseteq \mathcal{V}$ still containing $q_U^{-1}(q_U(\mathcal{Y}))$ and endowed with a finite surjective morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$, which, in general, is not equal to Π.*

Let $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$ be the above finite surjective U-morphism. The following diagram summarizes the situation.



Here $\mathcal{Z}$ is the closed sub-scheme defined by the equation $f = 0$. By assumption, $\mathcal{Z}$ is finite over U. Let $\mathcal{Y} = \Pi^{-1}(\Pi(\mathcal{Z} \cup \Delta(U)))$. Since $\mathcal{Z}$ and $\Delta(U)$ are both finite over U and since Π is a finite morphism of U-schemes, $\mathcal{Y}$ is also finite over U. Denote by $y_1, \ldots, y_m$ its closed points and let $S = \mathrm{Spec}(\mathcal{O}_{\mathcal{X}, y_1, \ldots, y_m})$. Set $T = \Delta(U) \subseteq S$. Further, let $G_U = \Delta^*(G_{\mathcal{X}})$ be as in the hypotheses of Theorem 3.6 and let G_{const} be the pull-back of G_U to $\mathcal{X}$. Finally, let $\varphi : G_{\mathrm{const}}|_T \to G_{\mathcal{X}}|_T$ be the canonical isomorphism. Recall that by assumption $\mathcal{X}$ is U-smooth, and thus S is regular.

By Proposition 5.1 there exists a finite étale covering $\theta_0 : \widetilde{S} \to S$, a section $\delta : T \to \widetilde{S}$ of θ_0 over T and an isomorphism

$$\Phi_0 : \theta_0^*(G_{\mathrm{const},S}) \to \theta_0^*(G_{\mathcal{X}}|_S)$$

such that $\delta^*\Phi_0 = \varphi$. Replacing $\widetilde{S}$ with a connected component of $\widetilde{S}$ which contains $\delta(T) = \delta(\Delta(U))$ we may and will assume that $\widetilde{S}$ is irreducible. We can extend these data to a neighborhood $\mathcal{V}$ of $\{y_1, \dots, y_n\}$ and get the diagram

$$\begin{array}{ccccccc} & & \widetilde{S} & \hookrightarrow & \widetilde{\mathcal{V}} & & \\ & \overset{\delta}{\nearrow} & \downarrow{\scriptstyle \theta_0} & & {\scriptstyle \theta}\downarrow & & \\ T & \hookrightarrow & S & \hookrightarrow & \mathcal{V} & \hookrightarrow & \mathcal{X} \end{array} \tag{12}$$

where $\pi : \widetilde{\mathcal{V}} \to \mathcal{V}$ finite étale, and an isomorphism $\Phi : \theta^*(G_{\mathrm{const}}) \to \theta^*(G_{\mathcal{X}})$.

Since T isomorphically projects onto U, it is still closed, viewed as a sub-scheme of $\mathcal{V}$. Note that since $\mathcal{Y}$ is semi-local and $\mathcal{V}$ contains all of its closed points, $\mathcal{V}$ contains $\Pi^{-1}(\Pi(\mathcal{Y})) = \mathcal{Y}$. By Lemma 5.5 there exists an open subset $\mathcal{W} \subseteq \mathcal{V}$ containing $\mathcal{Y}$ and endowed with a finite surjective U-morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$.

Let $\mathcal{X}' = \theta^{-1}(\mathcal{W})$, $f' = \theta^*(f)$, $q'_U = q_U \circ \theta$, and let $\Delta' : U \to \mathcal{X}'$ be the section of q'_U obtained as the composition of δ with Δ. We claim that the triple $(\mathcal{X}', f', \Delta')$ is a nice triple. Let us verify this. Firstly, the structure morphism $q'_U : \mathcal{X}' \to U$ coincides with the composition

$$\mathcal{X}' \xrightarrow{\theta} \mathcal{W} \hookrightarrow \mathcal{X} \xrightarrow{q_U} U.$$

Thus, it is smooth. The element f' belongs to the ring $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$, and the morphism Δ' is a section of q'_U. Each component of each fibre of the morphism q_U has dimension one, and the morphism $\mathcal{X}' \xrightarrow{\theta} \mathcal{W} \hookrightarrow \mathcal{X}$ is étale. Thus, each component of each fibre of the morphism q'_U is also of dimension one. Since $\{f = 0\} \subset \mathcal{W}$ and $\theta : \mathcal{X}' \to \mathcal{W}$ is finite, $\{f' = 0\}$ is finite over $\{f = 0\}$ and hence also over U. In other words, the $\mathcal{O}$-module $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/f' \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is finite. The morphism $\theta : \mathcal{X}' \to \mathcal{W}$ is finite and surjective. We have constructed above in Lemma 5.5 the finite surjective morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$. It follows that $\Pi^* \circ \theta : \mathcal{X}' \to \mathbf{A}^1 \times U$ is finite and surjective.

Clearly, the étale morphism $\theta : \mathcal{X}' \to \mathcal{X}$ is a morphism of nice triples, with $g = 1$.

Denote the restriction of Φ to $\mathcal{X}'$ simply by Φ. The equality $(\Delta')^*\Phi = \mathrm{id}_{G_U}$ holds by the very construction of the isomorphism Φ. Theorem 3.6 follows. □

6. A basic nice triple

Let k be an infinite field. Fix a smooth geometrically irreducible affine k-scheme X, and a finite family of points $x_1, x_2, \dots, x_n$ on X, and a non-zero function $\mathrm{f} \in k[X]$, which vanishes at each of the x_i for $i = 1, 2, \dots, n$. Let $\mathcal{O} = \mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$ be the semi-local ring of the family $x_1, x_2, \dots, x_n$ on X, $U = \mathrm{Spec}(\mathcal{O})$ and $\mathrm{can} : U \hookrightarrow X$ the canonical inclusion of schemes. The definition of a nice triple over U is given in Definition 3.1. The main aim of the present section is to prove the following.

Proposition 6.1. *One can shrink X such that $x_1, x_2, \dots, x_n$ are still in X and X is affine, and then construct a nice triple $(q_U : \mathcal{X} \to U, \Delta, f)$ over U and an essentially smooth morphism $q_X : \mathcal{X} \to X$ such that $q_X \circ \Delta = \mathrm{can}$, $f = q_X^*(\mathrm{f})$ and the set of closed points of $\Delta(U)$ is contained in the set of closed points of $\{f = 0\}$.*

Proof. By Proposition 2.3 there exist a Zariski open neighborhood X^0 of the family $\{x_1, \dots, x_n\}$ and an almost elementary fibration $p : X^0 \to S$, where S is an open sub-scheme of the projective space $\mathbf{P}^{\dim X - 1}$, such that

$$p|_{\{\mathrm{f}=0\} \cap X^0} : \{\mathrm{f} = 0\} \cap X^0 \to S$$

is finite surjective. Let $s_i = p(x_i) \in S$, for each $1 \leqslant i \leqslant n$. Shrinking S, we may assume that S is *affine* and still contain the family $\{s_1, s_2, \ldots, s_n\}$. Clearly, in this case $p^{-1}(S) \subseteq X^0$ contains the family $\{x_1, x_2, \ldots, x_n\}$. We replace X by $p^{-1}(S)$ and f by its restriction to this new X.

In this way we get an almost elementary fibration $p : X \to S$ such that

$$\{x_1, \ldots, x_n\} \subset \{\mathrm{f} = 0\} \subset X,$$

S is an open affine sub-scheme in the projective space $\mathbf{P}^{\dim X - 1}$, and the restriction $p|_{\{\mathrm{f}=0\}} : \{\mathrm{f} = 0\} \to S$ of p to the vanishing locus of f is a finite surjective morphism. In other words, $k[X]/(\mathrm{f})$ is finite as a $k[S]$-module.

As an open affine sub-scheme of the projective space $\mathbf{P}^{\dim X - 1}$ the scheme S is regular. By Proposition 2.4 one can shrink S in such a way that S is still affine and contain the family $\{s_1, s_2, \ldots, s_n\}$ and there exists a finite surjective morphism

$$\pi : X \to \mathbf{A}^1 \times S$$

such that $p = \mathrm{pr}_S \circ \pi$. Clearly, in this case $p^{-1}(S) \subseteq X$ contains the family $\{x_1, x_2, \ldots, x_n\}$. We replace X by $p^{-1}(S)$ and f by its restriction to this new X.

In this way we get an almost elementary fibration $p : X \to S$ such that

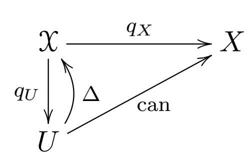
$$\{x_1, \ldots, x_n\} \subset \{\mathrm{f} = 0\} \subset X,$$

S is an open affine sub-scheme in the projective space $\mathbf{P}^{\dim X - 1}$, and the restriction $p|_{\{\mathrm{f}=0\}} : \{\mathrm{f} = 0\} \to S$ is a finite surjective morphism. Eventually we conclude that there exists a finite surjective morphism $\pi : X \to \mathbf{A}^1 \times S$ such that $p = \mathrm{pr}_S \circ \pi$.

Let $p_U = p \circ \mathrm{can} : U \to S$, where $U = \mathrm{Spec}(\mathcal{O})$ and $\mathrm{can} : U \hookrightarrow X$ are as above. Further, we consider the fibre product

$$\mathcal{X} := U \times_S X.$$

Then the canonical projections $q_U : \mathcal{X} \to U$ and $q_X : \mathcal{X} \to X$ and the diagonal morphism $\Delta : U \to \mathcal{X}$ can be included in the diagram



(13)

where

$$q_X \circ \Delta = \mathrm{can} \tag{14}$$

and

$$q_U \circ \Delta = \mathrm{id}_U. \tag{15}$$

Note that q_U is *a smooth morphism with geometrically irreducible fibres of dimension one*. Indeed, observe that q_U is a base change via p_U of the morphism p which has the desired properties. Note that $\mathcal{X}$ is irreducible. Indeed, U is irreducible and the fibre of q_U over the generic point of U is irreducible.

Taking the base change via p_U of the finite surjective morphism $\pi : X \to \mathbf{A}^1 \times S$, we get *a finite surjective morphism*

$$\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$$

such that $q_U = \mathrm{pr}_U \circ \Pi$, where $pr_U : \mathbf{A}^1 \times U \to U$ is the natural projection.

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Set $f := q_X^*(\mathrm{f})$. The $\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$-module $\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})/f\cdot\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is finite, since the $k[S]$-module $k[X]/\mathrm{f}\cdot k[X]$ is finite.

Now the data

$$(q_U : \mathcal{X} \to U, f, \Delta) \tag{16}$$

form an example of a *nice triple* as in Definition 3.1. Moreover, we have the following claim.

Claim 6.2. The schemes $\Delta(U)$ and $\{f = 0\}$ are both semi-local and the set of closed points of $\Delta(U)$ is contained in the set of closed points of $\{f = 0\}$.

This holds since the set $\{x_1, x_2, \dots, x_n\}$ is contained in the vanishing locus of the function f. The nice triple (16) together with the essentially smooth morphism q_X are the required one. Whence the proposition. □

7. Main construction

The main result of this section is Corollary 7.2.

Fix a k-smooth irreducible affine k-scheme X, a finite family of points $x_1, x_2, \dots, x_n$ on X, and set $\mathcal{O} := \mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$ and $U := \mathrm{Spec}(\mathcal{O})$. Let A be the Noetherian k-algebra from Theorem 1.1 and $T = \mathrm{Spec}(A)$. Further, consider a simple simply connected U-group scheme G and a principal G-bundle P over $\mathcal{O}\otimes_k A$ which is trivial over $K\otimes_k A$ for the field of fractions K of $\mathcal{O}$. We may and will assume that for certain $\mathrm{f}\in\mathcal{O}$ the principal G-bundle P is trivial over $\mathcal{O}_{\mathrm{f}}\otimes_k A$.

Shrinking X if necessary, we may secure the following properties.

(i) The points $x_1, x_2, \dots, x_n$ are still in X and X is affine.

(ii) The group scheme G is defined over X and it is a simple group scheme. We will often denote this X-group scheme by G_X and write G_U for the original G.

(iii) The principal G_U-bundle P is the restriction to $U\times_{\mathrm{Spec}(k)}T$ of a principal G_X-bundle P_X over $X\times_{\mathrm{Spec}(k)}T$ and $\mathrm{f}\in k[X]$. We often will write P_U for the original principal G_U-bundle P over $U\times_{\mathrm{Spec}(k)}T$.

(iv) The restriction P_{f} of the bundle P_X to the principal open subset $X_{\mathrm{f}}\times_{\mathrm{Spec}(k)}T$ is trivial and f vanishes at each x_i.

If we shrink X further such that property (i) is secured, then we automatically secure properties (ii) to (iv). For any such X we will write $\mathrm{can}: U\hookrightarrow X$ for the canonical embedding.

After substituting k by its algebraic closure $\widetilde{k}$ in $k[X]$, and T by $\widetilde{T} = \mathrm{Spec}(\widetilde{k})\times_{\mathrm{Spec}(k)}T$, we can assume that X is a $\widetilde{k}$-smooth geometrically irreducible affine $\widetilde{k}$-scheme. Note that $U\times_{\mathrm{Spec}(\widetilde{k})}\widetilde{T}\cong U\times_{\mathrm{Spec}(k)}T$ as U-schemes, and the same holds for X instead of U. To simplify the notation, we will continue to denote this new $\widetilde{k}$ by k and $\widetilde{T}$ by T.

In particular, we are given now the smooth geometrically irreducible affine k-scheme X, the finite family of points $x_1, x_2, \dots, x_n$ on X, and the non-zero function $\mathrm{f}\in k[X]$ vanishing at each point x_i. We may shrink X further securing property (i) and construct the nice triple $(q_U : \mathcal{X}\to U, \Delta, f)$ over U and the essentially smooth morphism $q_X : \mathcal{X}\to U$ as in Proposition 6.1. Since property (i) is secured properties (ii) to (iv) are secured too. Consider the $\mathcal{X}$-group scheme $G_{\mathcal{X}} := (q_X)^*(G_X)$. Note that the U-group scheme $\Delta^*(G_{\mathcal{X}})$ coincides with G_U from item (ii) since $\mathrm{can} = q_X\circ\Delta$ by Proposition 6.1. Consider one more $\mathcal{X}$-group scheme, namely

$$G_{\mathrm{const}} := (q_U)^*(\Delta^*(G_{\mathcal{X}})) = (q_U)^*(G_U).$$

By Theorem 3.6 there exists a morphism of nice triples

$$\theta : (q'_U : \mathcal{X}' \to U, f', \Delta') \to (q_U : \mathcal{X} \to U, f, \Delta)$$

and an isomorphism

$$\Phi : \theta^*(G_{\mathrm{const}}) \to \theta^*(G_{\mathcal{X}}) =: G_{\mathcal{X}'} \tag{17}$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = \mathrm{id}_{G_U}$. Set

$$q'_X = q_X \circ \theta : \mathcal{X}' \to X. \tag{18}$$

Recall that

$$q'_U = q_U \circ \theta : \mathcal{X}' \to U, \tag{19}$$

since θ is a morphism of nice triples.

Note that, since by Claim 6.2 f vanishes on all closed points of $\Delta(U)$, and θ is a morphism of nice triples, f' vanishes on all closed points of $\Delta'(U)$ as well. Therefore, the nice triple $(q'_U : \mathcal{X}' \to U, f', \Delta' : U \to \mathcal{X}')$ is subject to Theorem 3.4.

By Theorem 3.4 there exists a finite surjective morphism $\sigma : \mathcal{X}' \to \mathbf{A}^1 \times U$ of U-schemes satisfying (1) to (3) from that theorem. In particular, one has equalities of closed subsets

$$\sigma^{-1}(\sigma(\{f' = 0\})) = \{\sigma^*(N(f')) = 0\} = \{f' = 0\} \sqcup \{g_{f',\sigma} = 0\}$$

with $N(f')$ and $g_{f',\sigma}$ defined in item (2) of Theorem 3.4. Thus, replacing for brevity $g_{f',\sigma}$ by g', one gets the following elementary distinguished square in the category of U-smooth schemes (see item 2 of Theorem 3.4).

$$\begin{array}{ccc} (\mathcal{X}')^0_{N(f')} = (\mathcal{X}')^0_{f'g'} & \xrightarrow{\mathrm{inc}} & (\mathcal{X}')^0_{g'} \\ \downarrow{\scriptstyle \sigma^0_{f'g'}} & & \downarrow{\scriptstyle \sigma^0_{g'}} \\ (\mathbf{A}^1 \times U)_{N(f')} & \xrightarrow{\mathrm{inc}} & \mathbf{A}^1 \times U \end{array} \tag{20}$$

By Remark 3.5 we may and will assume that $(\mathcal{X}')^0$ and $(\mathcal{X}')^0_{g'}$ are affine schemes. Thus the scheme $(\mathcal{X}')^0_{g'}$ is of the form $\mathrm{Spec}(B)$ for a finitely generated étale $\mathcal{O}[t]$-algebra B. By Lemma 3.3 the top coefficient of the polynomial $N(f') \in \mathcal{O}[t]$ is a unit in $\mathcal{O}$. By item (3) of Theorem 3.4 the inclusion $(\sigma^0_{g'})^* : \mathcal{O}[t] \hookrightarrow B$ induces a ring isomorphism

$$\mathcal{O}[t]/(N(f')) \to B/N(f')B = B/f'B.$$

Hence we are under the assumptions of Lemma A.2. Lemma A.2 shows that the triple

$$(\mathcal{O}[t] \otimes_k A,\ (\sigma^0_{g'})^* \otimes \mathrm{id} : \mathcal{O}[t] \otimes_k A \to B \otimes_k A,\ s = N(f') \otimes 1 \in \mathcal{O}[t] \otimes_k A) \tag{21}$$

is subject to the assumptions of [CO92, Proposition 2.6(iv)].

Below, we use this to construct principal G_U-bundles over $(\mathbf{A}^1 \times U) \times_{\mathrm{Spec}(k)} T$ out of the following initial data: a principal G_U-bundle over $(\mathcal{X}')^0_{g'} \times_{\mathrm{Spec}(k)} T$, the trivial principal G_U-bundle over $(\mathbf{A}^1 \times U)_{N(f')} \times_{\mathrm{Spec}(k)} T$, and a principal G_U-bundle isomorphism of their pull-backs to $(\mathcal{X}')^0_{N(f')} \times_{\mathrm{Spec}(k)} T$.

Set

$$Q'_X = q'_X \times \mathrm{id}_T : \mathcal{X}' \times_{\mathrm{Spec}(k)} T \to X \times_{\mathrm{Spec}(k)} T,$$
$$Q'_U = q'_U \times \mathrm{id}_T : \mathcal{X}' \times_{\mathrm{Spec}(k)} T \to U \times_{\mathrm{Spec}(k)} T.$$

Consider $(Q'_X)^*(P_X)$ as a principal $(q'_U)^*(G_U) = \theta^*(G_{\mathrm{const}})$-bundle via the isomorphism Φ. Recall that P_X is trivial as a principal G_X-bundle over $X_{\mathrm{f}} \times_{\mathrm{Spec}(k)} T$. Therefore, $(Q'_X)^*(P_X)$ is trivial as a principal $\theta^*(G_{\mathcal{X}})$-bundle over $\mathcal{X}'_{f'} \times_{\mathrm{Spec}(k)} T$. So, $(Q'_X)^*(P_X)$ is trivial over $\mathcal{X}'_{f'} \times_{\mathrm{Spec}(k)} T$, when regarded as a principal $(q'_U)^*(G_U) = \theta^*(G_{\mathrm{const}})$-bundle via the isomorphism Φ.

Thus, regarded as a principal G_U-bundle, the bundle $(Q'_X)^*(P_X)$ over $\mathcal{X}' \times_{\mathrm{Spec}(k)} T$ becomes trivial over $\mathcal{X}'_{f'} \times_{\mathrm{Spec}(k)} T$, and *a fortiori* over $(\mathcal{X}')^0_{N(f')} \times_{\mathrm{Spec}(k)} T$. Take the trivial G_U-bundle over $(\mathbf{A}^1 \times U)_{N(f')} \times_{\mathrm{Spec}(k)} T$ and an isomorphism

$$\psi : G_U \times_U [(\mathcal{X}')^0_{N(f')} \times_{\mathrm{Spec}(k)} T] \to (Q'_X)^*(P_X)|_{[(\mathcal{X}')^0_{N(f')} \times_{\mathrm{Spec}(k)} T]} \tag{22}$$

of the principal G_U-bundles. As mentioned above, the triple (21) is subject to the assumptions of [CO92, Proposition 2.6(iv)]. The latter statement implies that one can find *a principal G_U-bundle $\mathcal{G}_t$ over* $(\mathbf{A}^1 \times U) \times_{\mathrm{Spec}(k)} T$ such that the following hold.

(1) $\mathcal{G}_t|_{[(\mathbf{A}^1 \times U)_{N(f')} \times_{\mathrm{Spec}(k)} T]} = G_U \times_U [(\mathbf{A}^1 \times U)_{N(f')} \times_{\mathrm{Spec}(k)} T]$.

(2) There is an isomorphism $\varphi : [(\sigma^0_{g'}) \times \mathrm{id}_T]^*(\mathcal{G}_t) \to (Q'_X)^*(P_X)|_{[(\mathcal{X}')^0_{g'} \times_{\mathrm{Spec}(k)} T]}$ of the principal G_U-bundles, where $(Q'_X)^*(P_X)$ is regarded as a principal G_U-bundle via the $\mathcal{X}'$-group scheme isomorphism Φ from (17).

Finally, form the following diagram.

$$\begin{array}{ccccc}
(\mathbf{A}^1 \times U) \times_{\mathrm{Spec}(k)} T & \xleftarrow{\sigma^0_{g'} \times \mathrm{id}} & (\mathcal{X}')^0_{g'} \times_{\mathrm{Spec}(k)} T & \xrightarrow{Q'_X = q'_X \times \mathrm{id}} & X \times_{\mathrm{Spec}(k)} T \\
& \searrow^{\mathrm{pr}_U \times \mathrm{id}} & q'_U \times \mathrm{id} \downarrow \uparrow \Delta' \times \mathrm{id} & \nearrow^{\mathrm{can} \times \mathrm{id}} & \\
& & U \times_{\mathrm{Spec}(k)} T & &
\end{array} \tag{23}$$

This diagram is well defined, since by item (4) of Theorem 3.4 the image of the morphism Δ' lands in $(\mathcal{X}')^0_{g'}$.

Theorem 7.1. *The principal G_U-bundle $\mathcal{G}_t$ over $(\mathbf{A}^1 \times U) \times_{\mathrm{Spec}(k)} T$, the monic polynomial $N(f') \in \mathcal{O}[t]$, the diagram (23), and the isomorphism Φ from (17) constructed above, satisfy the following conditions (1*)–(6*).*

(1*) $q'_U = \mathrm{pr}_U \circ \sigma^0_{g'}$.

(2*) $\sigma^0_{g'}$ *is étale.*

(3*) $q'_U \circ \Delta' = \mathrm{id}_U$.

(4*) $q'_X \circ \Delta' = \mathrm{can}$.

(5*) *The restriction of $\mathcal{G}_t$ to $(\mathbf{A}^1 \times U)_{N(f')} \times_{\mathrm{Spec}(k)} T$ is a trivial G_U-bundle.*

(6*) *$(\sigma^0_{g'} \times \mathrm{id})^*(\mathcal{G}_t)$ and $(Q'_X)^*(P_X)$ are isomorphic as G_U-bundles over $(\mathcal{X}')^0_{g'} \times_{\mathrm{Spec}(k)} T$. Here $(Q'_X)^*(P_X)$ is regarded as a principal G_U-bundle via the group scheme isomorphism Φ from (17).*

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Proof. By the choice of σ it is an U-scheme morphism, which proves property (1*). By the choice of $(\mathcal{X}')^0 \hookrightarrow \mathcal{X}'$ in Theorem 3.4, the morphism σ is étale on this sub-scheme, hence one gets property (2*). Property (3*) holds for Δ' since $(q'_X : \mathcal{X}' \to U, f', \Delta')$ is a nice triple and, in particular, Δ' is a section of q'_U. Property (4*) can be established as follows:

$$q'_X \circ \Delta' = (q_X \circ \theta) \circ \Delta' = q_X \circ \Delta = \text{can}.$$

The first equality here holds by the definition of q'_X, the second one holds since ρ_S is a morphism of nice triples; the third one follows from equality (14). Property (5*) is just property (1) in the above construction of $\mathcal{G}_t$. Property (6*) is precisely property (2) in the construction of $\mathcal{G}_t$. □

The composition

$$s' := \sigma^0_{g'} \circ \Delta' : U \to \mathbf{A}^1 \times U$$

is a section of the projection pr_U by properties (1*) and (3*). Recall that G_U over U is the original group scheme G introduced in the very beginning of this section. Since U is semi-local, we may assume that *s' is the zero section of the projection $\mathbf{A}^1_U \to U$*. Furthermore, making an affine transformation of $\mathbf{A}^1_U \to U$, we may assume that *$N(f')(1) \in \mathcal{O}$ is invertible.*

Corollary 7.2 (Theorem 1.2). *The principal G_U-bundle $\mathcal{G}_t$ over $\mathbf{A}^1_{[U\times_{\mathrm{Spec}(k)}T]}$ and the monic polynomial $N(f') \in \mathcal{O}[t]$ are subject to the following conditions.*

(i) *The restriction of $\mathcal{G}_t$ to $[(\mathbf{A}^1 \times U)_{N(f')} \times_{\mathrm{Spec}(k)} T]$ is a trivial G_U-bundle.*
(ii) *The restriction of $\mathcal{G}_t$ to $\{0\} \times U \times_{\mathrm{Spec}(k)} T$ is the original G_U-bundle P_U.*
(iii) *$N(f')(1) \in \mathcal{O}$ is invertible.*

Proof. Property (i) is just property (5*) above. Now by property (6*) the G_U-bundles

$$\mathcal{G}_t|_{\{0\}\times U\times_{\mathrm{Spec}(k)}T} = (s' \times \mathrm{id})^*(\mathcal{G}_t) = (\Delta' \times \mathrm{id})^*((\sigma^0_{g'} \times \mathrm{id})^*(\mathcal{G}_t)) \quad \text{and}$$
$$(\Delta' \times \mathrm{id})^*(Q'_X)^*(P_X) = (\text{can} \times \mathrm{id})^*(P_X)$$

are isomorphic, since $\Delta'^*(\Phi) = \mathrm{id}_{G_U}$. It remains to recall that the principal G_U-bundle $(\text{can} \times \mathrm{id})^*(P_X)$ is the original G_U-bundle P_U by the choice of P_X. Whence the corollary. □

8. Group of points of an isotropic simple group

In this section we establish several results concerning groups of points of simple groups, including, in particular, Lemma 8.2, Proposition 8.5 and Lemma 8.6, which play a crucial role in the rest of the paper.

Definition 8.1. Let G be a reductive group scheme over a commutative ring A. Assume that G has a proper parabolic subgroup $P = P^+$ over A, and denote by U^+ its unipotent radical. By [SGA3, Exp. XXVI, Corollary 2.3, Theorem 4.3.2] there exists a parabolic A-subgroup P^- of G opposite to P^+. Let U^- be the unipotent radical of P^-. For any commutative A-algebra B we define the P-elementary subgroup $E_P(B)$ of the group $G(B)$ as follows:

$$E_P(B) = \langle U^+(B), U^-(B)\rangle.$$

Note that $E_P(B)$ does not depend on the choice of P^-, since by [SGA3, Exp. XXVI, Corollary 1.8] any two such subgroups are conjugate by an element of $U^+(A)$.

Lemma 8.2. *Let $B \to \bar{B}$ be a surjective A-algebra homomorphism. Then the induced homomorphism of elementary groups $E_P(B) \to E_P(\bar{B})$ is also surjective.*

Proof. By [SGA3, Exp.XXVI, Corollary 2.5] the A-schemes U^+ and U^- are isomorphic to A-vector bundles of finite rank. Thus, the maps $U^{\pm}(B) \to U^{\pm}(\bar{B})$ are surjective. □

Let l be a field and G_l be an isotropic simple simply connected l-group scheme. Recall that an isotropic scheme contains an l-split rank-1 torus $\mathbb{G}_{m,l}$. Choose and fix two opposite parabolic subgroups $P_l = P_l^+$ and P_l^- of the l-group scheme G_l. Let U_l^+ and U_l^- be their unipotent radicals. We will be interested mostly in the group of points $G_l(l(t))$. The following definition originates from [Tit64, Main theorem].

Definition 8.3. Let L be a field extension of l. Define $G_l(L)^+$ as the subgroup of the group $G_l(L)$ generated by L-points of unipotent radicals of all parabolic subgroups of G_l defined over the field l.

By definition the group $G_l(L)^+$ is generated by unipotent radicals of *all* l-parabolic subgroups, and thus contains the elementary group $E_{P_l}(L)$, introduced in Definition 8.1. In fact they coincide.

Proposition 8.4. *The group $G_l(L)^+$ is generated by L-points of unipotent radicals of any two opposite parabolic subgroups of the l-group scheme G_l. In particular, one has the equality*

$$G_l(l(t))^+ = \langle U_l^+(l(t)), U_l^-(l(t))\rangle = E_{P_l}(l(t)). \tag{24}$$

Proof. Set $G_L = G_l \times_{\operatorname{Spec} l} \operatorname{Spec} L$. The group $G_l(L)^+$ is contained in the subgroup of $G_l(L) = G_L(L)$ generated by L-points of unipotent radicals of all parabolic subgroups of the group scheme G_L defined over the field L. By [BT73, Proposition 6.2.(v)] the latter group is generated by L-points of unipotent radicals of any two opposite parabolic subgroups of G_L, in particular, by L-points of U_L^+ and U_L^-. Since $U_L^{\pm}(L) = U_l^{\pm}(L)$, the claim follows. □

The following result is crucial for the what follows.

Proposition 8.5. *One has the equality*

$$G_l(l(t)) = G_l(l(t))^+ \cdot G_l(l), \tag{25}$$

where $G_l(l(t))^+$ is the group defined in Definition 8.3.

Proof. This is proved in [Gil07, Théorème 5.8]. □

Let $f(t) \in l[t]$ be a monic polynomial of degree $n = \deg(f)$. We consider a polynomial $\hat{f}$ in t^{-1} defined as follows

$$\hat{f}(t^{-1}) := f(t)/t^n \in l[t^{-1}].$$

Clearly, $\hat{f}(0) = 1$. If $f_1, f_2 \in l[t]$ are two monic polynomials, then $(f_1 f_2)\hat{} = \hat{f}_1 \hat{f}_2$. Also $\hat{t}^n = 1$. Clearly, for any two monic polynomials $f(t), h(t) \in l[t]$ there are the following l-algebra inclusions

$$l[t]_f \subseteq l[t]_{fh} \subseteq l[t]_{tfh} = l[t, t^{-1}]_{fh} = l[t^{-1}, t]_{\hat{f}\hat{h}}. \tag{26}$$

This is used in the following Lemma.

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LEMMA 8.6. *Let P_l be an arbitrary parabolic l-subgroup of G_l. Let $f(t) \in l[t]$ be a monic polynomial. For each $\alpha \in G_l(l[t]_f)$ one can find a monic polynomial $h(t) \in l[t]$, and elements*

$$\beta \in G_l(l), \quad u \in E_{P_l}(l[t]_{tfh}) = E_{P_l}(l[t^{-1}, t]_{\hat{f}\hat{h}}),$$

such that

$$\alpha = u\beta \quad \text{in} \quad G_l(l[t]_{tfh}) = G_l(l[t^{-1}, t]_{\hat{f}\hat{h}}). \tag{27}$$

Proof. Consider $\alpha \in G_l(l[t]_f)$ as an element of $G_l(l(t))$. The equalities (25) and (24) imply that there exist a monic polynomial $h(t) \in l[t]$, and elements

$$u \in E_{P_l}(l[t]_{tfh}), \quad \beta \in G_l(l),$$

such that $\alpha = u\beta$ in $G_l(l[t]_{tfh})$. By (26) one has $G_l(l[t]_{tfh}) = G_l(l[t^{-1}, t]_{\hat{f}\hat{h}})$ and $E_{P_l}(l[t]_{tfh}) = E_{P_l}(l[t^{-1}, t]_{\hat{f}\hat{h}})$. □

9. Principal G-bundles on a projective line

The main result of the present section is Corollary 9.8, which implies Theorem 1.3.

Let B be a Noetherian commutative ring, and let $\mathbf{A}^1_B$ and $\mathbf{P}^1_B$ be the affine line and the projective line over B, respectively. Usually we identify the affine line with a sub-scheme of the projective line as follows: $\mathbf{A}^1_B = \mathbf{P}^1_B - (\{\infty\} \times \mathrm{Spec}(B))$, where $\infty = [0:1] \in \mathbf{P}^1$. Let G be a semi-simple B-group scheme, let P a principal G-bundle over $\mathbf{A}^1_B$, and let $p : P \to \mathbf{A}^1_B$ be the corresponding canonical projection.

For a monic polynomial

$$f = f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 \in B[t]$$

we set $P_f = p^{-1}((\mathbf{A}^1_B)_f)$. Clearly, it is a principal G-bundle over $(\mathbf{A}^1_B)_f$. Further, we denote by

$$F(T_0, T_1) = T_1^n + a_{n-1}T_1^{n-1}T_0 + \cdots + a_0 T_0^n$$

the corresponding homogeneous polynomial in two variables. Note that the intersection of the principal open set in $\mathbf{P}^1_B$ defined by $F \neq 0$ with the affine line $\mathbf{A}^1_B$ equals the principal open subset $(\mathbf{A}^1_B)_f$. As in the previous section consider the polynomial $\hat{f}(t^{-1}) \in B[t^{-1}]$ in t^{-1} defined as $\hat{f}(t^{-1}) = f(t)/t^n$. Clearly, $\hat{f}(0) = 1$. If $f_1, f_2 \in B[t]$ are two monic polynomials, then $(f_1 f_2)\hat{} = \hat{f}_1 \hat{f}_2$. Also $\hat{t^n} = 1$.

DEFINITION 9.1. Let $f(t) \in B[t]$ be a monic polynomial. Let $\varphi : G_{(\mathbf{A}^1_B)_f} \to P_f$ be a principal G-bundle isomorphism. We write $P(\varphi, f)$ for the principal G-bundle over the projective line $\mathbf{P}^1_B$ obtained by gluing P and $G_{(\mathbf{P}^1_B)_F}$ over $(\mathbf{A}^1_B)_f$ via the principal G-bundle isomorphism φ.

LEMMA 9.2. *For any φ and f as in Definition 9.1, we have the following.*

(i) *The principal G-bundles $P(\varphi, f)$ and $P(\varphi, fg)$ coincide for each monic polynomial $g \in B[t]$.*
(ii) *For any monic polynomial $h(t) \in B[t]$ and any $\beta \in G(B[t^{-1}]_{\hat{f}\hat{h}})$, the principal G-bundles $P(\varphi, f)$ and $P(\varphi \circ \beta, tfh)$ are isomorphic.*

Proof. The first assertion is clear. To prove the second assertion, note that

$$\infty \times \mathrm{Spec}(B) \subset (\mathbf{P}^1_B)_{T_1 FH} = \mathrm{Spec}(B[t^{-1}]_{\hat{f}\hat{h}}).$$

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By the first assertion, the principal G-bundles $P(\varphi, f)$ and $P(\varphi, fth)$ coincide. It remains to note that the principal G-bundles $P(\varphi\circ\beta, fth)$ is isomorphic to the principal G-bundles $P(\varphi, fth)$. An isomorphism is given by the identity map $\mathrm{id}: P \to P$ over $\mathbf{A}^1_B$ and by the map $r_\beta : G_{(\mathbf{P}^1_B)_{T_1FH}} \to G_{(\mathbf{P}^1_B)_{FT_1H}}$, which is the right translation via the element $\beta \in G(B[t^{-1}]_{\hat{f}\hat{h}})$, over $\mathrm{Spec}(B[t^{-1}]_{\hat{f}\hat{h}})$. □

Lemma 9.3. *Let l be a field and G_l be a semi-simple l-group scheme. Let $f \in l[t]$ be a non-constant monic polynomial. Let P be a principal G_l-bundle over $\mathbf{A}^1_l$ such that P_f is trivial over $\mathbf{A}^1_f$. Let $\varphi : G_{\mathbf{A}^1_f} \to P_f$ be a principal G_l-bundle isomorphism. Let $P(\varphi, f)$ be the corresponding principal G-bundle over $\mathbf{P}^1_l$. Then there exists an $\alpha \in G(l[t]_f)$ such that the principal G-bundle $P(\varphi\circ\alpha, f)$ is trivial over $\mathbf{P}^1_l$.*

Proof. By [CO92, Proposition 2.2] one has

$$\ker[\mathrm{H}^1_{\text{ét}}(l[t], G_l) \to \mathrm{H}^1_{\text{ét}}(l(t), G_l)] = *.$$

So, we may assume that there is an isomorphism $G_{\mathbf{A}^1_l} = P$ over $\mathbf{A}^1_l$. In this case the above isomorphism φ coincides with the right multiplication by an element $\beta \in G_l(l[t]_f)$. Clearly, $P(\beta\circ\beta^{-1}, f)$ is trivial over $\mathbf{P}^1_l$. Thus, $P(\varphi\circ\alpha, f)$ is trivial for $\alpha = \beta^{-1}$. □

Corollary 9.4. *Let l be a field, and let G_l be an isotropic simply connected simple l-group scheme with a parabolic l-subgroup Q_l. Let P be a G_l-bundle over $\mathbf{A}^1_l$. Further, let $f(t) \in l[t]$ be a non-constant monic polynomial, $\varphi : G_{\mathbf{A}^1_f} \to P_{\mathbf{A}^1_f}$ be a principal G_l-bundle isomorphism and let $P(\varphi, f)$ be the corresponding principal G_l-bundle on the projective line $\mathbf{P}^1_l$. Then there exist a monic $h(t) \in l[t]$ and $u \in E_{Q_l}(l[t]_{tfh})$ such that the principal G_l-bundle $P(\varphi\circ u, tfh)$ is trivial over $\mathbf{P}^1_l$.*

Proof. By Lemma 9.3 there exists an $\alpha \in G_l(l[t]_f)$ such that the principal G_l-bundle $P(\varphi\circ\alpha, f)$ is trivial. By Lemma 8.6 there exist a monic polynomial $h(t) \in l[t]$ and elements

$$u \in E_{Q_l}(l[t^{-1}, t]_{\hat{f}\hat{h}}) = E_{Q_l}(l[t]_{tfh}), \quad \beta \in G_l(l)$$

such that

$$\alpha = u\beta \in G_l(l[t^{-1}, t]_{\hat{f}\hat{h}}) = G_l(l[t]_{tfh}). \tag{28}$$

The following chain of principal G_l-bundle isomorphisms completes the proof:

$$G_l \times_{\mathrm{Spec}(l)} \mathbf{P}^1_l = P(\varphi\circ\alpha, f) = P(\varphi\circ\alpha,\ tfh) = P(\varphi\circ u\circ\beta, tfh) \cong P(\varphi\circ u,\ tfh).$$

Here all the equalities are obvious. The last isomorphism holds by Lemma 9.2, since $\beta \in G_l(l[t^{-1}]_{\hat{f}\hat{h}})$. □

Let B' be a Noetherian semi-local ring. Let G be a simple simply connected B'-group scheme. Let $\mathfrak{m}_i \subseteq B'$, $i = 1, 2, \ldots, n$, be all maximal ideals of B'. Let J be the intersection of all $\mathfrak{m}_i$, $1 \leqslant i \leqslant n$. Then

$$l := B'/J = l_1 \times l_2 \times \cdots \times l_n,$$

where $l_i = B'/\mathfrak{m}_i$. Let $G_l = G \otimes_{B'} l$ be the fibre of G over $\mathrm{Spec}(l)$. In what follows, we write $\mathbf{P}^1$ and $\mathbf{A}^1$ for $\mathbf{P}^1_{B'}$ and $\mathbf{A}^1_{B'}$ respectively, whereas $\mathbf{P}^1_l$ and $\mathbf{A}^1_l$ denote the projective line and the affine line over l.

Let $f \in B'[t]$ be a monic polynomial, and let P be a principal $G_{B'}$-bundle over $\mathbf{A}^1$ such that $P_{\mathbf{A}^1_f}$ is trivial. Let $\varphi : G_{\mathbf{A}^1_f} \to P_{\mathbf{A}^1_f}$ be a principal G-bundle isomorphism, and let $P(\varphi, f)$ be the corresponding principal G-bundle on $\mathbf{P}^1$ (see Definition 9.1).

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Proposition 9.5. *Assume that the group scheme* G *over* B' *is isotropic, simple and simply connected. Then there exist a monic polynomial* $h(t) \in B'[t]$ *and an element* $\alpha \in G(B'[t]_{tfh})$ *such that the principal* G*-bundle* $P(\varphi \circ \alpha, tfh)$ *satisfies the following condition.*

(i) $P(\varphi \circ \alpha, tfh)|_{\mathbf{P}^1_l}$ *is a trivial principal* G_l*-bundle over the projective line* $\mathbf{P}^1_l$.

Proof. We denote by $\overline{f}$ the image of f in $l[t]$, by $\overline{P}$ the restriction of P to $\mathbf{A}^1_l$, by $\overline{P}(\overline{\varphi}, \overline{f})$ the restriction of $P(\varphi, f)$ to the projective line $\mathbf{P}^1_l$, etc. Let Q be a parabolic B'-subgroup of G. By Corollary 9.4 there exist a monic polynomial $\overline{h}(t) \in l[t]$ and an element

$$u \in E_{Q_l}(l[t]_{t\overline{f}\,\overline{h}}) \subseteq G_l(l[t]_{t\overline{f}\,\overline{h}})$$

such that the principal G_l-bundle $\overline{P}(\overline{\varphi} \circ u, t\overline{f}\,\overline{h})$ is trivial over $\mathbf{P}^1_l$.

Choose a monic polynomial $h(t) \in B'[t]$ of degree equal to the degree of $\overline{h}(t)$ and such that $h(t)$ modulo J coincides with $\overline{h}(t)$. Clearly, the homomorphism of B'-algebras $B'[t]_{tfh} \to l[t]_{t\overline{f}\,\overline{h}}$ is surjective. By Lemma 8.2 it induces a surjective group homomorphism

$$E_Q(B'[t]_{tfh}) \to E_Q(l[t]_{t\overline{f}\,\overline{h}}) = E_{Q_l}(l[t]_{t\overline{f}\,\overline{h}}).$$

Thus, there exists an $\alpha \in E_Q(B'[t]_{tfh}) \subseteq G(B'[t]_{tfh})$ such that α equals u modulo J; we write $\overline{\alpha} = u$.

Consider the G-bundle $P(\varphi \circ \alpha, tfh)$. We claim that its restriction to the projective line $\mathbf{P}^1_l$ is trivial. Indeed, one has the following chain of equalities of principal G_l-bundles over $\mathbf{P}^1_l$:

$$\overline{P}(\overline{\varphi \circ \alpha}, \overline{tfh}) = \overline{P}(\overline{\varphi} \circ \overline{\alpha}, t\overline{f}\,\overline{h}) = \overline{P}(\overline{\varphi} \circ u, t\overline{f}\,\overline{h}),$$

where the principal G_l-bundle $\overline{P}(\overline{\varphi} \circ u, t\overline{fh})$ is trivial over $\mathbf{P}^1_l$. □

We keep the same notation as introduced before Proposition 9.5.

Proposition 9.6. *Assume that the Noetherian semi-local ring* B' *contains a field* k*. Let* G *be a not necessarily isotropic simple simply connected* B'*-group scheme. Let* E *be a principal* G*-bundle over* $\mathbf{P}^1$ *whose restriction to the closed fibre* $E_{\mathbf{P}^1_l}$ *is trivial. Then* E *is of the form* $E = \mathrm{pr}^*(E_0)$*, where* E_0 *is a principal* G*-bundle over* $\mathrm{Spec}(B')$ *and* $\mathrm{pr} : \mathbf{P}^1 \to \mathrm{Spec}(B')$ *is the canonical projection.*

Proof. See Appendix A.2. □

Let us state an important corollary of the above propositions.

Corollary 9.7. *Let* k *be a field, and let* B' *be a semi-local Noetherian algebra over* k*. Let* G *be an isotropic simple simply connected* B'*-group scheme. Further, let* P *be a principal* G*-bundle over* $\mathbf{A}^1$*. Assume that there exists a monic polynomial* $f \in B'[t]$ *such that the principal* G*-bundle* $P_{\mathbf{A}^1_f}$ *is trivial. Then the principal* G*-bundle* P *is trivial.*

Proof. Let $f \in B'[t]$ be a monic polynomial such that the principal G-bundle $P_{\mathbf{A}^1_f}$ is trivial. Choose a principal G-bundle isomorphism $\varphi : G_{\mathbf{A}^1_f} \to P_{\mathbf{A}^1_f}$. By Proposition 9.5 there exists a monic polynomial $h(t) \in B'[t]$ and an element $\alpha \in G(B'[t]_{tfh})$ such that the restriction $P(\varphi \circ \alpha, tfh)|_{\mathbf{P}^1_l}$ of the principal G-bundle $P(\varphi \circ \alpha, tfh)$ to the projective line $\mathbf{P}^1_l$ is a trivial principal G_l-bundle.

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By Proposition 9.6 the principal G-bundle $P(\varphi\circ\alpha, tfh)$ is of the form: $P(\varphi\circ\alpha, tfh) = \mathrm{pr}^*(P_0)$, where P_0 is a principal G-bundle over $\mathrm{Spec}(B')$. Note that

$$G|_{\{\infty\}\times\mathrm{Spec}(B')} \cong P(\varphi\circ\alpha, tfh)|_{\{\infty\}\times\mathrm{Spec}(B')},$$

that is the restriction of $P(\varphi\circ\alpha, tfh)$ to $\{\infty\}\times\mathrm{Spec}(B')$ is trivial. Thus

$$G_{\mathbf{P}^1} \cong P(\varphi\circ\alpha, tfh).$$

Since the original principal G-bundle P over $\mathbf{A}^1$ is isomorphic to $P(\varphi\circ\alpha, tfh)|_{\mathbf{A}^1}$, it follows that P is trivial. This finishes the proof. □

Corollary 9.8 (Theorem 1.3). *Let k be a field and let B be a Noetherian k-algebra. Assume that a group scheme G over B is simple, simply connected and isotropic. Further, let P be a principal G-bundle over $\mathbf{A}^1_B$. Assume that there exists a monic polynomial $f\in B[t]$ such that the principal G-bundle $P_{(\mathbf{A}^1_B)_f}$ is trivial and $f(1)\in B$ is invertible. Then the principal G-bundle P is trivial.*

Proof. It is routine to prove that there is a closed B-group scheme embedding $G\hookrightarrow \mathrm{GL}_{N,B}$ for an $N>0$. Since $f(1)$ is invertible, the principle G-bundle P is trivial at the closed sub-scheme $\{1\}\times\mathrm{Spec}(B)\subset\mathbf{A}^1_B$. By Corollary 9.7, for any maximal ideal m of B, the bundle $P_{\mathbf{A}^1_{B_m}}$ is trivial too. Now [Mos08, Korollar 3.5.2] completes the proof of Corollary 9.8. □

10. Proofs of Theorems 1.1 and 1.6, and of Corollary 1.7

Proof of Theorem 1.1. Substitute to Theorem 1.3 $B = \mathcal{O}\otimes_k A$, $P_t := \mathcal{G}_t$, $h(t) = f(t)\otimes 1$ from Theorem 1.2. By Theorem 1.3 the G-bundle $\mathcal{G}_t$ is trivial. Now by item (ii) of Theorem 1.2 the original G-bundle $\mathcal{G}$ is trivial. □

Proof of Theorem 1.6. Let $\mathcal{G}$ be a principal G-bundle over $R\otimes_{\mathbb{Z}} A$ that becomes trivial over $K\otimes_{\mathbb{Z}} A$. Clearly, there is a non-zero $f\in R$ such that $\mathcal{G}$ is trivial over $R_f\otimes_{\mathbb{Z}} A$.

Let k' be the prime subfield of R. It follows from Popescu's theorem [Pop86, Swa98] that R is a filtered inductive limit of smooth k'-algebras R_α. Then there exist an index α, a reductive group scheme G_α over R_α, a principal G_α-bundle $\mathcal{G}_\alpha$ over $R_\alpha\otimes_{\mathbb{Z}} A$, and an element $f_\alpha\in R_\alpha$ such that $G = G_\alpha\times_{\mathrm{Spec}(R_\alpha)}\mathrm{Spec}(R)$, $\mathcal{G}\cong\mathcal{G}_\alpha\times_{\mathrm{Spec}(R_\alpha\otimes_{\mathbb{Z}}A)}\mathrm{Spec}(R\otimes_{\mathbb{Z}} A)$ as principal G-bundles, f is the image of f_α under the map $\varphi_\alpha : R_\alpha\to R$, and $\mathcal{G}_\alpha$ is trivial over $(R_\alpha)_{f_\alpha}\otimes_{\mathbb{Z}} A$.

If the field k' is infinite, then for each maximal ideal m_i in R ($i = 1,\dots,n$) set $p_i = \varphi_\alpha^{-1}(m_i)$. The map φ_α induces a map of semi-local rings $(R_\alpha)_{p_1,\dots,p_n}\to R$. Since the principal G_α-bundle $\mathcal{G}_\alpha$ is trivial over $(R_\alpha)_{f_\alpha}\otimes_{\mathbb{Z}} A\cong(R_\alpha)_{f_\alpha}\otimes_{k'}(k'\otimes_{\mathbb{Z}} A)$, by Theorem 1.1 the bundle $\mathcal{G}_\alpha$ is trivial over $(R_\alpha)_{p_1,\dots,p_n}\otimes_{k'}(k'\otimes_{\mathbb{Z}} A)\cong(R_\alpha)_{p_1,\dots,p_n}\otimes_{\mathbb{Z}} A$. Whence the G-bundle $\mathcal{G}$ is trivial over $R\otimes_{\mathbb{Z}} A$.

Now consider the case where the field k' is finite. Since R contains an infinite field by the assumption of the theorem, R also contains a field $k'(t)$ of rational functions in one variable t over k'. Set $R'_\alpha = R_\alpha\otimes_{k'} k'(t)$, then the map φ_α can be decomposed as follows:

$$R_\alpha\to R_\alpha\otimes_{k'} k'(t) = R'_\alpha\xrightarrow{\psi_\alpha} R.$$

Set $G'_\alpha = G_\alpha\times_{\mathrm{Spec}(R_\alpha)}\mathrm{Spec}(R'_\alpha)$, $\mathcal{G}'_\alpha = \mathcal{G}_\alpha\times_{\mathrm{Spec}(R_\alpha\otimes_{\mathbb{Z}}A)}\mathrm{Spec}(R'_\alpha\otimes_{\mathbb{Z}} A)$, $f'_\alpha = f_\alpha\otimes 1\in R'_\alpha$. Then R'_α is a smooth $k'(t)$-algebra, and the principal G'_α-bundle $\mathcal{G}'_\alpha$ is trivial over $(R'_\alpha)_{f'_\alpha}\otimes_{\mathbb{Z}} A$. Arguing exactly as in the previous case with the field $k'(t)$ instead of k', we conclude, by means of Theorem 1.1, that the G-bundle $\mathcal{G}$ is trivial over $R\otimes_{\mathbb{Z}} A$. □

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Proof of Corollary 1.7. We settle the case $n = 1$, $t_1 = t$. The general case follows by induction, since, if R is a regular domain containing $\mathbb{Q}$, the ring $R[t_1, \dots, t_{n-1}]$ is also a regular domain containing $\mathbb{Q}$.

Consider the following commutative diagram.

$$\begin{array}{ccc} H^1_{\text{ét}}(R[t], G) & \xrightarrow{t=0} & H^1_{\text{ét}}(R, G) \\ \downarrow & & \downarrow \\ H^1_{\text{ét}}(K[t], G) & \xrightarrow{t=0} & H^1_{\text{ét}}(K, G) \end{array} \tag{29}$$

Since K is perfect, the bottom arrow is bijective by the main result of [RR84]. Therefore, any element $\xi \in H^1_{\text{ét}}(R[t], G)$ having trivial image in $H^1_{\text{ét}}(R, G)$ also has trivial image in $H^1_{\text{ét}}(K[t], G)$. By Theorem 1.6, for any maximal ideal $m \subseteq R$ the map

$$H^1_{\text{ét}}(R_m[t], G) \to H^1_{\text{ét}}(K[t], G)$$

has trivial kernel. Therefore, for any maximal ideal m, the image of ξ in $H^1_{\text{ét}}(R_m[t], G)$ is trivial as well. By [Mos08, Korollar 3.5.2] this implies that ξ is trivial. □

Lemma 10.1. *Corollary 1.7 (and then Theorem 1.1) is wrong without the isotropy condition.*

Proof. In fact, let $\mathbb{R}[t_1, t_2]$ be the polynomial ring in two variables over the real field. Let $\mathbb{H}$ be the quaternions. Let P be the non-free rank-1 left projective $\mathbb{H}[t_1, t_2]$-module from [OS71, Proposition 1]. The isomorphism class $\xi \in H^1_{\text{ét}}(\mathbb{R}[t_1, t_2], \mathrm{GL}_{1,\mathbb{H}})$ of P can be lifted to a class $\widetilde{\xi} \in H^1_{\text{ét}}(\mathbb{R}[t_1, t_2], \mathrm{SL}_{1,\mathbb{H}})$. Indeed, $H^1_{\text{ét}}(\mathbb{R}[t_1, t_2], \mathbb{G}_m) = *$. Let $\widetilde{\xi}_0 \in H^1_{\text{ét}}(\mathbb{R}, \mathrm{SL}_{1,\mathbb{H}})$ be the evaluation of the class $\widetilde{\xi}$ at $t_1 = t_2 = 0$. The class $\widetilde{\xi}$ can be chosen such that the class $\widetilde{\xi}_0$ is trivial. Thus the kernel of the evaluation map $H^1_{\text{ét}}(\mathbb{R}[t_1, t_2], \mathrm{SL}_{1,\mathbb{H}}) \xrightarrow{t_1=t_2=0} H^1_{\text{ét}}(\mathbb{R}, \mathrm{SL}_{1,\mathbb{H}})$ is non-trivial. Whence the lemma. □

Using results of [Par86, Rag89] one can find plenty of other examples of anisotropic simple simply-connected groups G over some fields k such that the kernel of the evaluation map $H^1_{\text{ét}}(k[t_1, t_2], G) \xrightarrow{t_1=t_2=0} H^1_{\text{ét}}(k, G)$ is non-trivial.

11. Semi-simple case

In the present section we show how Theorem 1.1 extends to the case of semi-simple simply connected groups; this is Theorem 11.1 below. One readily sees that Theorem 1.6 and Corollary 1.7 extend to semi-simple simply connected groups as well, once we substitute the isotropy condition imposed in these statements by the same one as in Theorem 11.1.

By [SGA3, Exp. XXIV 5.3, Proposition 5.10] the category of semi-simple simply connected group schemes over a Noetherian domain R is semi-simple. In other words, each object has a unique decomposition into a product of indecomposable objects. Indecomposable objects can be described as follows. Take a domain R' such that $R \subseteq R'$ is a finite étale extension and a simple simply connected group scheme G' over R'. Now, applying the Weil restriction functor $\mathrm{R}_{R'/R}$ to the R-group scheme G' we get a simply connected R-group scheme $\mathrm{R}_{R'/R}(G')$, which is an indecomposable object in the above category. Conversely, each indecomposable object can be constructed in this way.

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Theorem 11.1. *Let k be an infinite field. Let $\mathcal{O}$ be the semi-local ring of finitely many closed points on a smooth irreducible k-variety X and let K be its field of fractions. Let G be a semi-simple simply connected $\mathcal{O}$-group scheme all of whose indecomposable factors are isotropic. Then for any Noetherian k-algebra A the map*

$$H^1_{\text{ét}}(R \otimes_k A, G) \to H^1_{\text{ét}}(K \otimes_k A, G),$$

induced by the inclusion R into K, has trivial kernel.

Proof. Take a decomposition of G into indecomposable factors $G = G_1 \times G_2 \times \cdots \times G_r$. Clearly, it suffices to check that for each index i the kernel of the map

$$H^1_{\text{ét}}(R \otimes_k A, G_i) \to H^1_{\text{ét}}(K \otimes_k A, G_i)$$

is trivial. We know that there exists a finite étale extension R'_i/R such that R'_i is a domain and the Weil restriction $\mathrm{R}_{R'_i/R}(G'_i)$ coincides with G_i.

Take H to be an affine R-group scheme. Since the Weil restriction functor commutes with the fibre products, the isomorphism

$$\mathrm{adj}_H : \mathrm{Mor}_R(H, \mathrm{R}_{R'_i/R}(G'_i)) \to \mathrm{Mor}_{R'_i}(H \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R'_i), G'_i)$$

takes R-group scheme homomorphisms to R'_i-group scheme homomorphisms. Particularly, this holds for $H = \mathbb{G}_{m,R}$. By assumption, the group G_i is isotropic, i.e. contains $\mathbb{G}_{m,R}$. By the adjunction there is a non-trivial R'_i-group scheme homomorphism $\varphi : \mathbb{G}_{m,R'_i} \to G'_i$. Since R'_i is a domain, the kernel of φ is of the form μ_{n,R'_i}. Applying the isomorphism adj_H to the closed subgroup $H = \mu_{n,R}$ of $\mathbb{G}_{m,R}$ we conclude that $n = 1$. Thus φ is a closed embedding. Whence G'_i is isotropic.

The Faddeev–Shapiro Lemma [SGA3, Exp. XXIV, Proposition 8.4] states that there is a canonical isomorphism

$$H^1_{\text{ét}}(R \otimes_k A, \mathrm{R}_{R'_i/R}(G'_i)) \cong H^1_{\text{ét}}(R' \otimes_k A, G_i)$$

that preserves the distinguished point. To complete the proof, it only remains to apply Theorem 1.6 to the semi-local regular ring R'_i, its fraction field K_i, and the isotropic R'_i-group scheme G'_i. □

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Appendix

A.1 Almost elementary fibration

In this section we prove Propositions 2.3 and 2.4.

Proof of Proposition 2.3. The proof almost literally follows the proof of the original Artin's result [Art73, Exp. XI, Proposition 3.3]. Shrinking X, one may assume that $X \subset \mathbf{A}^r_k$ is affine and still contains the points $x_1, x_2, \ldots, x_n$. Set $x := \coprod_{j=1}^n x_i$. Let X_0 be the closure of X in $\mathbf{P}^r_k$. Let $\bar{X}$ is the normalization of X_0 and set $Y = \bar{X} - X$ with the induced reduced structure. Let $Z \subset X$ be a subset of $\bar{X}$ consisting of all non-regular points of $\bar{X}$. By [EGAIV, Corollary 6.12.5]

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the set Z is Zariski closed in $\bar{X}$. Since $\bar{X}$ is normal we conclude that $\dim Z \leqslant n-2$. Since X is k-smooth one has an inclusion $Z \subset Y$. Summarizing one has:

(i) $Z \subset Y$;
(ii) $\dim \bar{X} = \dim X = n$;
(iii) $\dim Y = n-1$;
(iv) $\dim Z \leqslant n-2$.

Shrinking X and following Artin's procedure one can construct a diagram of the form

$$\begin{array}{ccccc} X & \xrightarrow{\;j\;} & \overline{X}' & \xleftarrow{\;i\;} & Y' \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \bar{p}} & \swarrow{\scriptstyle q} & \\ & & S & & \end{array} \tag{A.1}$$

subject to conditions (i) and (ii) of Definition 2.1 and such that $x \subset X$ and S is an affine open subset in $\mathbf{P}^{n-1}_k$ and i is a closed imbedding and Y' is a regular scheme. Moreover, the restriction of $q \otimes_k \bar{k} : Y' \otimes_k \bar{k} \to S \otimes_k \bar{k}$ to the reduced sub-scheme $(Y' \otimes_k \bar{k})_{\mathrm{red}}$ is a finite étale morphism all of whose fibres are non-empty (here $\bar{k}$ is the algebraic closure of k). In this case for each irreducible component Y'_r of Y' the restriction $q|_{Y'_r} : Y'_r \to S$ is a finite surjective morphism. Since Y'_r, S are regular irreducible schemes of the same dimension, the morphism $q|_{Y'_r}$ is finite flat (see Grothendieck [Eis95, Corollary 17.18]). Thus q is subject to the condition (iii) of Definition 2.1. Finally, the ideal sheaf $I_{Y'}$ defining the closed sub-scheme Y' in $\overline{X}'$ is locally principal. In fact, S is regular and $\bar{p}$ is smooth. So, $\overline{X}'$ is regular. The closed sub-scheme Y' is regular of pure codimension one in $\overline{X}'$. Thus $I_{Y'}$ is locally principal. Whence the Proposition. $\square$

Proof of Proposition 2.4. To prove this Proposition it suffices to construct a finite surjective S-morphism

$$\bar{\pi} : \overline{X} \to \mathbf{P}^1 \times S$$

such that $Y_{\mathrm{red}} = \bar{\pi}^{-1}(\{\infty\} \times S)$ set-theoretically. To do that, we first note that, under the hypotheses of the Proposition, the closed sub-scheme Y of $\overline{X}$ is a locally principal divisor. We will construct a desired $\bar{\pi}$ using two sections t_0 and t_1 of the sheaf $\mathcal{O}(nY)$ for a sufficiently large n. Assume that t_0 and t_1 are such that the vanishing locus of t_0 is nY and the vanishing locus of t_1 does not intersect Y. Then the pair t_0, t_1 defines a regular map $\varphi := [t_0 : t_1] : \overline{X} \to \mathbf{P}^1$. Set $\bar{\pi} = (\varphi, \bar{p}) : \overline{X} \to \mathbf{P}^1 \times S$. Clearly, $\bar{\pi}$ is an S-morphism of the S-schemes. It is a projective morphism since both S-schemes are projective S-schemes. It is a quasi-finite surjective morphism. In fact, for each point $s \in S$ the morphism $\bar{\pi}$ induces a non-constant morphism $\overline{X}_s \to \mathbf{P}^1_s$ of two $k(s)$-smooth geometrically irreducible projective $k(s)$-curves. Thus $\bar{\pi}$ is finite surjective as a quasi-finite projective morphism. It remains to find an appropriate integer n and two sections t_0 and t_1 with the above properties.

Firstly, for each point s of the scheme S set $\overline{X}_s := (\bar{\pi})^{-1}(s)$ scheme-theoretically, and note that $\overline{X}_s$ is a $k(s)$-smooth geometrically irreducible projective $k(s)$-curve. The morphism $\bar{\pi}$ is smooth. In particular, it is flat. Whence the function $s \mapsto \chi(\overline{X}_s, \mathcal{O}_{\overline{X}_s})$ is constant by [Mum70, ch. II, § 5, Corollary 1]. The latter means that the genus $g(\overline{X}_s)$ is the same for all points $s \in S$. Set $g = g(\overline{X}_s)$. By the assumption, Y is finite flat over S and S is semi-local. Let r be the rank of the free $\Gamma(S, \mathcal{O}_S)$-module $\Gamma(Y, \mathcal{O}_Y)$.

Assume that $n \geqslant 2g-1$. Then $h^0(\overline{X}_s, \mathcal{O}_{\overline{X}_s}(nY_s)) = \chi(\overline{X}_s, \mathcal{O}_{\overline{X}_s}(nY_s)) = rn - g + 1$. Let $\mathcal{E}_n := \bar{p}_*(\mathcal{O}_{\overline{X}_s}(nY_s))$. By [Mum70, ch. II, § 5, Corollary 1] and [Mum70, ch. II, § 5, Lemma 1] the sheaf $\mathcal{E}_n$ on S is locally free of rank $rn-g+1$, and for each point $s\in S$ the canonical map $\mathcal{E}_n\otimes_{\mathcal{O}_S}k(s)\xrightarrow{\text{can}}H^0(\overline{X}_s,\mathcal{O}_{\overline{X}_s}(nY_s))$ is an isomorphism.

Let $s=\coprod s_i$, where s_i are all closed points of the semi-local scheme S. Let $k(s)=\prod k(s_i)$, where $k(s_i)$ denotes the residue field of the point s_i. Consider the commutative diagram

$$\begin{array}{ccc} H^0(S,\mathcal{E}_n) & \xrightarrow{\text{id}} & H^0(\overline{X},\mathcal{O}_{\overline{X}}(nY)) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \mathcal{E}_n\otimes_{\mathcal{O}_S}k(s) & \xrightarrow{\text{can}} & H^0(\overline{X}_s,\mathcal{O}_{\overline{X}_s}(nY_s)) \end{array} \tag{A.2}$$

where α, β, and can are the canonical homomorphisms. As mentioned in the previous paragraph, the map can is an isomorphism. The map α is surjective, since $s=\coprod s_i$ is a closed sub-scheme of the affine scheme S. Whence the map β is surjective.

For each $s_i\in s$ the curve $\overline{X}_{s_i}$ is a $k(s_i)$-smooth geometrically irreducible $k(s_i)$-curve of genus g. Whence there exists an integer n_0 such that for each $n\geqslant n_0$ and each $s_i\in s$ has $H^1(\overline{X}_{s_i},\mathcal{O}((n-1)Y_{s_i}))=0$. Thus there exists for any i a section $t_{1,i}$ of $\mathcal{O}_{\overline{X}_{s_i}}(nY_{s_i})$ that does not vanish on Y_{s_i}. By the surjectivity of β we may choose a section t_1 of $\mathcal{O}_{\overline{X}}(nY)$ such that $\beta(t_1)|_{\overline{X}_{s_i}}=t_{1,i}$ for any i. The vanishing locus of t_1 does not intersect Y_s, whence it does not intersect Y. Clearly, t_1 is the desired section of $\mathcal{O}_{\overline{X}}(nY)$. It remains to take for t_0 a section of $\mathcal{O}_{\overline{X}}(nY)$ with the vanishing locus nY. □

A.2 Proof of the Horrocks-type Theorem 9.6

In this section we give a proof of Theorem 9.6. This proof is rather standard, and for the most part follows [Rag94]. However, our group scheme G does not come from the ground field k. Therefore, we have to somewhat modify Raghunathan's arguments. We will use the following lemma.

Lemma A.1. *Let W be a semi-local irreducible Noetherian scheme over an arbitrary field k. Let H and H' be two reductive group schemes over W, such that H is a closed W-subgroup scheme of H', and denote by $j:H\hookrightarrow H'$ the corresponding embedding. Denote by $\mathbf{P}^1_W$ the projective line over W.*

Let $F\in H^1(\mathbf{P}^1_W,H)$ be a principal H-bundle, and let $M:=j_*(F)\in H^1(\mathbf{P}^1_W,H')$ be the corresponding principal H'-bundle. If M is a trivial H'-bundle, then there exists a principal H-bundle F_0 over W such that $\mathrm{pr}^*(F_0)\cong F$, where $\mathrm{pr}:\mathbf{P}^1_W\to W$ is the canonical projection.

Proof. Set $X=H'/j(H)$. Locally in the étale topology on W this scheme is isomorphic to the W-scheme $W\times_{\mathrm{Spec}(k)}H'_{0,k}/H_{0,k}$, where $H_{0,k}$ and $H'_{0,k}$ are the split reductive k-group schemes of the same types as H and H' respectively. By results of Haboush [Hab75] and Nagata [Nag64] (see Nisnevich [Nis77, Corollary]) the k-scheme $H'_{0,k}/H_{0,k}$ is an affine k-scheme. Thus X is an affine W-scheme. Consider the long exact sequence of pointed sets

$$1\to H(\mathbf{P}^1_W)\xrightarrow{j_*}H'(\mathbf{P}^1_W)\to X(\mathbf{P}^1_W)\xrightarrow{\partial}H^1_{\text{ét}}(\mathbf{P}^1_W,H)\xrightarrow{j_*}H^1_{\text{ét}}(\mathbf{P}^1_W,H').$$

Since $j_*(F)$ is trivial, there is $\varphi\in X(\mathbf{P}^1_W)$ such that $\partial(\varphi)=F$.

The W-morphism $\varphi : \mathbf{P}^1_W \to X$ is a W-morphism of a W-projective scheme to a W-affine scheme. Thus φ is 'constant', that is, there exists a section $s : W \to X$ such that $\varphi = s \circ \mathrm{pr}$. Consider another long exact sequence of pointed sets, this time the one corresponding to the scheme W, and the morphism of the first sequence to the second one induced by the projection pr. We get a big commutative diagram. In particular, we get the following commutative square.

$$\begin{array}{ccc} X(W) & \xrightarrow{\partial} & H^1_{\text{ét}}(W,H) \\ \downarrow{\scriptstyle \mathrm{pr}^*_W} & & \downarrow{\scriptstyle \mathrm{pr}^*_W} \\ X(\mathbf{P}^1_W) & \xrightarrow{\partial} & H^1_{\text{ét}}(\mathbf{P}^1_W,H) \end{array} \tag{A.3}$$

We have $\mathrm{pr}^*_W(s) = \varphi$. Hence

$$F = \partial(\varphi) = \partial(\mathrm{pr}^*_W(s)) = \mathrm{pr}^*_W(\partial(s)).$$

Setting $F_0 = \partial(s)$ we see that $F = \mathrm{pr}^*_W(F_0)$. The lemma is proved. □

Proof of Proposition 9.6. It is routine to prove that there is a closed B'-group scheme embedding $j : G \hookrightarrow \mathrm{GL}_{N,B'}$ for an $N > 0$. By the assumption of the theorem, the G-bundle E is trivial on $\mathbf{P}^1_l \subset \mathbf{P}^1$. Hence the $\mathrm{GL}_{N,B'}$-bundle $j_*(E)$ over $\mathbf{P}^1$ is trivial over $\mathbf{P}^1_l$.

The B'-group scheme $\mathrm{GL}_{N,B'}$ is just the ordinary general linear group. Thus $j_*(E)$ corresponds to a vector bundle M over $\mathbf{P}^1$. Moreover, this vector bundle is trivial on $\mathbf{P}^1_l$. Using the equality $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) = 0$ and [EGAIII, Corollary 4.6.4], we see that M is of the form $M = \mathrm{pr}^*(M_0)$ for a vector bundle M_0 over $\mathrm{Spec}(B')$. Since B' is semi-local, M_0 is trivial over $\mathrm{Spec}(B')$. Thus M is trivial on $\mathbf{P}^1$. Thus $j_*(F)$ is a trivial $\mathrm{GL}_{N,B'}$-bundle.

Now, applying Lemma A.1 to the embedding $j : G \hookrightarrow \mathrm{GL}_{N,B'}$, we see that $E = \mathrm{pr}^*(E_0)$ for some $E_0 \in H^1(B', G)$. Theorem 9.6 is proved. □

The following lemma shows that the square (20) and equally some of its base changes can be used for gluing modules and principal bundles as in [Bha88] and [CO92, Proposition 5.2].

Lemma A.2. *Let R be a Noetherian ring and let $\varphi : R[t] \hookrightarrow R'$ be a ring homomorphism making R' into an étale $R[t]$-algebra. Let $h \in R[t]$ be a polynomial whose top coefficient is a unit in R and such that the induced map $R[t]/(h) \to R'/(\varphi(h))$ is an isomorphism. Then the triple $(R[t], \varphi : R[t] \to R', h)$ is subject to the assumptions of [CO92, Proposition 5.2]. Moreover, for any Noetherian R-algebra S the triple*

$$(S[t], \varphi \otimes \mathrm{id} : R[t] \otimes_R S = S[t] \to S' = R' \otimes_R S, h \otimes 1)$$

is subject to the assumptions of [CO92, Proposition 5.2] too. In particular, if R contains a field k, then for any Noetherian k-algebra A the triple

$$(S[t], \varphi \otimes \mathrm{id} : R[t] \otimes_k A \to R' \otimes_k A, h \otimes 1)$$

is subject to the assumptions of [CO92, Proposition 5.2].

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Proof. The top coefficient of $h \otimes 1 \in S[t]$ is a unit. Thus $h \otimes 1$ is not a zero divisor in $S[t]$. The $S[t]$-algebra S' is étale by the assumption of the lemma. Particularly, it is a flat $S[t]$-algebra. Thus $\varphi(h) \otimes 1 \in S'$ is non-zero and it is not a zero divisor in S'. Clearly, the induced map $S[t]/(h \otimes 1) \to S'/(\varphi(h) \otimes 1)$ is an isomorphism. Whence the lemma. □

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