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SEMIPRIME RINGS WITH HYPERCENTRAL DERIVATIONS

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ABSTRACT. Let R be a semiprime ring with a derivation d, λ a left ideal of R and k, n two positive integers. Suppose that $[d(x^n), x^n]_k = 0$ for all $x \in \lambda$. Then $[\lambda, R]d(R) = 0$. That is, there exists a central idempotent $e \in U$, the left Utumi quotient ring of R, such that d vanishes identically on eU and $\lambda(1-e)$ is central in $(1-e)U$.

In [16] Posner proved a result on derivations in prime rings. The well-known result states that if R is a prime ring with a nonzero derivation d such that $[x, d(x)] = xd(x) - d(x)x$ is central for all $x \in R$, then R is commutative. Recently, many authors studied Posner's theorem in more generalized versions. Vukman proved in [18] the theorem: Let R be a prime ring, char $R \neq 2, 3$, and d a nonzero derivation of R such that $\big[[d(x), x], x\big]$ is central for all $x \in R$. Then R is commutative. In [7] Deng and Bell generalized Vukman's result to the semiprime case with more weak conditions. More precisely, they proved the theorem: Let R be a semiprime ring with a nonzero derivation d and let λ be a left ideal of R such that $d(\lambda) \neq 0$. Then R must contain a nonzero central ideal if one of the following holds: (i) $\big[[d(x), x], x\big]$ is central for all $x \in \lambda$, where R is a 6-torsion free ring; (ii) $[d(x), x^n]$ is central for all $x \in \lambda$, where n is a fixed positive integer and R is an $n!$-torsion free ring. More related results had been obtained in [5], [6], [14], [12], [13] and [15].

The goal of this paper is to prove a common generalization of these results. To give its statement we first fix some notation. Let R be a ring. For $x, y \in R$, the commutator of x and y, denoted by $[x, y]$, is defined to be $xy - yx$. For each $k \geq 1$, the k-th commutator $[x, y]_k$ is defined inductively as follows: $[x, y]_1 = [x, y]$ and $[x, y]_{k+1} = \big[[x, y]_k, y\big]$. Now the result we want to prove in this paper is the following

MAIN THEOREM. *Let R be a semiprime ring with a derivation d, λ a left ideal of R and k, n two positive integers. Suppose that $[d(x^n), x^n]_k = 0$ for all $x \in \lambda$. Then $[\lambda, R]d(R) = 0$. That is, there exists a central idempotent e of U, the left Utumi quotient ring of R, such that d vanishes identically on eU and $\lambda(1-e)$ is central in $(1-e)U$.*

We remark that the Main Theorem gives a common generalization of the following theorems: [16; Theorem 2], [18; Theorem 1], [7; Theorems 1 and 2], [6; Theorem 1 and Main Theorem 1] and [12; Theorem 1]. Before proceeding the proof of the Main Theorem we first quote two results which will be used in this paper.

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THEOREM A (CHUANG AND LIN, [4]). *Let R be a ring without nonzero nil ideals and let k be a fixed positive integer. If $a \in R$ is such that, to each $y \in R$, there corresponds a positive integer $n = n(a,y)$ such that $[a, y^n]_k = 0$, then a is central.*

THEOREM B (CHUANG AND LEE, [6]). *Let R be a semiprime ring, λ a left ideal of R and d a derivation of R such that $d(x^n)$ is central, $n = n(x) \geq 1$, for all $x \in \lambda$, where the $n(x)'s$ are bounded. Then $[\lambda, R]d(R) = 0$.*

Throughout this paper for a semiprime ring R we denote by $U(R)$ the left Utumi quotient ring of R, by $Q(R)$ the two-sided Martindale quotient ring of R and by $C(R)$ the extended centroid of R. We begin this paper with a version of Theorem A in terms of derivations for the situation when R is a prime ring.

LEMMA 1. *Let R be a prime ring with a nonzero derivation d. Suppose that $[d(x^n), x^n]_k = 0$ for all $x \in R$, where k and n are two fixed positive integers. Then R is commutative.*

PROOF. Note that

$$[d(x), x^n]_{k+1} = \big[x, [d(x^n), x^n]_k\big] = 0 \tag{1}$$

for all $x \in R$. In particular, by (1) for $x, y \in R$ we have

$$[d(x)y + xd(y), (xy)^n]_{k+1} = 0. \tag{2}$$

If d is not $Q(R)$-inner, by Kharchenko's theorem [10] (2) implies that $[xz, (xy)^n]_{k+1} = 0$ for all $x, y, z \in R$. Applying [3] again, we obtain $[xz, (xy)^n]_{k+1} = 0$ for all $x, y, z \in U(R)$. Take $x = 1$. Then $[z, y^n]_{k+1} = 0$ for all $y, z \in R$. Therefore R is a prime PI-ring. In particular, R contains no nonzero nil ideals. Now Theorem A implies that R is commutative as desired. Therefore d can be assumed to be $Q(R)$-inner. That is, there exists an element $b \in Q(R)$ such that $d(x) = [b, x]$ for all $x \in R$. Then by assumption $[b, x^n]_{k+1} = 0$ for all $x \in R$ and hence for all $x \in U(R)$ [3]. Recall that $U(R)$ is also a prime ring. Also, $b \notin C(R)$ since $d \neq 0$. So $U(R)$ contains no nonzero nil ideals by [17, Ex. 7.6.3, p. 287], since $[b, X^n]_{k+1} = 0$ is a nontrivial GPI of $U(R)$. By Theorem A, either $b \in C(R)$ or $U(R)$ is commutative, a contradiction. This completes the proof.

LEMMA 2. *Let R be a prime ring and let λ be a left ideal of R. Suppose that $x^m \in \lambda$ for all $x \in R$, where m is a fixed positive integer. Then λ contains a nonzero ideal of R.*

PROOF. Let A be the additive subgroup of R generated by all elements x^m with $x \in R$. Then by Chuang's theorem [2] either x^m is central for all $x \in R$ or λ contains a noncentral Lie ideal L of R unless $R = M_2\big(\mathrm{GF}(2)\big)$, where GF(2) denotes the Galois field of two elements. Suppose first that x^m is central for all $x \in R$. In this case, R contains no nonzero nil ideals. Therefore by [9; Theorem 3.2.2] R is commutative and hence λ itself is a nonzero ideal of R. Suppose next that λ contains a noncentral Lie ideal L of R. Then by [11; Theorem 12] the subring of R generated by L contains a nonzero ideal of R unless $\operatorname{char} R = 2$ and $\dim_{C(R)} RC(R) = 4$. Thus it remains to check the case $\dim_{C(R)} RC(R) = 4$.

Note that in this case R contains a nonzero central element α. Then $\alpha^m R$ is a nonzero ideal of R contained in λ. Now the proof is complete.

Although the next lemma seems to be well-known, for completeness we will give the result here since it is a crucial step in the proof of the Main Theorem and has no suitable reference in the literature. Recall that the left Utumi quotient ring can be defined for any right faithful ring R (*i.e.*, $Ra = 0$, $a \in R$, implies $a = 0$). Note that a dense left ideal λ of a right faithful ring R itself is a right faithful ring. Moreover, $U(\lambda)$ coincides with $U(R)$. More precisely, there exists a ring isomorphism ϕ from $U(\lambda)$ onto $U(R)$ such that $\phi(x) = x$ for all $x \in \lambda$.

LEMMA 3. *Let R be a right faithful ring and λ be a dense left ideal of R. Suppose that δ is a derivation from λ into R, i.e., δ is an additive mapping from λ into R such that $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in \lambda$. Then δ can be uniquely extended to a derivation of $U(R)$.*

PROOF. Denote by $\phi\colon U(\lambda) \longrightarrow U(R)$ the ring isomorphism from $U(\lambda)$ onto $U(R)$ such that $\phi(x) = x$ for all $x \in \lambda$. Let i be the inclusion mapping from R into $U(R)$. Set $\bar{\delta} = \phi^{-1} \circ i \circ \delta$. Then $\bar{\delta}$ is a derivation from λ into $U(\lambda)$. By [14; Lemma 2] $\bar{\delta}$ can be uniquely extended to a derivation of $U(\lambda)$. We also denote the derivation by $\bar{\delta}$. Then it is clear that $\phi \circ \bar{\delta} \circ \phi^{-1}$ is a unique derivation of $U(R)$ extending the derivation $\delta\colon \lambda \longrightarrow R$. This completes the proof.

We are now in a position to handle the situation when R is a prime ring.

LEMMA 4. *The Main Theorem holds when R is a prime ring.*

PROOF. Assume first that d is an inner derivation of R, that is, there exists $a \in R$ such that $d(x) = [a, x]$ for all $x \in R$. Note that by hypothesis $[a, x^n]_{k+1} = 0$ for all $x \in \lambda$. Expanding it we get $x^{n(k+1)}a \in \lambda$ for all $x \in \lambda$. Consider the set $\lambda_0 = \{x \in \lambda \mid xa \in \lambda\}$, a left ideal of R. We remark that for any $t \in R$, $\lambda t = 0$ if and only if $\lambda_0 t = 0$. Indeed, if $\lambda_0 t = 0$, then $x^{n(k+1)}t = 0$ for all $x \in \lambda$ since $x^{n(k+1)} \in \lambda_0$ for all $x \in \lambda$. By [8; Theorem 2 with $n(x)$ bounded] $\lambda t = 0$ follows. Set $\bar{\lambda}_0 = \lambda_0 / \big(\lambda_0 \cap r_R(\lambda_0)\big)$ and $\bar{\lambda} = \lambda / \big(\lambda \cap r_R(\lambda)\big)$, where $r_R(\lambda_0)$ and $r_R(\lambda)$ denote the right annihilators of λ_0 and λ in R, respectively. It follows from the above that $\bar{\lambda}_0$ is a left ideal of the prime ring $\bar{\lambda}$ such that $\bar{x}^{n(k+1)} \in \bar{\lambda}_0$ for all $\bar{x} \in \bar{\lambda}$. By Lemma 2, $\bar{\lambda}_0$ contains a nonzero ideal of $\bar{\lambda}$. In particular, $\bar{\lambda}_0$ is a dense left ideal of $\bar{\lambda}$. Define the mapping $\bar{d}\colon \bar{\lambda}_0 \longrightarrow \bar{\lambda}$ by $\bar{d}(\bar{x}) = \overline{[a, x]}$ for $x \in \lambda_0$ and $\bar{x} = x + \big(\lambda \cap r_R(\lambda)\big)$. Then $\bar{d}$ is a well-defined derivation from $\bar{\lambda}_0$ into $\bar{\lambda}$ satisfying $[\bar{d}(\bar{x}^n), \bar{x}^n]_k = 0$ for all $\bar{x} \in \bar{\lambda}_0$. By Lemma 3, $\bar{d}$ can be uniquely extended to a derivation of $U(\bar{\lambda})$, denoted by $\bar{d}$ also. Note that ${}_{\bar{\lambda}}\bar{\lambda}_0$ is a dense left submodule of ${}_{\bar{\lambda}}U(\bar{\lambda})$. Applying [14; Theorem 2] we have $[\bar{d}(\bar{x}^n), \bar{x}^n]_k = 0$ for all $\bar{x} \in U(\bar{\lambda})$. Now Lemma 1 implies that either $\bar{d} = 0$ or $[U(\bar{\lambda}), U(\bar{\lambda})] = 0$.

If $\bar{d} = 0$, then in particular $\bar{d}(\bar{\lambda}_0) = 0$, which implies that $\lambda_0[a, \lambda_0] = 0$ or equivalently $\lambda[a, \lambda_0] = 0$. On the other hand, if $[U(\bar{\lambda}), U(\bar{\lambda})] = 0$, then $\lambda[\lambda, \lambda] = 0$. For $x \in \lambda_0$ and $y \in \lambda$ we have $xa \in \lambda$ and hence $\lambda\big((xa)y - y(xa)\big) = 0$. Since $\lambda(xy - yx) = 0$, this formula implies $\lambda x(ay - ya) = 0$. In particular, $x^2[a, y] = 0$ for all $x \in \lambda_0, y \in \lambda$. So

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$x^{2n(k+1)}[a,\lambda] = 0$ for all $x \in \lambda$. Applying Felzenszwalb's theorem [8] again, we obtain $\lambda[a,\lambda] = 0$.

In either case, $\lambda[a,\lambda_0] = 0$ always holds. Let $x \in \lambda$. Using this and $x^{n(k+1)} \in \lambda_0$ to expand $[a,x^{n(k+1)}]_{k+1} = 0$ we yield $[a,x^{n(k+1)}]x^{nk(k+1)} = 0$. On the other hand, $x^{nk(k+1)}[a,x^{n(k+1)}] = 0$. Thus $[a,x^{\ell}] = 0$, where $\ell = n(k+1)^2$. Now Theorem B implies that either a is central or $[\lambda, R] = 0$, which finishes the case.

For the general case we note first that λ and $U(R)\lambda$ satisfy the differential GPIs with coefficients in $U(R)$, since $R\lambda \subseteq \lambda$ and since R and $U(R)$ satisfy the same differential GPIs with coefficients in $U(R)$ by [14; Theorem 3]. Also, d can be uniquely extended to a derivation of $U(R)$. Replacing R and λ by $U(R)$ and $U(R)\lambda$ respectively we may assume that $R = U(R)$. In this case we have $R = Q(R) = U(R)$.

If d is $Q(R)$-inner, then we are done by the inner case. Therefore we may assume that d is not $Q(R)$-inner. We want to apply Kharchenko's theorem to handle this situation. As before, $[d(x),x^n]_{k+1} = 0$ for all $x \in \lambda$. Using the same argument given in the proof of Lemma 1, $[x,(yx)^n]_{k+1} = 0$ for all $x \in \lambda$ and $y \in R$. Applying the inner case we have $[Rx,R][x,R] = 0$, which implies that x is central. That is, λ is central. Then either $\lambda = 0$ or R is commutative. This completes the proof.

PROOF OF THE MAIN THEOREM. For simplicity, let $U = U(R)$. Since $U\lambda$ and λ satisfy the same differential GPIs with coefficients in U, we may assume that $R = U$. Denote by B the complete Boolean algebra of idempotents in $C(R)$ [1]. Fix a maximal ideal Δ of B. Then ΔU is a prime ideal of U invariant under all derivations of U [1]. Let $\bar{d}$ be the canonical derivation of $U/\Delta U$ induced by d. Then $[\bar{d}(\bar{x}^n),\bar{x}^n]_k = 0$ for all $\bar{x} \in \bar{\lambda} = (\lambda + \Delta U)/\Delta U$. Now Lemma 4 implies that $[\bar{\lambda},\bar{U}]d(\bar{U}) = 0$, where $\bar{U} = U/\Delta U$. That is, $[\lambda, U]d(U) \subseteq \Delta U$.

But since $\cap\{\Delta U \mid \Delta \text{ is any maximal ideal of } B\} = 0$ by [1], we obtain $[\lambda, U]d(U) = 0$ as desired. For the rest, it is a matter of routine by the theory of orthogonal completion for semiprime rings that $[\lambda, U]d(U) = 0$ is equivalent to the fact that there exists a central idempotent $e \in U$ such that $U = eU \oplus (1-e)U$ with $d = 0$ on eU and $\lambda(1-e)$ central in $(1-e)U$. This finishes the proof of the Main Theorem.

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