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A GROUP VARIETY DEFINED BY A SEMIGROUP LAW

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Abstract

A group variety defined by one semigroup law in two variables is constructed and it is proved that its free group is not a periodic extension of a locally soluble group.

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In group theory a law in variables $x_1, x_2, \dots, x_n$ is called a semigroup law if it can be represented in the form

$$u_1(x_1, \dots, x_n) = u_2(x_1, \dots, x_n)$$

where u_1 and u_2 are semigroup words, that is words which do not contain x_i^{-1} for $i = 1, \dots, n$.

Obviously every group of finite exponent satisfies a nontrivial semigroup law. It is established in [4] that nilpotent groups of a given class can be defined by a semigroup law. Therefore free groups of a product of a locally nilpotent variety and a periodic variety satisfy a nontrivial semigroup law. As shown in [4], a nontrivial semigroup law follows from the property of being Engel. In [1], conditions under which soluble group varieties have a nontrivial semigroup law are studied. It is proved in [1] that a finitely generated soluble group satisfies a nontrivial semigroup law if and only if it has a nilpotent subgroup of finite index.

In view of these facts, one may raise a question concerning the existence of a finitely generated group with a semigroup law which is not a nilpotent-by-periodic group. In [2], the question of whether a 2-generated group without free subsemigroups must be

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a periodic extension of a locally nilpotent group is posed. In this paper the following theorem is proved.

THEOREM. *There exists a nontrivial semigroup law such that some 2-generated group, which is not a periodic extension of a locally soluble group, satisfies this law.*

Thus, in particular, the negative answer to the question raised in [2] is obtained.

To prove the Theorem, we introduce some word in two variables and then study a 2-generated relatively free group of the variety defined by the word.

We put

$$v = v(x, y) = x^d y^d,$$

$$w(x, y) = v^{n+1} x v^{n+4} x \cdots v^{n+(h-1)^2} x v^{n+h^2} x v^{-n-(h+1)^2} x^{-1} \cdots v^{-n-(2h-1)^2} x^{-1} v^{-n+h^2(2h-3)} x^{-1},$$

where h, d and n are sufficiently large natural numbers. Note that both the sum of exponents of the word v and that of the letter x in the word $w(x, y)$ are equal to zero as the equation $1^2 + 2^2 + \cdots + k^2 = k(2k+1)(k+1)/6$ holds.

The study of the 2-generated relatively free group of the variety defined by the word $w(x, y)$ uses the technique, described in [3], of geometric interpretation for the deduction of consequences of defining relations. Following the patterns detailed in [3; 25.1] and [3; 29.3], we define groups $G(i)$ for every nonnegative integer i and the group $G(\infty)$ with the corresponding alterations. We assume that the alphabet of presentations of these groups consists of the letters a and b.

While [3] is the main source of information for references, in this paper we also use a few results obtained in [5].

LEMMA 1. *Let A be a simple word in rank i or a period of rank $j \le i$ and let some power A^f of the word A be conjugate in rank i to the value $v(X, \overline{Y})$ for words X and $\overline{Y}$ such that $w(X, \overline{Y}) \overset{i}{\neq} 1$. Then $1 \le |f| \le 100\zeta^{-1}$.*

PROOF. Notice that the words X and $\overline{Y}$ cannot be commutative in rank i since otherwise the equation $w(X, \overline{Y}) \overset{i}{=} 1$ would hold.

Suppose that $f = 0$, then $X^d \overline{Y}^d \overset{i}{=} 1$. Hence X^d and $\overline{Y}^d$ commute in rank i which implies the commutativity of X and $\overline{Y}$ in rank i by [3; Lemma 25.2] and [3; Lemma 25.12]. Therefore, $|f| \ge 1$.

Since the words X and $\overline{Y}$ are not commutative in rank i, the inequality $|f| \le 100\zeta^{-1}$ holds by [5; Lemma 3].

LEMMA 2. *In the notation of* [3; 30.2], *we assume that T is a word minimal in rank i such that $T \overset{i}{=} W^{-1} X W$. Then $|T| < d|A|$.*

PROOF. The word $X^d\overline{Y}^d$ is conjugate to A^f in rank i and we can turn a conjugacy diagram of the words $X^d\overline{Y}^d$ and A^f into a diagram Δ on a sphere with three holes and three cyclic segments q_1, q_2, q_3 of the contour with the labels $\varphi(q_1) \equiv C^m$, $\varphi(q_2) \equiv B^k$, $\varphi(q_3) \equiv A^{-f}$. By [3; Lemma 24.9] and [3; Lemma 22.2] applied to the diagram Δ, we have $|Z| < 2(|C^m|+|B^k|+|A^f|)$. Therefore, $|W| < \bar{\alpha}(|C^m|+|B^k|+4(|C^m|+|B^k|+|A^f|)+|A^f|) < 3(|C^m|+|B^k|+|A^f|)$ by [3; Lemma 25.4], hence $|T| < 7(|C^m|+|B^k|+|A^f|)$. Assume that $|T| \geq d|A|$. Then $|C^m|+|B^k| > 2\zeta^{-2}|A^f|$. If $|B^k| < \zeta|C^m|$, then Δ is a J-map which is impossible by [5; Lemma 2] and [3; Lemma 25.8]. Hence $|B^k| \geq \zeta|C^m|$ and we can consider Δ as an E-map. By [3; Lemma 24.6] and [3; Lemma 25.10], the segments q_1 and q_2 of the contour of the diagram Δ are compatible, whence the commutativity of the words X and $\overline{Y}$ in the rank i follows, which contradicts the inequality $w(X, \overline{Y}) \overset{i}{\neq} 1$.

LEMMA 3. *The word $T_{A,j}$ is not equal in rank i to any power of the word A.*

PROOF. If $T_{A,j}$ is conjugate to A^m in rank i, then X is conjugate to A^m too, where $m \neq 0$. Therefore, by Lemma 1, the diagram Δ considered in the proof of Lemma 2 is a J-map or an E-map, which is impossible by [5; Lemma 2] and [3; Lemma 25.8, Lemma 24.6, Lemma 25.10].

LEMMA 4. *The presentation $G(\infty)$ satisfies condition $R6$.*

PROOF. It follows from the equations $A^{a_1}SA^{b_1} \overset{i-1}{=} A^{c_1}TA^{d_1}$ and $A^{a_2}SA^{b_2} \overset{i-1}{=} A^{c_2}TA^{d_2}$ that $A^{a_1-c_1}SA^{b_1-d_1} \overset{i-1}{=} A^{a_2-c_2}SA^{b_2-d_2}$, which by [3; Lemma 25.18] and Lemma 3 implies the equations $a_1 - c_1 = a_2 - c_2$ and $b_1 - d_1 = b_2 - d_2$. Therefore, when proving that the presentation $G(\infty)$ satisfies condition $R6$, we may consider equations $A^{a_u}SA^{b_u} \overset{i-1}{=} A^{c_u}TA^{d_u}$ where $u = 1, 2, 3, 4$, $c_u = a_u + p$, $d_u = b_u + q$, $S \equiv T^{\pm 1}_{A,j}$, $T \equiv T^{\pm 1}_{A,t}$ and the words $A^{a_u}SA^{b_u}$ and $A^{c_u}TA^{d_u}$ are consecutive subwords of cyclic shifts of the words $R_{A,j}$ and $R^{\pm 1}_{A,t}$ such that

$$b_1 + a_2 = (-1)^r f(A, j)(n + (k-1)^2),$$
$$b_2 + a_3 = (-1)^r f(A, j)(n + k^2),$$
$$b_3 + a_4 = (-1)^r f(A, j)(n + (k+1)^2)$$

and

$$d_1 + c_2 = (-1)^s f(A, t)(n + (m-1)^2),$$
$$d_2 + c_3 = (-1)^s f(A, t)(n + m^2),$$
$$d_3 + c_4 = (-1)^s f(A, t)(n + (m+1)^2)$$

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or

$$d_1 + c_2 = (-1)^s f(A, t)(n + (m+1)^2),$$
$$d_2 + c_3 = (-1)^s f(A, t)(n + m^2),$$
$$d_3 + c_4 = (-1)^s f(A, t)(n + (m-1)^2)$$

for some positive integers k and m.

Then on one hand, for some number M we have

$$M = \frac{(b_1 + a_2) - (b_2 + a_3)}{(b_2 + a_3) - (b_3 + a_4)} = \frac{2k - 1}{2k + 1},$$

and on the other hand,

$$M = \frac{(b_1 + q + a_2 + p) - (b_2 + q + a_3 + p)}{(b_2 + q + a_3 + p) - (b_3 + q + a_4 + p)} = \frac{(d_1 + c_2) - (d_2 + c_3)}{(d_2 + c_3) - (d_3 + c_4)} = \left(\frac{2m - 1}{2m + 1}\right)^{\pm 1}$$

whence it follows that $k = m$ and the exponent of the expression $(2m - 1/2m + 1)^{\pm 1}$ is equal to 1. So the words $A^{c_u} T A^{d_u}$ are subwords of a cyclic shift of $R_{A,t}$ and not of $R_{A,t}^{-1}$. Hence

$$\begin{aligned} p + q &= (b_1 + q + a_2 + p) - (b_1 + a_2) \\ &= (d_1 + c_2) - (b_1 + a_2) \\ &= \pm((n + (k-1)^2) f(A, j) - (n + (k-1)^2) f(A, t)) \\ &= \pm(n + (k-1)^2)(f(A, j) - f(A, t)). \end{aligned}$$

Similarly, $(p + q) = \pm(n + k^2)(f(A, j) - f(A, t))$. Therefore, $f(A, j) = f(A, t)$ and $p + q = 0$. Thus $T \overset{i-1}{=} A^{-p} S A^p$ and the lemma is proved.

LEMMA 5. *The presentation $G(i)$ satisfies condition $R5$.*

The proof of Lemma 5 is analogous to that of [3; Lemma 29.2].

LEMMA 6. *The presentation $G(i)$ satisfies condition R.*

The proof of Lemma 6 consists of references to the previous lemmas and to the definition of the presentation of $G(i)$.

LEMMA 7. *The group $G(\infty)$ is a free group of the variety defined by the law $w(x, y) \equiv 1$.*

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The proof of Lemma 7 is similar to that of [3; Theorem 19.7].

Now we can prove the Theorem.

PROOF. Let us assume that the group $G(\infty)$ is a periodic extension of a locally soluble group. Then $G(\infty)$ contains a nontrivial normal locally soluble subgroup H as $G(\infty)$ is a torsionfree group by [3; Theorem 26.4]. Let K be a 2-generated subgroup of H. Since K is soluble, K is abelian by [3; Lemma 25.14]. Hence H is abelian. But any normal abelian subgroup of $G(\infty)$ is central by [3; Lemma 25.14]. Therefore, the subgroup generated by H and a is abelian and hence it is cyclic by [3; Theorem 26.5]. Since by [3; Lemma 25.12] if a nonzero power of some element of $G(\infty)$ is central, then this element is central too, the generator a is central. Therefore, $[a, b] \stackrel{i}{=} 1$ for some i, which contradicts [3; Lemma 23.16], Lemma 6 and the definition of groups whose presentation satisfies condition R.

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