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OSCILLATORY INTEGRALS WITH NONHOMOGENEOUS PHASE FUNCTIONS RELATED TO SCHRÖDINGER EQUATIONS

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ABSTRACT. In this paper we consider solutions to the free Schrödinger equation in $n+1$ dimensions. When we restrict the last variable to be a smooth function of the first n variables we find that the solution, so restricted, is locally in L^2, when the initial data is in an appropriate Sobolev space.

1**. Introduction.** Consider, for a fixed smooth function $t(x)$, the solution to the Schrödinger equation

$$\begin{cases} i\partial_t u(x,t) + \Delta_x u(x,t) = 0 & (x,t) \in \mathbb{R}^{n+1} \\ u(x,0) = f(x) \in L^2(\mathbb{R}^n) \end{cases}$$

at time $t = t(x)$—$u\big(x, t(x)\big)$. We obtain results of the form

$$\big\| u\big(\,\cdot\,, t(\,\cdot\,)\big) \big\|_{L^2(\mathbb{D}^n)} \leq C \, \| f \|_{H^s} \,,$$

where s depends on the smoothness of t. Here $H^s(\mathbb{R}^n)$ denotes the L^2-Sobolev space,

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \| f \|_{H^s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} < \infty \right\},$$

and $\mathbb{D}^n$ denotes the closed unit disk in $\mathbb{R}^n$.

This is motivated by a desire to understand the Schrödinger maximal operator,

$$u^*(x) = \sup_{|t| \leq 1} \big| u(x,t) \big|.$$

One would like to prove an estimate of the form

$$\| u^* \|_{L^2(\mathbb{D}^n)} \leq C \, \| f \|_{H^s} \,, \tag{1.1}$$

which in turn implies that $\lim_{t \to 0} u(x,t) = f(x)$ a.e., whenever $f \in H^s(\mathbb{R}^n)$. One way to prove an estimate as in (1.1) is to consider, for an arbitrary bounded measurable function $t(x)$, the operator $S_t \colon H^s \longrightarrow L^2(\mathbb{D}^n)$ defined by

$$S_t f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it(x)|\xi|^2} \hat{f}(\xi) \, d\xi = u\big(x, t(x)\big),$$

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and show that

$$\|S_t f\| = \big\|u\big(\,\cdot\,, t(\,\cdot\,)\big)\big\|_{L^2(\mathbb{D}^n)} \leq C\,\|f\|_{H^s}\,,$$

where C is *uniform* over the family of operators S_t. When $n = 1$ the definitive result is that (1.1) is satisfied for all $f \in H^s$ if and only if $s \geq 1/4$. There are no such sharp results when $n \geq 2$; $s \geq 1/4$ is always a necessary condition, while $s > 1/2$ is a sufficient condition when $n \geq 3$, and $s > 1/2 - \epsilon$ for some positive ϵ is a sufficient condition when $n = 2$. See [1], [2], [3], [8], [10] and [12].

When studying S_t one may first consider for $k = 0, 1, \ldots$ the family of operators $R_k \colon L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{D}^n)$ of the form,

$$R_k f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot y + t(x)|y|^2)} \theta_k(y) f(y)\,dy, \tag{1.2}$$

so as to reduce H^s estimates to L^2 estimates. Here $\{\theta_k\}_{k=0}^{\infty}$ is a partition of unity subordinate to dyadic intervals. This family of operators R_k is similar to another one-parameter family of operators, the so-called oscillatory integral operators, $T_\lambda \colon L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, of the form

$$T_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)} a(x,y) f(y)\,dy. \tag{1.3}$$

These operators are typically studied when the phase function, $\phi(x,y)$, is smooth, and the amplitude $a(x,y)$ is a smooth, compactly supported function ([4], [6], [7], [9], and [11])[1]. Strictly speaking, R_k is not an oscillatory integral operator, even if $t(x)$ is a smooth function, since the "phase function" $x \cdot y + t(x)|y|^2$ is not homogeneous. If we make the change of variables $y \longrightarrow 2^k y$, R_k cannot be put in the form T_λ. So even if we do assume that $t(x)$ is a smooth function, the fact that we will prove estimates of the form

$$\|R_k f\|_{L^2(\mathbb{D}^n)} \leq C 2^{s_0 k}\,\|f\|_2$$

for R_k in Section 3 is novel. Here s_0 is a number that depends on the smoothness of the function t.

We prove the following theorems in this paper.

THEOREM 1. *Suppose $t \in C^\infty$ is such that $\nabla t(x)$ does not vanish on $\mathbb{D}^n$. Then for any $s > 0$,*

$$\big\|u\big(\,\cdot\,, t(\,\cdot\,)\big)\big\|_{L^2(\mathbb{D}^n)} \leq C\,\|f\|_{H^s}\,,$$

where C may depend on s and t.

THEOREM 2. *Suppose that t has only non-degenerate critical points. Then for any $s > 0$,*

$$\big\|u\big(\,\cdot\,, t(\,\cdot\,)\big)\big\|_{L^2(\mathbb{D}^n)} \leq C\,\|f\|_{H^s}\,,$$

when $n = 1$ or $n = 2$.

[1] See [1] for an example of a non-smooth phase function.

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THEOREM 3. *There is a smooth function $t(x)$ such that an estimate*

$$\left\| u\left(\,\cdot\,,t(\,\cdot\,)\right)\right\|_{L^2(\mathbb{D}^n)} \leq C\,\|f\|_{H^s}$$

cannot hold for all $f \in H^s$ whenever $s < 1/4$.

2. **Preliminary Lemmas.** Throughout this paper we shall let $t(x)$ denote a fixed, given $C^\infty(\mathbb{D}^n)$ function. We will use a standard partition of unity subordinate to dyadic intervals $\{\theta_k\}_{k=0}^{\infty}$: $\theta_0 \in C_0^\infty(|y| \leq 2)$, $\theta_0(y) = 1$ when $|y| \leq 1$; $\theta_k(y) = \theta_0(2^{-k}y) - \theta_0(2^{1-k}y)$, when $k \geq 1$.

If $a(x,y)$ is a function of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, then denote by $\operatorname{supp}_x(a)$ the projection onto the x-coordinates of the support of a. Let $\nabla_y a(x,y)$ denote the gradient of a as a function of y with x held fixed.

The expression $x \lesssim y$ means that there is a constant C whose particular value is unimportant such that $x \leq Cy$.

Recalling the definition of R_k in (1.2), we begin with the following lemma whose purpose, as noted earlier, is to reduce H^s estimates to L^2 estimates. This approach is found in [1].

LEMMA 1. *Suppose there are constants C and s_0 such that*

$$\|R_k f\|_{L^2(\mathbb{D}^n)} \leq C2^{s_0 k}\,\|f\|_2\,.$$

Then for any $s > s_0$ there is a constant C_s depending on C and s such that

$$\left\| u\left(\,\cdot\,,t(\,\cdot\,)\right)\right\|_{L^2(\mathbb{D}^n)} \leq C_s\,\|f\|_{H^s}\,.$$

PROOF. Note that $R_k \hat{f} = R_k(\chi_{[\operatorname{supp}(\theta_k)]}\hat{f})$. Hence

$$\begin{aligned}\|R_k \hat{f}\|_{L^2(\mathbb{D}^n)} &\leq C2^{s_0 k}\left(\int_{\operatorname{supp}(\theta_k)} |\hat{f}(y)|^2\,dy\right)^{1/2} \\ &\lesssim (2^{-(s-s_0)})^k \left(\int_{\mathbb{R}^n} |y|^{2s}|\hat{f}(y)|^2\,dy\right)^{1/2} \leq C2^{-(s-s_0)k}\,\|f\|_{H^s}\,.\end{aligned}$$

Then by Minkowski's inequality,

$$\left\| u\left(\,\cdot\,,t(\,\cdot\,)\right)\right\|_{L^2(\mathbb{D}^n)} \leq \sum_{k=0}^{\infty} \|R_k \hat{f}\|_{L^2(\mathbb{D}^n)} \leq C\sum_{k=0}^{\infty} 2^{-(s-s_0)k}\;\|f\|_{H^s} = C_s\,\|f\|_{H^s}\,,$$

as desired.

REMARK. We may multiply R_k by a C_0^∞ function α which is unity on $\mathbb{D}^n$, if necessary, and all results about this "new" R_k will be the same as for that in (1.2). By abuse of notation R_k will denote either one.

We shall also need the converse of Lemma 1.

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LEMMA 2. *Suppose there are constants C and ρ, independent of k, such that, as an operator form $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{D}^n)$,*

$$\|R_k\| \geq C2^{\rho k}. \tag{2.1}$$

Then the map

$$S_t f(x) = u\big(x, t(x)\big) \tag{2.2}$$

is not a bounded map from $H^s(\mathbb{R}^n) \longrightarrow L^2(\mathbb{D}^n)$ for any $s < \rho$.

PROOF. The condition (2.1) means that for each $k = 1, 2, \ldots$ there exists a function $f_k \in L^2$ such that $\|f_k\|_2 = 1$, $\operatorname{supp}(f_k) \in \operatorname{supp}(\theta_k)$ and for which

$$\|R_k f_k\|_{L^2(\mathbb{D}^n)} \geq C2^{\rho k}.$$

Choose $g_k \in L^2$ such that $\hat{g}_k = f_k$. Fix an s for which S in (2.2) is a bounded map from $H^s(\mathbb{R}^n) \longrightarrow L^2(\mathbb{D}^n)$. On the one hand, by (2.1),

$$\|S_t g_k\|_{L^2(\mathbb{D}^n)} = \|R_k f_k\|_{L^2(\mathbb{D}^n)} \geq C2^{\rho k}.$$

On the other hand, by the choice of s,

$$\begin{aligned} \|S_{t(x)} g_k\|_{L^2(\mathbb{D}^n)} \leq C' \|g_k\|_{H^s} &= C' \left(\int |f_k(x)|^2 (1 + |x|^2)^s \, dx \right)^{1/2} \\ &\leq C'' 2^{sk} \|f_k\|_2 = C'' 2^{sk}. \end{aligned}$$

Hence $2^{(\rho - s)k} \lesssim 1$ for each k, which is only possible when $\rho \leq s$.

Our proof of Theorem 1 exploits the similarities between R_k and T_λ in (1.3). We use the fact that when the *mixed Hessian* of ϕ, the $n \times n$ matrix $H_\phi(x, y)$ defined by

$$\big(H_\phi(x, y)\big)_{i,j} = \frac{\partial^2 \phi}{\partial x_i \partial y_j}(x, y),$$

is non-singular on the support of a, the decay of $\|T_\lambda\|$ is as rapid as possible.

LEMMA 3. *Suppose that H_ϕ is non-singular on* $\operatorname{supp}(a)$ *and that the following quantities are uniformly bounded on* $\operatorname{supp}(a)$*:*

(i) $\|H_\phi^{-1}(x, y)\|$

(ii) $\|\nabla_y D_x^\alpha \phi\|_{L^\infty(X)}$ *for all α with $|\alpha| = 2$*

(iii) $\left\|\nabla_x D_y^\alpha \phi\right\|_{L^\infty(X)}$ *for all α with $|\alpha| \leq n + 2$.*

Then if $M = \max\{1, |\operatorname{supp}_x(a)|\}$ and

$$M_a = \|a\|_\infty \left(M |\operatorname{supp}_y(a)| \Big\{ \sum_{|\alpha| \leq n+1} \sup_{xyz} |D_y^\alpha a(x, y) \overline{a(x, z)}| \Big\}^{\frac{n}{n+1}} \right)^{1/2},$$

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then

$$\|T_\lambda f\|_2 \leq CM_a\lambda^{-n/2}\,\|f\|_2\,,$$

where C is bounded.

A proof of Lemma 3 may be found in [7]. See also [6], [11]. That the power of λ which appears in the conclusion of Lemma 3 is optimal is a consequence of the next lemma. The idea for this lemma, which we use to prove Theorem 3, is found in [6].

LEMMA 4. *Let T_λ be as in (1.3). Suppose that there are measurable sets $A \subset \operatorname{supp}_x(a)$ and $\tilde{A} \subset \operatorname{supp}_y(a)$ and measurable functions ϕ_1 and ϕ_2 such that*

$$\lambda|\phi(x,y) - \phi_1(x) - \phi_2(y)| < 1/2 \quad \text{when } (x,y) \in A \times \tilde{A}. \tag{2.3}$$

If $|a(x,y)| \geq c > 0$ when $(x,y) \in A \times \tilde{A}$, and

$$\left|\int_{\tilde{A}} a(x,y)\,dy\right| \geq 3/4 \int_{\tilde{A}} |a(x,y)|\,dy \tag{2.4}$$

then there is a positive constant C such that

$$\|T_\lambda\| \geq C\sqrt{|A|\,|\tilde{A}|}.$$

PROOF. Let $f(y) = e^{-i\lambda\phi_2(y)}\chi_{\tilde{A}}(y)$; then $\|f\|_2 = |\tilde{A}|^{1/2}$. When $x \in A$,

$$\begin{aligned} |T_\lambda f(x)| &= |e^{-i\lambda\phi_1(x)}T_\lambda f(x)| \\ &\geq \left|\int_{\tilde{A}} a(x,y)\,dy\right| - \left|\int_{\tilde{A}} (e^{i\lambda(\phi(x,y)-\phi_1(x)-\phi_2(y))} - 1)a(x,y)\,dy\right| \\ &= I + II. \end{aligned}$$

By condition (2.3)

$$|II| \leq 1/2 \int_{\tilde{A}} |a(x,y)|\,dy.$$

Thus (2.4) guarantees that

$$\int_A |T_\lambda f(x)|^2\,dx \geq C|A|\,|\tilde{A}|^2,$$

and dividing by $\|f\|_2^2$ gives the result.

When the mixed Hessian is degenerate having rank $n-1$, it may be beneficial to split coordinates. If $x, y \in \mathbb{R}^n$, write $x = (x', x_n)$ and $y = (y', y_n)$ where $x', y' \in \mathbb{R}^{n-1}$, the idea being that (after perhaps a change of variables) the mixed Hessian is nondegenerate in the x', y' variables. To execute this line of thinking we must recall the notion of frozen operators. For an operator of the form $Tf(x) = \int K(x,y)f(y)\,dy$, for each choice of x_n, y_n we define the frozen operator $T_{x_ny_n}\colon L^2(\mathbb{R}^{n-1}) \longrightarrow L^2(\mathbb{R}^{n-1})$ by $T_{x_ny_n}f(x') = \int K(x', x_n, y', y_n)f(y')\,dy'$. Lemma 5 and Lemma 6 are the main technical device regarding frozen operators that we make use of in the proof of Theorem 1. Their proofs are elementary and may be found in [7].

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LEMMA 5. *Let T be as above, T^* the adjoint of T. Then as operators from $L^2(\mathbb{R}^{n-1})$ to $L^2(\mathbb{R}^{n-1})$*

$$\|T_{x_n z_n}\| \le \left(\int_{-\infty}^{\infty} \|(T^*)_{z_n y_n}\|^2 \, dy_n\right)^{1/2} \left(\int_{-\infty}^{\infty} \|(T)_{x_n y_n}\|^2 \, dy_n\right)^{1/2}.$$

LEMMA 6. *Suppose there exists a measurable function $\eta(x_n, y_n)$ such that*

$$\|T_{x_n y_n} f\|_{L^2(\mathbb{R}^{n-1})} \le \eta(x_n, y_n) \, \|f\|_{L^2(\mathbb{R}^{n-1})},$$
$$\left\| \int \eta(x_n, y_n) h(y_n) \, dy_n \right\|_{L^2(\mathbb{R})} \le C \|h\|_{L^2(\mathbb{R})}.$$

Then

$$\|Tf\|_{L^2(\mathbb{R}^n)} \le C \, \|f\|_{L^2(\mathbb{R}^n)}.$$

Finally, our proofs rely on stationary phase estimates, which, while at times delicate, are standard. The reader is referred to [5], [7] or [11] for details.

3. **A Proof of Theorem 1.** Given Lemma 1 a proof of Theorem 1 follows from the appropriate estimate for R_k.

PROPOSITION 1. *Suppose $t \in C^\infty$ is such that $\nabla t(x) \neq 0$ for all $x \in \mathbb{D}^n$. Then there is a constant C, which is independent of k, such that*

$$\|R_k f\|_{L^2(\mathbb{D}^n)} \le C \, \|f\|_2.$$

PROOF. Since ∇t does not vanish we may assume that $\text{supp}_x(a)$ is a small neighborhood of the origin of $\mathbb{R}^n$ on which there is a C^∞ diffeomorphism ρ such that $t \circ \rho(x) = x_n$, and for which $D\rho(0) = \mathrm{I}$, the $n \times n$ identity matrix. Let $\lambda = 2^k$. After making a change of variables $\big(x \longrightarrow \rho(x), y \longrightarrow \lambda y\big)$ it suffices to show that

$$\|\tilde{R}_\lambda f\|_{L^2(\mathbb{D}^n)} \lesssim \lambda^{-n/2} \, \|f\|_2, \tag{3.1}$$

where

$$\tilde{R}_\lambda f(x) = \alpha(x) \int_{\mathbb{R}^n} e^{i(\lambda \rho(x) \cdot y + \lambda^2 x_n |y|^2)} \theta_1(y) f(y) \, dy.$$

Here α is a cut off function to $\mathbb{D}^n$. Write $a(x, y) = \alpha(x)\theta_1(y)$ and proceed. If

$$K(x, z) = \int_{\mathbb{R}^n} \exp\Big(i\big\{\lambda\big(\rho(x) - \rho(z)\big) \cdot y + \lambda^2 (x_n - z_n)|y|^2\big\}\Big) a(x, y)\overline{a(z, y)} \, dy,$$

then, letting $S_\lambda = \tilde{R}_\lambda \tilde{R}_\lambda^*$,

$$\begin{aligned} S_\lambda f(x) &= \int_{\mathbb{R}^n} f(z) K(x, z) \psi\left(\frac{x_n - z_n}{\epsilon}\right) dz + \int_{\mathbb{R}^n} f(z) K(x, z) \tilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) dz \\ &= S_\lambda^1 f(x) + S_\lambda^2 f(x), \end{aligned}$$

where $\psi \in C_0^\infty$ is such that $\psi \equiv 1$ near 0 and $\tilde{\psi} = 1 - \psi$.

We find that the frozen operators $(S^1_\lambda)_{x_n z_n}$ have the form

$$(S^1_\lambda)_{x_n z_n} f(x') = \psi\left(\frac{x_n - z_n}{\epsilon}\right)(\tilde{R}_\lambda \tilde{R}^*_\lambda)_{x_n z_n} f(x'). \tag{3.2}$$

For fixed x_n, since $x_n|y|^2$ is a function of y only, we may consider $(\tilde{R}_\lambda)_{x_n y_n}$ as an oscillatory integral operator with phase function $\rho(x', x_n) \cdot (y', y_n)$. Clearly, by the construction of ρ, the mixed Hessian of this phase function is non-degenerate on $\operatorname{supp}(a)$. Consequently by Lemma 3

$$\left\|(\tilde{R}_\lambda)_{x_n y_n} f\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \lambda^{-(n-1)/2} \|f\|_{L^2(\mathbb{R}^{n-1})}. \tag{3.3}$$

By (3.2), (3.3) and Lemma 5,

$$\begin{aligned}\left\|(S^1_\lambda)_{x_n z_n} f\right\|_{L^2(\mathbb{R}^{n-1})} &\lesssim \psi\left(\frac{x_n - z_n}{\epsilon}\right)\left\|(\tilde{R}_\lambda \tilde{R}^*_\lambda)_{x_n z_n} f(x')\right\|_{L^2(\mathbb{R}^{n-1})} \\ &\lesssim \psi\left(\frac{x_n - z_n}{\epsilon}\right)\lambda^{-(n-1)} \|f\|_{L^2(\mathbb{R}^{n-1})}.\end{aligned}$$

It follows then from Lemma 6 and the generalized Young's inequality that

$$\left\|S^1_\lambda f\right\|_2 \lesssim \lambda^{-n+1}\epsilon \|f\|_2 = \lambda^{-n} \|f\|_2, \tag{3.4}$$

if we take $\epsilon = \lambda^{-1}$. In what follows we shall take $\epsilon = \lambda^{-1}$, and in doing so we may assume, given the support properties of $\tilde{\psi}$, that $\lambda|x_n - z_n| \gtrsim 1$.

Now we turn our attention to S^2_λ. Note that

$$\lambda\big(\rho(x) - \rho(z)\big) \cdot y + \lambda^2(x_n - z_n)|y|^2 = \lambda^2(x_n - z_n)|y + F(x, z)|^2 - \frac{|\rho(x) - \rho(z)|^2}{4(x_n - z_n)},$$

where $F(x, z) = \frac{\rho(x)-\rho(z)}{2\lambda(x_n - z_n)}$. Let $A(x, z, y) = a(x, y)\overline{a(z, y)}$ and $\mu = \lambda^2(x_n - z_n)$. Then the kernel of S^2_λ is

$$\tilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) \exp\left(-i\frac{|\rho(x) - \rho(z)|^2}{4(x_n - z_n)}\right) \int_{\mathbb{R}^n} e^{i\mu|y|^2} A\big(x, z, F(x, z)\big)\, dy.$$

Here we have that

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{i\mu|y|^2} A\big(x, z, F(x, z)\big)\, dy \\ &\quad = \left(\frac{i\mu}{\pi}\right)^{-n/2}\left(A\big(x, z, F(x, z)\big) + \int_{\mathbb{R}^n} r_1(i|\xi|^2/4\mu) e^{-i\xi\cdot F(x,z)} \hat{A}(x, z, -\xi)\, d\xi\right),\end{aligned}$$

where r_1 is the remainder term in the first order Taylor expansion of e^x, and $\hat{A}$ denotes the Fourier transform in the last variable. So $(S^2_\lambda)_{x_n z_n}$ is the sum of two terms, $(S^2_\lambda)'_{x_n z_n}$ and $(S^2_\lambda)''_{x_n z_n}$ having kernels $K'(x', z')$ and $K''(x', z')$ respectively. Since

$$K'(x', z') = \left(\frac{i\mu}{\pi}\right)^{-n/2} \tilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) \exp\big(i\lambda'|\rho(x) - \rho(z)|^2\big) A\big(x, z, F(x, z)\big),$$

where $\lambda' = \frac{1}{4(x_n - z_n)}$, we may treat $(S_\lambda^2)'_{x_n z_n}$ as an oscillatory integral operator with phase function $|\rho(x', x_n) - \rho(z', z_n)|^2$ and amplitude $A\big(x, z, F(x, z)\big)$. And although this amplitude function does depend on λ, because $(\lambda|x_n - z_n|)^{-1} \lesssim 1$ we may uniformly bound finitely many z'-derivatives of A. Moreover, since $|x|, |z| \le 2$, then $|\operatorname{supp}_{z'} A| \lesssim 1$. So by Lemma 3,

$$\left\|(S_\lambda^2)'_{x_n z_n} f\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \tilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) \lambda^{-n} |x_n - z_n|^{-1/2} \|f\|_{L^2(\mathbb{R}^{n-1})}. \tag{3.5}$$

Also $(S_\lambda^2)''_{x_n z_n}$ may be treated as an oscillatory integral operator as

$$\begin{aligned} K''(x', z') = (-i\pi)^{n/2} \mu^{-n/2-1} \tilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) \exp\big(i\lambda' |\rho(x) - \rho(z)|^2\big) \\ \times \mu \int_{\mathbb{R}^n} r_1(i|\xi|^2/4\mu) e^{-i\xi \cdot F(x,z)} \hat{A}(x, z, -\xi)\, d\xi. \end{aligned}$$

The phase function is the same as in the previous case, but the amplitude is different. To apply Lemma 3 we must consider z'-derivatives and the volume of the z'-support of this amplitude,

$$\mu \int_{\mathbb{R}^n} r_1(i|\xi|^2/4\mu) e^{-i\xi \cdot F(x,z)} \hat{A}(x, z, -\xi)\, d\xi,$$

and find L^∞ bounds on these quantities which are independent of λ. Since $|x|, |z| \le 2$ when this amplitude does vanish, and by consideration of stationary phase estimates it suffices to show that for $s > n/2$

$$\sum_{|\alpha| \le 2+s} \left| D_\xi^\alpha D_{z'}^\beta \big(e^{-i\xi \cdot F(x,z)} \hat{A}(x, z, -\xi)\, d\xi\big) \right|_{L^2(d\xi)} \lesssim 1,$$

for all $|\beta| \le n$, and this is easily seen to be so given that $|\lambda(x_n - z_n)| \ge 1$. Then Lemma 3 shows that

$$\left\|(S_\lambda^2)''_{x_n z_n} f\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \tilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) \lambda^{-n-2} |x_n - z_n|^{-3/2} \|f\|_{L^2(\mathbb{R}^{n-1})}. \tag{3.6}$$

Using (3.5) and (3.6) and the fact that $|x_n - z_n|\lambda \gtrsim 1$ on $\operatorname{supp} \tilde{\psi}$ we see that

$$\left\|(S_\lambda^2)_{x_n z_n}\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \lambda^{-n} |x_n - z_n|^{-1/2} \|f\|_{L^2(\mathbb{R}^{n-1})}. \tag{3.7}$$

Then Lemma 6 and the generalized Young's inequality imply

$$\|S_\lambda f\|_2 \lesssim \lambda^{-n} \|f\|_2,$$

and this implies (3.1).

4. **Nondegenerate Critical Points.** The case when $\nabla t \ne 0$ represents the easiest to treat using the methods of Theorem 1. When ∇t vanishes, the situation is more complicated. However the case when the Hessian of t is non-singular whenever ∇t vanishes—*i.e.*, t has non-degenerate critical points—is treated below. We limit ourselves to the case when $n = 1$ or $n = 2$. Theorem 2 will follow from Lemma 1 once we prove the following.

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PROPOSITION 2. *Suppose that $t(x)$ has only non-degenerate critical points. Then*

$$\|R_k f\| \le Ck^{n/2} \, \|f\|_2 \, ,$$

when $n = 1$ or $n = 2$.

Before giving the proof of Proposition 2, we state a technical lemma whose proof is given at the end of this section.

LEMMA 7. *Let $n = 1$ or $n = 2$, and suppose that A is an $n \times n$ diagonal matrix whose eigenvalues are ± 1. If $\mathrm{A}(x)$ denotes the quadratic form $\mathrm{A}\, x \cdot x$, then*

$$\sup_{|z|\le 1} \int_{\mathbb{D}^n} \frac{dx}{\big(1 + \lambda^2 |\,\mathrm{A}(x) - \mathrm{A}(z)|\big)^{n/2}} \lesssim \left(\frac{\ln(\lambda)}{\lambda}\right)^n . \tag{4.1}$$

PROOF OF PROPOSITION 2. We know that t only has finitely many isolated critical points in $\mathbb{D}^n$. Away from these critical points $|\nabla t| \ge c > 0$. Near a given critical point we may change variables in such a way that t is a quadratic form. After a partition of unity, an application of Theorem 1 and a change of variables, we may assume that R_k is of the form

$$R_k f(x) = \lambda^{n/2} \int_{\mathbb{R}^n} \exp\Big(i\big[\lambda\rho(x) \cdot y + \lambda^2 A(x)|y|^2\big]\Big) a(x, y) f(y)\, dy,$$

where $\lambda = 2^k$, A is as in Lemma 7, ρ is a C^∞ diffeomorphism and $a \in C_0^\infty(\mathbb{D}^n \times \mathbb{D}^n)$. As always $R_k R_k^*$ has a kernel K of the form

$$K(x, z) = \lambda^n \int_{\mathbb{R}^n} \exp\Big(i\Big[\lambda\big(\rho(x) - \rho(z)\big) \cdot y + \lambda^2\big(\mathrm{A}(x) - \mathrm{A}(z)\big)|y|^2\Big]\Big) a(x, y)\overline{a(z, y)}\, dy.$$

In general $|K(x, z)| \lesssim \lambda^n$, while by stationary phase $|K(x, z)| \lesssim \lambda^n \Big(\lambda^2\big(\mathrm{A}(x) - \mathrm{A}(z)\big)\Big)^{-n/2}$. Then an application of the generalized Young's inequality and Lemma 7 yields the desired result.

We restrict ourselves to the case $n = 1, 2$ because the estimate in (4.1) is no longer valid for larger n. The estimate that one does get for $n \ge 3$ is not good enough to prove results that are better than those already found in [10] and [12].

PROOF OF LEMMA 6. We consider the cases of when $n = 1$ and $n = 2$ separately.

Case 1. $n = 1$.

After a change of variables, $x \mapsto x/\lambda$ it suffices to show that

$$\sup_{|z|\le\lambda} \int_0^\lambda \frac{dx}{(1 + |x^2 - z^2|)^{1/2}} \lesssim \ln(\lambda).$$

We calculate, for fixed $|z| \le \lambda$, that

$$\begin{aligned}\int_0^\lambda \frac{dx}{(1 + |x^2 - z^2|)^{1/2}} &= \int_0^{|z|} \frac{dx}{(1 + z^2 - x^2)^{1/2}} + \int_{|z|}^\lambda \frac{dx}{(1 + x^2 - z^2)^{1/2}} \\ &= \arcsin\left(\frac{z}{\sqrt{1 + z^2}}\right) + \ln\left(\frac{\lambda + \sqrt{1 - z^2 + \lambda^2}}{|z| + 1}\right) \lesssim \ln(\lambda).\end{aligned}$$

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Case 2. $n = 2$ and $A = \pm \mathrm{I}$ (say $A = \mathrm{I}$).

Again we change variables as before, so it suffices to show that

$$\sup_{|z|\le\lambda} \int_0^\lambda \frac{r\,dr}{\left(1 + \left|r^2 - |z|^2\right|\right)} \lesssim \ln(\lambda)^2. \tag{5.3.2}$$

We make a further change of variables, $s = r^2$ so that the left-hand side of (5.3.2) is equal to (modulo a constant factor)

$$\begin{aligned}\int_0^{\lambda^2} \frac{dr}{1 + |r - |z|^2|} &= \int_0^{|z|^2} \frac{dr}{1 + |z|^2 - r} + \int_{|z|^2}^{\lambda^2} \frac{dr}{1 + r - |z|^2} \\ &= \ln(1 + |z|^2) + \ln(1 + \lambda^2 - |z|^2) \lesssim \ln(\lambda).\end{aligned}$$

Case 3. $n = 2$ and $A = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We must consider, where $c = Az \cdot z$,

$$\int_{B(0,1)} \frac{dx\,dy}{1 + \lambda|x^2 - y^2 - c|}.$$

After the change of variables $u = x + y$, $v = x - y$ and a dilation, we may consider

$$\int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} \frac{dx\,dy}{1 + |xy - c|} \quad |c| \le \lambda^2.$$

In fact it is clear that we only have to consider

$$\int_1^{\lambda} \int_1^{\lambda} \frac{dx\,dy}{1 + |xy - c|} \quad |c| \le \lambda^2.$$

By changing variables the above is equal to

$$\int_1^{\lambda} \frac{1}{y} \left(\int_y^{\lambda y} \frac{dx}{1 + |x - c|} \right) dy \lesssim \ln(\lambda)^2,$$

and this completes the proof.

5**. Lower Bounds.** It is not possible that we may always get estimates as in Theorems 1 and 2 for *all* $s > 0$, as Theorem 3 shows. Lemma 2 tells us that we need to find a lower bound for R_k. Here again we take advantage of the similarity between R_k and the general oscillatory integral operator T_λ: we let our phase function be $\phi(x, y) = x \cdot y + \lambda t(x)|y|^2$; the fact that it depends on the parameter λ does not worry us in this case, as Lemma 4 is still applicable.

Theorem 3 follows from Lemma 2 and the next result.

PROPOSITION 3. *There is a smooth function $t(x)$ such that for any $\epsilon > 0$ we may find a constant C_ϵ such that*

$$\|R_k\| \ge C_\epsilon 2^{k/4-\epsilon}.$$

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PROOF. We define a function, τ, of a single variable, r, locally and extend using a standard construction. For $j = 1, 2, \ldots$ let $r_j = 1/j$ and notice that the distance between two consecutive points in this sequence is

$$r_j - r_{j-1} = \frac{1}{j(j+1)} \sim \frac{1}{j^2}.$$

Define

$$\tau_j(r) = 2^{-j}(r - r_j).$$

Let ψ_j be a sequence of C_0^∞ functions with $0 \leq \psi_j \leq 1$ such that $\psi_j \equiv 1$ when $\|r - r_j\| \leq (10j)^{-2}$, and $\operatorname{supp} \phi_i \cap \operatorname{supp} \phi_j$ is empty when $|i - j| \geq 2$. Then $\tau(r) = \sum \psi_j(r)\tau_j(r)$ and $t(x) = \tau(x_n)$.

Make the change of variables $y \longrightarrow 2^k y$. We have to show that

$$\|\tilde{R}_k\| \geq C_\epsilon 2^{k(-n/2+1/4-\epsilon)}$$

where

$$\tilde{R}_k f(x) = \int_{\mathbb{R}^n} e^{i2^k\phi(x,y)}\theta_1(y) f(y)\, dy,$$

and

$$\phi(x, y) = x \cdot y + 2^k t(x)|y|^2.$$

Let $x_k = (0, \ldots, 0, r_k)$, and $y_k = (0, \ldots, 0, -1/2)$ and define

$$\Phi(x, y) = \phi(x, y) - \phi(x_k, y) - \phi(x, y_k) + \phi(x_k, y_k).$$

In the language of Lemma 4, $\phi_1(x) = \phi(x, y_k) + \phi(x_k, y_k)$ and $\phi_2(y) = \phi(x_k, y)$. Let A be the rectangle $|x'| \leq C2^{-k/2}$, $|x_n - r_k| \leq Ck^{-2}$ and let $\tilde{A}$ be the rectangle $|y'| \leq C2^{-k/2}$, $|y_n + 1/2| \leq C2^{-k/2}$, where C is a small (absolute) constant. In this region we have that

$$\Phi(x, y) = x' \cdot y' + (x_n - r_k)(y_n + 1/2)^2.$$

We see on $A \times \tilde{A}$ that $2^k|\Phi(x, y)| < 1/2$ for a proper choice of C. An application of Lemma 4 shows that

$$\|R_k\| \gtrsim \sqrt{|A||\tilde{A}|} \gtrsim 2^{-n/2+1/4}k^{-2} \geq C_\epsilon 2^{k(-n/2+1/4-\epsilon)}.$$

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