Linear Maps on Selfadjoint Operators Preserving Invertibility, Positive Definiteness, Numerical Range

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Abstract. Let H be a complex Hilbert space, and $\mathcal{H}(H)$ be the real linear space of bounded selfadjoint operators on H. We study linear maps $\phi \colon \mathcal{H}(H) \to \mathcal{H}(H)$ leaving invariant various properties such as invertibility, positive definiteness, numerical range, *etc.* The maps ϕ are not assumed *a priori* continuous. It is shown that under an appropriate surjective or injective assumption ϕ has the form $X \mapsto \xi TXT^*$ or $X \mapsto \xi TX^t T^*$, for a suitable invertible or unitary T and $\xi \in \{1, -1\}$, where X^t stands for the transpose of X relative to some orthonormal basis. Examples are given to show that the surjective or injective assumption cannot be relaxed. The results are extended to complex linear maps on the algebra of bounded linear operators on H. Similar results are proved for the (real) linear space of (selfadjoint) operators of the form $\alpha I + K$, where α is a scalar and K is compact.

1 Introduction

In the last few decades, many researchers have devoted their effort in studying *linear preserver problems* on the algebra M_n of $n \times n$ complex matrices. These problems concern the characterization of linear maps $\phi \colon M_n \to M_n$ leaving invariant various properties or functions of matrices such as invertibility, positive definiteness, the spectral norm, *etc.* Those *linear preservers* ϕ typically have simple structures, namely, ϕ has the form

$$X \mapsto MXN \quad \text{or} \quad X \mapsto MX^t N,$$

for some $M, N \in M_n$ with special properties; here X^t is the transpose of X. Many interesting techniques have been developed to prove such results; see [25] for some general background.

In the last few years, many researchers have studied linear preservers ϕ on more general algebras. In particular, a lot of attention has been paid to Kaplansky's problem [17] of characterization of linear maps preserving invertibility, and to the closely related problem concerning spectrum preserving maps [1]–[5], [8]–[10], [16], [27]. We refer to [3] for some historical remarks. A recent result of Sourour [27] states that every bijective invertibility preserving linear map on $\mathcal{B}(X)$, the algebra of all bounded

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linear operators on a Banach space X, is a Jordan automorphism of $\mathcal{B}(X)$ composed by a left multiplication with an invertible operator.

In this paper we develop some techniques in proving linear preserver results on the real linear space $\mathcal{H}(H)$ of selfadjoint operators acting on a complex Hilbert space H. The results are extended to linear maps on $\mathcal{B}(H)$. Similar results are proved for the complex linear space $\mathcal{B}_{\mathcal{K}}(H) \subset \mathcal{B}(H)$ of operators of the form $\alpha I + K$, where α is a scalar and K is compact, and for the real linear space $\mathcal{H}_{\mathcal{K}}(H)$ of selfadjoint operators in $\mathcal{B}_{\mathcal{K}}(H)$.

Let $\mathbf{V}$ be one of the spaces $\mathcal{H}(H)$, $\mathcal{B}(H)$, $\mathcal{H}_{\mathcal{K}}(H)$, or $\mathcal{B}_{\mathcal{K}}(H)$. When we say that $\phi \colon \mathbf{V} \to \mathbf{V}$ is linear, it naturally means that ϕ is *real linear* if $\mathbf{V} = \mathcal{H}(H)$ or $\mathcal{H}_{\mathcal{K}}(H)$, and ϕ is *complex linear* if $\mathbf{V} = \mathcal{B}(H)$ or $\mathcal{B}_{\mathcal{K}}(H)$.

In Section 2, we study linear maps $\phi \colon \mathbf{V} \to \mathbf{V}$ preserving the set of orthogonal projections, the spectrum, and the numerical range. Special cases of the results have been mentioned or used by other researchers. We organize and extend the results so that they can be used to treat other problems including those in our later sections. In Section 3, we characterize injective linear maps $\phi \colon \mathbf{V} \to \mathbf{V}$ that map the set of positive definite operators onto itself. This extends a result of Schneider [26] to the infinite dimensional setting. In Section 4, we study linear maps on $\mathcal{H}(H)$ preserving invertibility. Note that there is an essential difference between our problem and the problem of characterization of invertibility preserving maps on associative algebras. Namely, in the latter case one can always assume that an invertibility preserving map ϕ is unital, since otherwise one can replace ϕ by $X \to \phi(1)^{-1}\phi(X)$. Such a reduction is not possible in our problem because $\mathcal{H}(H)$ is not closed under multiplication. Examples are given to show that the injective or surjective assumptions imposed on ϕ in our theorems are essential.

2 Orthogonal Projections, Spectrum, Numerical Range Preservers

Given $X \in \mathcal{B}(H)$, let $\sigma(X)$ be the spectrum of X, and let

$$W(X) = \{(Xv, v) : v \in H, (v, v) = 1\}$$

be the *numerical range* of X. The closure of $W(X)$ is denoted by $\overline{W(X)}$. It is well known that if $X \in \mathcal{H}(H)$, then $\overline{W(X)}$ is the convex hull of $\sigma(X)$. A selfadjoint operator $X \in \mathcal{B}(H)$ will be called *positive definite* (resp., *negative definite*) if $\min\{\lambda : \lambda \in \sigma(X)\} > 0$ (resp., $\max\{\lambda : \lambda \in \sigma(X)\} < 0$).

Lemma 1 *Let $\mathbf{V}$ be $\mathcal{H}(H)$, $\mathcal{B}(H)$, $\mathcal{H}_{\mathcal{K}}(H)$, or $\mathcal{B}_{\mathcal{K}}(H)$. The following conditions are equivalent for a linear map $\phi \colon \mathbf{V} \to \mathbf{V}$:*

(i) $\overline{W\big(\phi(X)\big)} = \overline{W(X)}$ *for all $X \in \mathbf{V}$.*
(ii) *For any selfadjoint $X \in \mathbf{V}$ and $t \in \mathbf{R}$, the operator $tI - X$ is positive definite if and only if $tI - \phi(X)$ is.*

Proof Suppose (i) holds. Then, clearly, $\phi(I) = I$ and for every $X \in \mathbf{V}$ the operator $\phi(X)$ is selfadjoint if and only if X is selfadjoint. For any selfadjoint $X \in \mathbf{V}$, $\overline{W(X)}$

is the convex hull of $\sigma(X)$; $tI - X$ is positive definite if and only if $t > \max \sigma(X)$; $-tI + X$ is positive definite if and only if $t < \min \sigma(X)$. Since $\overline{W(X)} = \overline{W(\phi(X))}$, condition (ii) follows. Conversely, suppose (ii) holds. If $\mathbf{V} = \mathcal{H}(H)$ or $\mathcal{H}_{\mathcal{K}}(H)$, then condition (ii) certainly ensures $\sigma(X)$ and $\sigma(\phi(X))$ have the same convex hull. Suppose $\mathbf{V} = \mathcal{B}(H)$ or $\mathcal{B}_{\mathcal{K}}(H)$. Let $A = X + iY$ with selfadjoint X and Y. Observe that the compact convex set $\overline{W(A)}$ lies in the half plane $\{x + iy \in \mathbf{C} : x, y \in \mathbf{R}, y \leq kx + b\}$, where $k, b \in \mathbf{R}$, if and only if the selfadjoint operator $tI + kX - Y$ is positive definite for any $t > b$. By condition (ii), this is true if and only if $tI + k\phi(X) - \phi(Y)$ is positive definite for all $t > b$. Thus, $\overline{W(\phi(A))}$ and $\overline{W(A)}$ always belong to the same half planes, and therefore the two convex sets must be equal. Hence, condition (i) holds.
∎

We have the following linear preserver result.

Theorem 2   *Let* $\mathbf{V}$ *be one of* $\mathcal{H}(H)$, $\mathcal{B}(H)$, $\mathcal{H}_{\mathcal{K}}(H)$, *or* $\mathcal{B}_{\mathcal{K}}(H)$. *The following statements are equivalent for a surjective linear map* $\phi \colon \mathbf{V} \to \mathbf{V}$:

(a) ϕ *maps the set of selfadjoint elements into itself and*

$$\sigma(\phi(X)) = \sigma(X) \quad \text{for all selfadjoint } X \in \mathbf{V}.$$

(b) ϕ *is continuous (in the norm topology), maps the set of selfadjoint elements into itself, and is such that* $\sigma(\phi(X)) = \sigma(X)$ *for all orthogonal projections* $X \in \mathbf{V}$.

(c) $W(\phi(X)) = W(X)$ *for all* $X \in \mathbf{V}$.

(d) $\overline{W(\phi(X))} = \overline{W(X)}$ *for all* $X \in \mathbf{V}$.

(e)   *There is a unitary* $U \in \mathcal{B}(H)$ *such that* ϕ *is of the form*

$$(1) \qquad\qquad X \mapsto UXU^*, \quad or \quad X \mapsto UX^tU^*,$$

where X^t *is the transpose of* X *with respect to some fixed orthonormal basis.*

The equivalence of (a) and (e) in Theorem 2 extends a result of Marcus and Moyls (Theorem 4 in [19]) to the infinite dimensional case. Note that Omladič [22] also studied linear mappings ϕ on $\mathcal{B}(H)$ that preserve numerical ranges. The surjectivity assumption in Theorem 2 is essential. Otherwise, we can identify $H \oplus H$ with H, and define $\phi \colon \mathbf{V} \to \mathbf{V}$ by $\phi(X) = X \oplus X$ acting on H. Then ϕ satisfies (a)–(d), but not (e). On the other hand, many parts of the theorem are valid without the surjectivity assumption, as it will become apparent in the proof, namely,

$$(2) \qquad (e) \Rightarrow (a), (b), (c), (d); \quad (a) \Rightarrow (b) \Rightarrow (d); \quad (c) \Rightarrow (d).$$

Note also that if H is finite dimensional, then one can omit the surjectivity hypothesis in Theorem 2. To verify this, use (2) and observe that if (d) holds true, then ϕ is injective, and therefore (because $\dim H < \infty$) also surjective.

Proof   The only non-trivial implication in (2) is (b) $\Rightarrow$ (d). We prove (b) $\Rightarrow$ (d) $\Rightarrow$ (e).

Suppose (b) holds. Then it is clear that $\phi(I) = I$. For any selfadjoint $X \in \mathbf{V}$ with finite spectrum, we can write $X = \sum_{j=1}^{n} t_j P_j$, where $t_1 \geq \cdots \geq t_n$ are the eigenvalues of X and $P_i \neq 0$ are orthogonal projections such that they sum up to I and $P_i P_j = P_j P_i = 0$ whenever $i \neq j$. It follows that

$$\phi(X) = \sum_{j=1}^{n} t_j \phi(P_j) = t_n I + \sum_{j=1}^{n} (t_j - t_n)\phi(P_j) \geq (t_1 - t_n)\phi(P_1) + t_n I,$$

and since $\phi(P_1) \neq 0$, we have $\max \sigma\big(\phi(X)\big) \geq t_1$. Now, for any $t \geq t_1$,

$$tI - \phi(X) = \phi(tI - X) = \sum_{j=1}^{n} (t - t_j)\phi(P_j)$$

is positive semidefinite. Thus, $\max \sigma\big(\phi(X)\big) \leq t_1$. As a result, we see that $\max \sigma\big(\phi(X)\big) = t_1 = \max \sigma(X)$. Similarly, we can show that $\min \sigma\big(\phi(X)\big) = \min \sigma(X)$. Hence $\overline{W(X)} = \overline{W\big(\phi(X)\big)}$. Since the set of all selfadjoint operators with a finite spectrum is dense in the set of selfadjoint operators in $\mathbf{V}$, by the continuity of ϕ, we have $\overline{W\big(\phi(X)\big)} = \overline{W(X)}$ for all $X \in \mathcal{H}(H)$ or $\mathcal{H}_{\mathcal{K}}(H)$. If $\mathbf{V} = \mathcal{B}(H)$ or $\mathcal{B}_{\mathcal{K}}(H)$, by Lemma 1, we again have $\overline{W\big(\phi(X)\big)} = \overline{W(X)}$ for all $X \in \mathbf{V}$. Hence condition (d) holds.

Suppose now (d) holds. Since

$$\|X\| = \max\{|\lambda| : \lambda \in \overline{W(X)}\} = \max\{|\lambda| : \lambda \in \sigma(X)\} \quad \text{for } X \in \mathcal{H}(H),$$

we see that ϕ is continuous. Since $W(X) = \{1\}$ if and only if $X = I$, we see that ϕ is unital. Suppose $\mathbf{V} = \mathcal{H}(H)$ or $\mathcal{H}_{\mathcal{K}}(H)$. We extend ϕ to the complex linear map on $\mathcal{B}(H)$ or on $\mathcal{B}_{\mathcal{K}}(H)$ by

$$\phi(X + iY) = \phi(X) + i\phi(Y), \quad X, Y \in \mathcal{H}(H) \text{ or } X, Y \in \mathcal{H}_{\mathcal{K}}(H).$$

Clearly, $\phi(X^*) = \big(\phi(X)\big)^*$. Since ϕ preserves the closure of the numerical range for selfadjoint operators, by Lemma 1 the extended map ϕ will preserve the closure of the numerical range on $\mathcal{B}(H)$ or $\mathcal{B}_{\mathcal{K}}(H)$. Note that the closure of the numerical range of $X \in \mathcal{B}(H)$ or $\mathcal{B}_{\mathcal{K}}(H)$ is just the norm numerical range considered in [24]. By Theorem 3.1 in [24], there exists a unitary $U \in \mathcal{B}(H)$ such that ϕ has the form (1). ∎

If $\mathbf{V} = \mathcal{H}(H)$ or $\mathcal{B}(H)$, the continuity assumption in (b) can be dropped. We need the following lemma to prove this claim. Define $\rho = 8$ if H is infinite dimensional, $\rho = 9$ if H is finite dimensional of even dimension, and $\rho = 10$ if H is finite dimensional of odd dimension.

Lemma 3 *For every operator $T \in \mathcal{H}(H)$, and for every $\delta > 0$, there exist ρ orthogonal projections $P_j \in \mathcal{H}(H)$, $j = 1, \ldots, \rho$, such that T admits a representation*

$$\tag{3} T = \sum_{j=1}^{\rho} \alpha_j P_j, \quad \alpha_j \in \mathbf{R},$$

where

$$\max_{1 \le j \le \rho} |\alpha_j| \le (15 + \delta)\|T\|.$$

Proof We assume (without loss of generality) that T is a nonzero contraction, *i.e.*, $0 < \|T\| \le 1$.

Assume first that H is infinite dimensional. We compile arguments from [15] and [23]. The proof of Theorem 1 in [15] shows that for every selfadjoint contraction A there exist $B_1, B_2 \in \mathcal{B}(H)$ such that

$$\tag{4} A = (B_1^* B_1 - B_1 B_1^*) + (B_2^* B_2 - B_2 B_2^*) \text{ and } \|B_1\|^2 \le \|A\|, \quad \|B_2\|^2 \le \|A\|.$$

Now we follow the proof of Theorem 3 in [23]. We use the notation of that proof except that we replace H by M since in our paper the symbol H denotes the underlying Hilbert space. Using (4) we choose first $\lambda > \sqrt{2}\|T\|^{1/2} \ge \|K + L\|^{1/2}$ and then $\mu > \|T\| + 2\lambda^2 \ge \|M\|$. Applying the inequalities

$$\|C\| \le \|T\| + 2\lambda^2 + 2\mu, \quad \|C_1\| \le \|C\|, \ \|C_2\| \le \|C\|,$$

we finally choose

$$\sigma > \|T\| + 2\lambda^2 + 2\mu.$$

As shown in [23], for every such choice of λ, μ, ν, representation (3) holds with

$$\alpha_j \in \{\pm\lambda^2, \pm\sigma, \pm\mu\}.$$

Clearly, $\max_{1 \le j \le \rho} |\alpha_j| = \sigma$. Now for a fixed $\epsilon > 0$, let $\lambda = \sqrt{2}\|T\|^{1/2} + \epsilon$, $\mu = \|T\| + 2(\sqrt{2}\|T\|^{1/2} + \epsilon)^2 + \epsilon$, $\sigma = (15 + \delta)\|T\|$. Then

$$\sigma > \|T\| + 2(\sqrt{2}\|T\|^{1/2} + \epsilon)^2 + 2\big(\|T\| + 2(\sqrt{2}\|T\|^{1/2} + \epsilon)^2 + \epsilon\big) = \|T\| + 2\lambda^2 + 2\mu$$

for sufficiently small ϵ.

Now assume that $\dim H = n$ is finite. By a result of Fong [13], (4) holds with $B_1 = B_2$ provided the trace of A is zero. Thus, if trace $T = 0$ is zero, and n is even, we repeat the arguments for the infinite dimensional H. If the trace of T is not zero, we subtract (trace $T/n)I$ from T, and reduce the problem to the already solved case at the expense of increasing ρ by 1. If n is odd, we write $T = \lambda x^* x + T_0$, where x is a normalized eigenvector of T corresponding to an eigenvalue λ, and T_0 is a restriction of T to the orthogonal complement of Span x; thereby the problem is reduced to an even dimensional H, again at the expense of increasing ρ by 1. ∎

Corollary 4 *If ϕ is a linear map on $\mathcal{B}(H)$ or on $\mathcal{H}(H)$ that maps the set of selfadjoint elements into itself and is such that ϕ is uniformly bounded on the set of orthogonal projections, then ϕ is continuous. Moreover, $\|\phi\| \leq 15\rho Q$ if ϕ is defined on $\mathcal{H}(H)$, and $\|\phi\| \leq 30\rho Q$ if ϕ is defined on $\mathcal{B}(H)$, where*

$$Q = \sup\{\|\phi(X)\| : X \text{ orthogonal projection}\}.$$

Proof If ϕ is defined on $\mathcal{H}(H)$, everything follows from Lemma 3. If ϕ is defined on $\mathcal{B}(H)$, write $C = A + iB \in \mathcal{B}(H)$, where $A, B \in \mathcal{H}(H)$, and using Lemma 3 we obtain:

$$\|\phi(A + iB)\| \leq \|\phi(A) + i\phi(B)\| \leq \|\phi(A)\| + \|\phi(B)\|$$
$$\leq 15\rho Q\|A\| + 15\rho Q\|B\| \leq 30\rho Q\|C\|. \qquad \blacksquare$$

Note that the result of Corollary 4 fails for linear maps on $\mathcal{B}_{\mathcal{K}}(H)$ or on $\mathcal{H}_{\mathcal{K}}(H)$. Indeed, assume that H is an infinite dimensional Hilbert space. Write $\mathcal{H}_{\mathcal{K}}(H) = \mathcal{H}_{\mathcal{F}}(H) \oplus V$, where $\mathcal{H}_{\mathcal{F}}(H)$ is the linear space of operators of the form $tI + F$ with t a real scalar and F a finite rank selfadjoint operator, and where the linear space V is any complement of $\mathcal{H}_{\mathcal{F}}(H)$ in $\mathcal{H}_{\mathcal{K}}(H)$. Define a real linear map $\phi\colon \mathcal{H}_{\mathcal{K}}(H) \to \mathcal{H}_{\mathcal{K}}(H)$ by $\phi(A) = A$ whenever $A \in \mathcal{H}_{\mathcal{F}}(H)$ and $\phi(A) = -A$ whenever $A \in V$. This linear map is obviously discontinuous. However, ϕ maps orthogonal projections in $\mathcal{H}_{\mathcal{K}}(H)$ into themselves. A similar example can be set up for $\mathcal{B}_{\mathcal{K}}(H)$.

3 Positive Definiteness Preservers

Denote by $\mathcal{B}(H)^+$ the convex cone of positive definite operators on H. We have the following result extending a theorem of Schneider [26] to the infinite dimensional case.

Theorem 5 *Let $\mathbf{V}$ be one of $\mathcal{B}(H)$, $\mathcal{H}(H)$, $\mathcal{B}_{\mathcal{K}}(H)$, or $\mathcal{H}_{\mathcal{K}}(H)$, and let $\phi\colon \mathbf{V} \to \mathbf{V}$ be a linear map. Then ϕ is injective and satisfies $\phi\big(\mathbf{V} \cap \mathcal{B}(H)^+\big) = \mathbf{V} \cap \mathcal{B}(H)^+$ if and only if there exists an invertible $T \in \mathcal{B}(H)$, with the additional property that $TT^* \in \mathcal{H}_{\mathcal{K}}(H)$ if $\mathbf{V}$ is one of $\mathcal{B}_{\mathcal{K}}(H)$ or $\mathcal{H}_{\mathcal{K}}(H)$, such that ϕ has the form*

$$(5) \qquad\qquad X \mapsto TXT^* \quad or \quad X \mapsto TX^t T^*,$$

where X^t is the transpose of X with respect to some fixed orthonormal basis of H.

Notice that *a priori* ϕ is not assumed to be continuous. It turns out that under the hypotheses of the theorem, continuity of ϕ is guaranteed. The assumption that ϕ is injective is essential in Theorem 5. Otherwise, one can consider $\phi\colon \mathbf{V} \to \mathbf{V}$ defined by $\phi(X) = S^*XS$, where $S\colon H \to H$ is the shift operator defined by $Se_j = e_{j+1}$ for $j = 1, 2, \ldots$, for a fixed orthonormal basis $\{e_1, e_2, \ldots, \}$ of H (assuming that H is separable and infinite dimensional). Then $\phi\big(\mathbf{V} \cap \mathcal{B}(H)^+\big) = \mathbf{V} \cap \mathcal{B}(H)^+$, but ϕ is not of the form (5).

If H is finite dimensional, then the condition $\phi\big(\mathbf{V} \cap \mathcal{B}(H)^+\big) = V \cap \mathcal{B}(H)^+$ is equivalent to the condition that ϕ maps the set of positive semidefinite matrices onto itself (this latter condition was used in [26]), and every such linear map is automatically bijective and of course continuous.

Proof The "if" direction is clear. For the converse, first note that the cases when $\mathbf{V} = \mathcal{B}(H)$ or $\mathbf{V} = \mathcal{B}_{\mathcal{K}}(H)$ are easily reduced to the cases when $\mathbf{V} = \mathcal{H}(H)$ or $\mathbf{V} = \mathcal{H}_{\mathcal{K}}(H)$, respectively, by considering the restriction of ϕ to $\mathcal{H}(H)$ or to $\mathcal{H}_{\mathcal{K}}(H)$, as the case may be.

We first show that ϕ is surjective. For any $Y \in \mathbf{V}$, we can write $Y = B_1 - B_2$ for some $B_1, B_2 \in \mathbf{V} \cap \mathcal{B}(H)^+$. Then there exists $A_1, A_2 \in \mathbf{V} \cap \mathcal{B}(H)^+$ such that $\phi(A_1) = B_1$ and $\phi(A_2) = B_2$. Thus, for $X = A_1 - A_2$, we have $\phi(X) = Y$. As a result, we see that ϕ is bijective. Hence $A \in \mathbf{V}$ is positive definite if and only if $\phi(A)$ is positive definite.

Let $P = \phi(I)$. We may assume that $P = I$; otherwise, replace ϕ by the mapping $A \mapsto P^{-1/2}\phi(A)P^{-1/2}$. Notice that the spectral theorem for compact selfadjoint operators easily implies that

$$P \in \mathcal{H}_{\mathcal{K}}(H) \cap \mathcal{B}(H)^+ \Rightarrow P^{-1/2} \in \mathcal{H}_{\mathcal{K}}(H) \cap \mathcal{B}(H)^+.$$

Now, for any operator $A \in \mathbf{V}$, $tI - A$ is positive definite if and only if $tI - \phi(A)$ is positive definite. By Lemma 1, ϕ preserves the closure of numerical range of $A \in \mathbf{V}$, *i.e.*, the property (d) of Theorem 2 holds. By the equivalence of (d) and (e) of Theorem 2, the result follows. ∎

4 Invertibility preservers

Theorem 6   *Let $\phi\colon \mathcal{H}(H) \to \mathcal{H}(H)$ be a bijective real linear operator that maps the set of invertible selfadjoint operators into itself. Then there exist an invertible $T \in \mathcal{B}(H)$, and a number $\xi \in \{1, -1\}$ such that ϕ has the form*

$$(6) \qquad\qquad X \mapsto \xi TXT^* \quad or \quad X \mapsto \xi TX^tT^*,$$

where X^t denotes the transpose of X with respect to some fixed orthonormal basis of H.

Proof Since ϕ is surjective there exists $A \in \mathcal{H}(H)$ such that $\phi(A) = I$. We first show that there is no loss of generality in assuming that A is invertible. Indeed, if this is not the case, then by the spectral theorem for selfadjoint operators there exists a projection $P \in \mathcal{H}(H)$ such that $A_1 = PAP$ is positive semidefinite, $A_2 = (I-P)A(I-P)$ is negative semidefinite, and $A = A_1 + A_2$. If $B = P - (I - P)$, then $A + \varepsilon B$ is invertible for every positive real number ε. For an ε small enough the operator $\phi(A+\varepsilon B) = I + \varepsilon\phi(B) = R$ is positive definite. Replacing ϕ by $X \mapsto R^{-1/2}\phi(X)R^{-1/2}$ we get a bijective linear invertibility preserving map ϕ such that $\phi(A) = I$ for an invertible $A \in \mathcal{H}(H)$.

Next, we prove that ϕ is continuous. We continue to assume that $A \in \mathcal{H}(H)$ is invertible such that $\phi(A) = I$. Let X be any operator from $\mathcal{H}(H)$ and t a real number

with $|t| > \|X\|\|A^{-1}\|$. Then a (not necessarily selfadjoint) operator $tI - XA^{-1}$ is invertible, and consequently, $tA - X$ is invertible. By the invertibility preserving property $tI - \phi(X)$ is invertible. Hence, the spectrum of $\phi(X)$ is contained in the interval $\left[-\|X\|\,\|A^{-1}\|, \|X\|\,\|A^{-1}\|\right]$, which further yields that $\|\phi\| \leq \|A^{-1}\|$.

Since $\phi(I)$ is invertible it is either positive definite, or negative definite, or there exists a unitary operator $U \colon H \to H$ such that $U^*\phi(I)U$ has a matrix representation

$$(7) \qquad\qquad \begin{bmatrix} P & 0 \\ 0 & -Q \end{bmatrix}$$

with respect to a nontrivial orthogonal decomposition of H, where P and Q are positive definite. We will show that the third possibility cannot occur.

In order to do this we argue by contradiction, and suppose that $U^*\phi(I)U$ has the form (7). We first prove that if invertible $A \in \mathcal{H}(H)$ satisfies $\phi(A) = I$, then A is neither positive definite nor negative definite. Assume on the contrary that A is positive definite. Then $\phi\big(tA + (1-t)I\big)$ is a continuous path from $U \begin{bmatrix} P & 0 \\ 0 & -Q \end{bmatrix} U^*$ to I, such that every point on this path is an invertible selfadjoint operator. Since $\sigma(I) = \{1\}$, the result from [21] on the variation of the spectrum implies that every operator on this path is positive definite, contradicting the fact that the endpoint of this path is $U \begin{bmatrix} P & 0 \\ 0 & -Q \end{bmatrix} U^*$. In the same way we see that A is not negative definite. Hence, there exists an invertible $S \in \mathcal{B}(H)$ such that $A = S \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} S^*$. Then $\psi \colon \mathcal{H}(H) \to \mathcal{H}(H)$ defined by $\psi(X) = \phi(SXS^*)$ is a bijective real linear operator preserving invertibility and having the property that $\psi\left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\right) = I$. For any $\begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix}$, the operator

$$\begin{bmatrix} I & X^* \\ X & -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I - XX^* \end{bmatrix} \begin{bmatrix} I & X^* \\ 0 & I \end{bmatrix}$$

is invertible. It follows that

$$I + t\psi\left(\begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix}\right) = \psi\left(\begin{bmatrix} I & tX^* \\ tX & -I \end{bmatrix}\right)$$

is invertible for every real number t which further yields that the spectrum of $\psi\left(\begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix}\right)$ contains only 0, or equivalently, $\psi\left(\begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix}\right) = 0$, contradicting the bijectivity assumption. Thus, A cannot be indefinite either.

Hence, we have proved that $\phi(I)$ is either positive definite or negative definite. Applying the fact that every positive definite operator has a positive definite square root we may assume without loss of generality that either $\phi(I) = I$, or $\phi(I) = -I$, and after multiplying ϕ by -1, if necessary, we may assume that $\phi(I) = I$. If P is any orthogonal projection, then $tI - P$ is invertible whenever $t \neq 0, 1$, and so is $tI - \phi(P)$. It follows that $\phi(P)$ is an orthogonal projection. Since $\phi(P) \notin \{0, I\}$ if $P \notin \{0, I\}$, condition (b) of Theorem 2 holds. By the equivalence of (b) and (e) in Theorem 2, the result follows. ∎

The following result analogous to Theorem 6 holds for $\mathcal{H}_{\mathcal{K}}(H)$.

Theorem 7   *Let* $\phi\colon \mathcal{H}_{\mathcal{K}}(H) \to \mathcal{H}_{\mathcal{K}}(H)$ *be a bijective real linear operator that maps invertible selfadjoint operators in* $\mathcal{H}_{\mathcal{K}}(H)$ *into invertible operators. Then there exist a bounded invertible linear operator* $T\colon H \to H$ *such that* $TT^* \in \mathcal{H}_{\mathcal{K}}(H)$, *and a number* $\xi \in \{1, -1\}$, *such that* ϕ *has the form*

$$X \mapsto \xi TXT^* \quad or \quad X \mapsto \xi TX^t T^*$$

where X^t *denotes the transpose of* X *with respect to some fixed orthonormal basis of* H.

Proof First, we note the following easily proved properties:

(a) If $X \in \mathcal{B}_{\mathcal{K}}(H)$ is invertible in $\mathcal{B}(H)$, then $X^{-1} \in \mathcal{B}_{\mathcal{K}}(H)$;
(b) If $X \in \mathcal{H}_{\mathcal{K}}(H)$ is positive definite, then $\sqrt{X} \in \mathcal{H}_{\mathcal{K}}(H)$.

The proof of Theorem 7 proceeds by essentially repeating the arguments from the proof of Theorem 6, with obvious changes, and using the properties (a) and (b) above. The only non-obvious part is the claim proved in the first paragraph of the proof of Theorem 6, namely, that we can assume that $A = \phi^{-1}(I)$ is invertible. Since $A \in \mathcal{H}_{\mathcal{K}}(H)$, the spectral theorem for compact selfadjoint operators shows that $A + \varepsilon I$ is invertible for infinitely many real values of ε in every neighborhood of zero. Now choose ε small enough so that $A + \varepsilon I$ is invertible and $\phi(A + \varepsilon I) = I + \varepsilon \phi(I)$ is positive definite.

The condition $TT^* \in \mathcal{H}_{\mathcal{K}}(H)$ follows from the property that $TXT^* \in \mathcal{H}_{\mathcal{K}}(H)$ for every $X \in \mathcal{H}_{\mathcal{K}}(H)$.   ∎

Note that the assumption that ϕ is surjective is essential in Theorems 6 and 7. Otherwise, assuming H is infinite dimensional, we can identify H with $H \oplus H$, and define ϕ by $\phi(X) = X \oplus X$ (acting on H). Then ϕ maps the set of invertible selfadjoint operators into itself, but ϕ is not of the form (6). The surjectivity is essential if $H = \mathbf{C}^2$; the linear map

$$\phi \begin{bmatrix} t & x \\ x^* & s \end{bmatrix} = \begin{bmatrix} t & x \\ x^* & -t \end{bmatrix}, \quad t, s \in \mathbf{R}, \ x \in \mathbf{C}.$$

will map the set of invertible selfadjoint operators on $\mathbf{C}^2$ into itself, but ϕ is not of the form (6). If $\dim H \geq 3$ is finite, then Theorem 6 holds without the bijectivity assumption by the result in [6, 18]; see [6, Theorem 10] for the odd dimension cases, and see [6, Lemma 7] and [18, Theorem 2.2] for the even dimension cases. In this regards the following open problem is of interest.

Problem 8   Suppose $\dim H$ is infinite. Are Theorems 6 and 7 valid under the weaker hypothesis that ϕ is surjective, rather than bijective?

Problem 8 has an affirmative answer for Theorem 6 under some additional assumptions on ϕ or H.

Theorem 9   *Let* $\phi\colon \mathcal{H}(H) \to \mathcal{H}(H)$ *be a surjective linear operator that maps the set of invertible selfadjoint operators into itself. Then* ϕ *has the form* (6) *if any one of the following conditions holds.*

(a) $\phi(I)$ *is positive definite or negative definite.*
(b) *H is separable.*

We need a lemma to prove Theorem 9.

Lemma 10 *Let $H = H_1 \oplus H_1^{\perp}$ be an orthogonal decomposition of the Hilbert space H, and assume that one of the subspaces H_1 or $H_1^{\perp}$ is finite dimensional. Then, if an invertible operator $T \in \mathcal{H}(H)$ is partitioned $\begin{bmatrix} X_1 & X^* \\ X & X_2 \end{bmatrix}$ conformally with the orthogonal decomposition $H = H_1 \oplus H_1^{\perp}$, then there exists a bounded linear operator $Y \colon H_1 \to H_1^{\perp}$ such that*

$$(8) \qquad \begin{bmatrix} X_1 & Y^* \\ Y & -X_2 \end{bmatrix}$$

is invertible as well.

Proof Say, $H_1^{\perp}$ has dimension $n < \infty$ (if H_1 is finite dimensional, just replace (8) with its negative). Thus, X_2 is in fact a Hermitian $n \times n$ matrix. There exists an invertible matrix S such that

$$(9) \qquad S^* X_2 S = \begin{bmatrix} X_{22} & 0 \\ 0 & \pm I \end{bmatrix},$$

where X_{22} is a $p \times p$ Hermitian matrix which is congruent to $-X_{22}$, and I is the $(n-p) \times (n-p)$ identity matrix. Partition $S^* X = \begin{bmatrix} X_{11} \\ X_{12} \end{bmatrix}$, where $X_{11} \colon H_1 \to \mathbf{C}^p$, $X_{12} \colon H_1 \to \mathbf{C}^{n-p}$. So, we may assume that

$$T = \begin{bmatrix} X_1 & X_{11}^* & X_{12}^* \\ X_{11} & X_{22} & 0 \\ X_{12} & 0 & \pm I \end{bmatrix},$$

where the third row and the third column may be absent. We will consider only the case that the $(3,3)$-entry is I.

Now let R be invertible such that $R^* X_{22} R = -X_{22}$. Then

$$\begin{bmatrix} I & 0 & 0 \\ 0 & R^* & 0 \\ 0 & 0 & I \end{bmatrix} T \begin{bmatrix} I & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} X_1 & X_{11}^* R & X_{12}^* \\ R^* X_{11} & -X_{22} & 0 \\ X_{12} & 0 & I \end{bmatrix}$$

is invertible and we are done if the third row and the third column are absent. If this is not the case, then denote

$$Z = \begin{bmatrix} X_1 & X_{11}^* R \\ R^* X_{11} & -X_{22} \end{bmatrix} \quad \text{and} \quad W = [X_{12} \quad 0].$$

Schur complements show that $Z - W^* W$ is invertible. Consider the analytic operator valued function $F(z) = Z + z W^* W$, $z \in \mathbf{C}$. The values of $F(z)$ are Fredholm operators, and $F(-1)$ is invertible. By Gohberg's theorem [14], $F(z)$ is invertible for all

z excepting a discrete set of values. In particular, $F(\alpha)$ is invertible for some $\alpha > 0$. Applying Schur complements once again we see that

$$\begin{bmatrix} X_1 & X_{11}^* R & \sqrt{\alpha} X_{12}^* \\ R^* X_{11} & -X_{22} & 0 \\ \sqrt{\alpha} X_{12} & 0 & -I \end{bmatrix}$$

is invertible, as desired.   ∎

Proof of Theorem 9   We first consider (a). Without loss of generality, we may assume that $\phi(I)$ is positive definite; otherwise, replace ϕ by the mapping $X \mapsto -\phi(X)$. Furthermore, we may assume that $\phi(I) = I$; otherwise, replace ϕ by the mapping $X \mapsto \phi(I)^{-1/2}\phi(X)\phi(I)^{-1/2}$. We need to show that ϕ is injective. So, we may assume that H is infinite-dimensional. As in the proof of Theorem 6 we see that ϕ is continuous and maps orthogonal projections into orthogonal projections. Using an idea from [7] we will show that $\phi(X^2) = \phi(X)^2$ for every $X \in \mathcal{H}(H)$ with a finite spectrum. Such an X can be written as $X = \sum_{k=1}^n t_i P_i$ where the t_i's are real numbers and P_i are orthogonal projections summing up to I and satisfying $P_i P_j = P_j P_i = 0$ whenever $i \neq j$. Since $P_i + P_j$ is a projection if $i \neq j$, we have $\big(\phi(P_i) + \phi(P_j)\big)^2 = \phi(P_i) + \phi(P_j)$. This yields $\phi(P_i)\phi(P_j) + \phi(P_j)\phi(P_i) = 0$. Using this equation we get the desired relation $\phi(X^2) = \phi(X)^2$. The set of all selfadjoint operators with a finite spectrum is dense in $\mathcal{H}(H)$, and so, by the continuity of ϕ we have $\phi(X^2) = \phi(X)^2$ for every $X \in \mathcal{H}(H)$. So, ϕ is a Jordan homomorphism. Using a result of Martindale [20] we can extend ϕ to a ring homomorphism (additive and multiplicative map) $\varphi \colon \mathcal{B}(H) \to \mathcal{B}(H)$. In particular, $\varphi(iI)\varphi(H) = \varphi(iH) = \varphi(H)\varphi(iI)$ for every selfadjoint operator H. Thus, $\varphi(iI)$ commutes with every selfadjoint operator and is therefore a scalar operator. It is now easy to conclude that $\varphi(iI) = \pm iI$. Hence, φ is linear or conjugate-linear. Using also the fact that the range of φ contains all selfadjoint operators we see that φ is surjective. Obviously, it is continuous. We prove that φ is injective in the following.

Assume on the contrary that the kernel of φ is a nontrivial closed ideal in $\mathcal{B}(H)$. All nontrivial closed ideals in $\mathcal{B}(H)$ have the form $\mathcal{J}_\alpha$, where α is any infinite cardinal less than or equal to the dimension of $\mathcal{B}(H)$, which is defined as follows: $\mathcal{J}_\alpha$ consists of those $X \in \mathcal{B}(H)$ whose range does not contain any closed subspaces of H of dimension α (see [12] for details). Thus, $\mathrm{Ker}\,\varphi = \mathcal{J}_\alpha$ for some α, and φ induces a continuous $*$-isomorphism (or conjugate linear isomorphism) between Banach algebras $\mathcal{B}(H)$ and $\mathcal{B}(H)/\mathcal{J}_\alpha$. If $\alpha = \aleph_0$, then $\mathcal{J}_{\aleph_0}$ is the ideal of compact operators, and we have a contradiction, because the set of invertibles in $\mathcal{B}(H)$ is connected, whereas the set of invertibles in $\mathcal{B}(H)/\mathcal{J}_{\aleph_0}$ is disconnected. If $\alpha > \aleph_0$, we also have a contradiction, because for every closed ideal $\mathcal{J}/\mathcal{J}_\alpha$ of the algebra $\mathcal{B}(H)/\mathcal{J}_\alpha$ the invertibles in the factor algebra $\mathcal{B}(H)/\mathcal{J}$ are connected (by the result of Corollary 3 of [11]), whereas the algebra $\mathcal{B}(H)$ has a closed ideal, namely $\mathcal{J}_{\aleph_0}$, such that the invertibles in the factor algebra $\mathcal{B}(H)/\mathcal{J}_{\aleph_0}$ form a disconnected set. Our proof of (a) is complete.

Next, we turn to (b). Repeating the first part of the proof of Theorem 6, we may assume that $\phi(A) = I$ for some invertible $A \in \mathcal{H}(H)$. By (a), it suffices to show that $\phi(I)$ is not indefinite. Arguing by contradiction, as in the proof of Theorem 6 we

obtain that there exist a nontrivial orthogonal decomposition $H = H_1 \oplus H_1^\perp$, and a surjective linear map $\psi\colon \mathcal{H}(H) \to \mathcal{H}(H)$ that preserves invertibility and is such that $\psi\left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\right) = I$, and

$$(10) \qquad \psi\left(\begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix}\right) = 0 \quad \text{for every } X\colon H_1 \to H_1^\perp.$$

Clearly, $\dim H_1 \neq \dim H_1^\perp$; otherwise, an invertible operator of the form $\begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix}$ would exist, a contradiction with the invertibility preserving property of ψ. Hence, one of H_1 or $H_1^\perp$ must be finite dimensional.

Define $\tilde{\psi}\colon \mathcal{H}(H) \to \mathcal{H}(H)$ by

$$\tilde{\psi}\left(\begin{bmatrix} X_1 & X \\ X^* & X_2 \end{bmatrix}\right) = \psi\left(\begin{bmatrix} X_1 & 0 \\ 0 & -X_2 \end{bmatrix}\right).$$

Then $\tilde{\psi}(I) = I$. Because of (10), and because of the surjectivity of ψ, the linear map $\tilde{\psi}$ is surjective as well. Since ψ maps invertible operators into invertible operators, by Lemma 10, the map $\tilde{\psi}$ also has this property. Now by the result in part (a), $\tilde{\psi}$ must be bijective, a contradiction. ∎

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