ON LATTICE EMBEDDINGS FOR PARTIALLY ORDERED SETS

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1. Introduction. Let P be a set partially ordered by a (reflexive, anti-symmetric, and transitive) binary relation $\prec$. Let $\mathfrak{K}$ be the family of all subsets K of P having the property that $x \in P$ and $y \in K$ and $y \prec x$ imply $x \in K$. Our principal object is to prove and apply the following:

THEOREM. *With respect to the partial ordering of* $\mathfrak{K}$ *by inclusion* ($K_1 \prec K_2$ *means* $K_1 \supset K_2$),

(1) *P is isomorphically embedded in $\mathfrak{K}$ preserving all suprema that exist in P, and*

(2) *$\mathfrak{K}$ is a complete distributive lattice.*

COROLLARY. *Every partially ordered set can be embedded in a complete distributive lattice, preserving suprema.*

This corollary is also a consequence of a two-stage embedding construction of MacNeille's (**2**, §11, 12) consisting of an initial completion by cuts, preserving both suprema and infima, followed by a certain complete-distributive-lattice embedding which preserves suprema and distributive infima. Our construction is much simpler than MacNeille's but does not in general preserve infima, even when they are distributive.

Following some related remarks concerning lattices of topologies in §3, an application of this theorem is indicated in §4. The author is indebted to the referee for suggestions leading to the recasting of results in essentially their present form, and to E. E. Floyd for a simplifying observation.

2. Proof of the theorem. We see at once that every subfamily $\mathfrak{K}_1$ of $\mathfrak{K}$ has an infimum (supremum) in $\mathfrak{K}$, namely the union (intersection) of the sets of the family $\mathfrak{K}_1$. That is, $\mathfrak{K}$ is a complete lattice. And now, since $\mathfrak{K}$ is a sublattice of the Boolean algebra of all subsets of P, it is obvious that $\mathfrak{K}$ is distributive.

It is easily seen that the correspondence

$$K(x) = \{y : x \prec y\} \qquad (x \in P)$$

is an isomorphism of P into $\mathfrak{K}$. We verify that it preserves suprema. Take any family $\{x_\alpha\}$ of elements of P having a supremum

$$x = \bigvee_\alpha x_\alpha$$

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in P. For each α, $x_\alpha \prec x$, so that

$$\{y\colon x \prec y\} \subset \bigcap_\alpha \{y\colon x_\alpha \prec y\}.$$

That is,

$$K(\bigvee_\alpha x_\alpha) \subset \bigvee_\alpha K(x_\alpha). \tag{2.1}$$

Now take any

$$z \in \bigvee_\alpha K(x_\alpha) = \bigcap_\alpha \{y\colon x_\alpha \prec y\}.$$

For each α, $x_\alpha \prec z$, whence

$$x = \bigvee_\alpha x_\alpha \prec z.$$

Thus

$$z \in K(\bigvee_\alpha x_\alpha).$$

That is,

$$K(\bigvee_\alpha x_\alpha) \supset \bigvee_\alpha K(x_\alpha),$$

which with (2.1) yields

$$K(\bigvee_\alpha x_\alpha) = \bigvee_\alpha K(x_\alpha)$$

as desired, completing the proof.

To see that this embedding does not always preserve distributive infima, let P be the rationals of the closed unit interval [0, 1], partially ordered by $\leqslant$. The family $\{x_n\}$ of positive such rationals has the distributive infimum

$$0 = \bigwedge_n x_n,$$

so that

$$K(\bigwedge_n x_n) = \{y\colon 0 \leqslant y\} = [0, 1].$$

On the other hand,

$$\bigwedge_n K(x_n) = \bigcup_n \{y\colon x_n \leqslant y\} = (0, 1].$$

3. Lattices of topologies. We review some well-known facts. A topology T on a set S may be specified in any of several equivalent ways: in particular, by a closure function $C(X)$ on 2^S to 2^S such that

$$C(\phi) = \phi \qquad (\phi = \text{empty set}), \tag{3.1}$$

$$C(X) \cup C(Y) = C(X \cup Y), \tag{3.2}$$

$$X \subset C(X), \tag{3.3}$$

$$C(C(X)) = C(X). \tag{3.4}$$

The various topologies on S form a lattice $L_T(S)$ under the partial order $T_1 \prec T_2$ defined by the requirement

$$C_1(X) \supset C_2(X), \qquad X \in S. \tag{3.5}$$

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This lattice is not in general distributive (**4**, p. 134). The definitive statement in this connection is very simple but seems not to have been elsewhere recorded:

(3.6) *Given a set S these statements are equivalent:*

(α) *$L_T(S)$ is modular.*

(β) *$L_T(S)$ is distributive.*

(γ) *The cardinality of S is <3.*

Proof. If (γ) holds, $L_T(S)$ has at most four elements and so is distributive by (**1**, p. 134, Theorem 2). That (β) implies (α) is trivial. To see that whenever (γ) fails (α) fails, assume $|S| \geqslant 3$, fix distinct points x and y of S, and consider these three closure topologies on S:

$$T_1: \; C_1(X) = X \text{ if } x \notin X; \; C_1(X) = X \cup \{y\}, \qquad x \in X,$$
$$T_2: \; C_2(X) = X \cup \{x\}, \qquad (X \neq \phi),$$
$$T_3: \; C_3(X) = X \cup \{y\}, \qquad (X \neq \phi).$$

One verifies easily that

$$(T_1 \vee T_2) \wedge T_3 = T_3 \prec T_1 = T_1 \vee (T_2 \wedge T_3).$$

Since $|S| \geqslant 3$ implies $T_1 \neq T_3$, this contradicts modularity.

By dropping requirement (3.4) on closure functions, Wada (**5**) arrived at the larger lattice $L_A(S)$ of what he termed the "additive topologies" on S; and by dropping (3.3) as well he obtained the still larger lattice $L(S)$ of Tukey topologies (**3**, p. 24). He observed that $L(S)$ is complete and distributive and that it embeds $L_A(S)$ as a sublattice and $L_T(S)$ as a partially ordered set, preserving suprema.

4. Channel structures. Application of our embedding theorem to $L_A(S)$ yields a suprema-preserving embedding of $L_A(S)$ in a complete distributive lattice $L_C(S)$ whose elements, termed *channel structures* on S, are of considerable intrinsic interest. The notion of channel structure is due, in its original somewhat different form, to McShane[1]. Here we content ourselves with a very brief indication of this original form.

We first note (cf. **3**, p. 19, Theorem 3.14) that an additive topology on a set S can be equivalently defined by a neighbourhood function $\mathfrak{N}$ associating with each point $x \in S$ a non-empty class $\mathfrak{N}(x)$ of subsets of S such that

(4.1) $x \in N$ for each $N \in \mathfrak{N}(x)$;

(4.2) if $S \supset M \supset N$ and $N \in \mathfrak{N}(x)$, then $M \in \mathfrak{N}(x)$; if $M, N \in \mathfrak{N}(x)$, then $M \cap N \in \mathfrak{N}(x)$.

[1]Channel structures will form the subject of a forthcoming joint study by E. J. McShane, E. E. Floyd, and the present author.

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The partial order (3.5) on $L_A(S)$ is then equivalently defined (**3**, p. 24) by the requirement

$$\text{for each } y \in S,\ \mathfrak{N}_1(x) \subset \mathfrak{N}_2(x). \tag{4.3}$$

Suppose now we define a *channel to* $x \in S$ as a non-empty class $\mathfrak{N}(x)$ of subsets of S satisfying (4.1) and (4.2). It is then not difficult to see that each channel structure (= element of $L_C(S)$) on S consists essentially of a function $\mathcal{N}$ which assigns to each $x \in S$ a collection $\mathcal{N}(x)$ of channels to x with the following property: if $\mathfrak{N}_1(x)$ and $\mathfrak{N}_2(x)$ are channels to x and $\mathfrak{N}_1(x) \in \mathcal{N}(x)$ and $\mathfrak{N}_2(x) \supset \mathfrak{N}_1(x)$, then $\mathfrak{N}_2(x) \in \mathcal{N}(x)$. We conclude by remarking that because the lattice $L_A(S)$ has a unit I (the discrete topology on S), none of these collections $\mathcal{N}(x)$ can be empty.

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